

Dynamical chiral symmetry breaking on a brane in reduced QED

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Reduced gauge theories are theories in which while gauge fields propagate in a bulk, fermion fields are localized on a brane. We study dynamical chiral symmetry breaking on a 2-brane and a 1-brane in reduced QED_{3+1} , and on a 1-brane in reduced QED_{2+1} . Since, unlike higher dimensional gauge theories, QED_{3+1} and QED_{2+1} are well defined, their reduced versions can serve as a laboratory for studying dynamics in a higher dimensional brane world. The analysis of the Schwinger-Dyson (SD) equations in these theories reveals rich and quite nontrivial dynamics in which the conformal symmetry and its breakdown play a crucial role. Explicit solutions of the SD equations in the near-critical regime are obtained and the character of the corresponding phase transition is described.

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I. INTRODUCTION

Dynamics in a brane world has recently attracted considerable interest. In most cases, it has been studied in higher dimensional theories (for a recent review, see Ref. [1]). The aim of this paper is to consider dynamics on a brane not in higher dimensions but in a $(3+1)$ - and $(2+1)$ -dimensional world. More precisely, we study dynamical chiral symmetry breaking in the reduced $(3+1)$ -dimensional QED (QED_{3+1}) and QED_{2+1} . The term “reduced” implies here that while massless gauge fields propagate in a $[(3+1)$ - or $(2+1)$ -dimensional] bulk, fermion fields are localized on a brane. We will consider the cases of a 2-brane and a 1-brane in QED_{3+1} and a 1-brane in QED_{2+1} . We would like also to emphasize that though we use the conventional term “QED” for a $U(1)$ Abelian gauge theory with fermions, we do not specify the origin of the gauge field: it is not necessarily an electromagnetic field.

Motivations for considering this type of models are rather obvious. It is well known that relativistic field models can serve as effective theories for the description of long wavelength excitations in condensed matter systems [2]. The reduced QED describes the situation when while fermions are localized on a plane (say, on a Cu–O plane in a high- T_c superconductor) or on a string (polymer like systems), interactions between them are provided by a bulk gauge field. In addition to that, reduced QED_{3+1} can be relevant for the dynamics of cosmological strings [3]. At last, reduced QED has been recently considered in higher dimensions for the description of the mechanism of (quasi)localization of a photon field on a 3-brane [4].

Another, more practical, reason is using the reduced

QED_{3+1} and QED_{2+1} as a laboratory for studying dynamical chiral symmetry breaking in the brane world. Unlike higher dimensional gauge theories, QED_{3+1} and QED_{2+1} are well defined. While QED_{3+1} is renormalizable and therefore well defined in perturbative theory, QED_{2+1} is superrenormalizable and therefore asymptotically free.

As we will see, the dynamics of chiral symmetry breaking in reduced QED is quite nontrivial. In particular, the conformal symmetry (and its breakdown) plays a crucial role in the dynamics.

The paper is organized as follows. In Sec. II general features of reduced QED are described. In Sec. III we study dynamical chiral symmetry breaking on a 2-brane in reduced QED_{3+1} . In Sec. IV chiral symmetry breaking on a 1-brane in reduced QED_{3+1} is considered. In particular, we discuss subtleties connected with spontaneous breakdown of continuous symmetries on a 1-brane. In Sec. V chiral symmetry breaking in the reduced QED_{2+1} with a 1-brane is studied. In Sec. VI we summarize our results. An analysis of the Schwinger-Dyson equation in the reduced QED_{2+1} with a 1-brane is done in Appendix A.

II. REDUCED QED: GENERAL FEATURES

In this section, general features of the reduced QED will be described. The $\text{QED}_{(D-1)+1}$ action in Euclidean space reads $[X=(x_D, x_1, \dots, x_{D-1})]$

$$S = \int d^D X \left(\frac{1}{4e^2} F_{ab}^2 + A_a J^a - \frac{1}{2e^2 \xi} (\partial_a A^a)^2 \right), \quad (1)$$

where ξ is a gauge parameter and J^a is a fermion current. We will consider the chiral limit (no fermion bare mass term) and, for convenience, consistently omit the kinetic term for fermions in the action, restoring it only when it is necessary. Integrating over A_a , we get

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$$S = \frac{1}{2} \int d^D X d^D Y J^a(X) \tilde{D}_{ab}^{(0)}(X-Y) J^b(Y), \quad (2)$$

where

$$\tilde{D}_{ab}^{(0)} = e^2 \int \frac{d^D K}{(2\pi)^D} \exp(iK(X-Y)) \left(\delta_{ab} - (1-\xi) \frac{K_a K_b}{K^2} \right) \frac{1}{K^2} \quad (3)$$

with $K = (k_D, k_1, \dots, k_{D-1})$. In reduced QED, with a d -brane, we assume that the fermion current has the following form:

$$\begin{aligned} J^a(X) &= 0 \quad \text{for } a = d+1, d+2, \dots, D-1, \\ J^a(X) &= j^a(x_D, x_1, \dots, x_d) \delta^{D-d-1}(\bar{x}) \quad \text{for } a \\ &= D, 1, \dots, d, \end{aligned} \quad (4)$$

where $\bar{x} \equiv (x_{d+1}, \dots, x_{D-1})$. Integrating over \bar{x} and \bar{y} in Eq. (2), we obtain the reduced $(d+1)$ -dimensional action

$$\tilde{S}_{[Dd]eff} = \frac{1}{2} \int d^{d+1} x d^{d+1} y j^\mu(x) D_{[Dd]\mu\nu}^{(0)}(x-y) j^\nu(y), \quad (5)$$

where

$$\begin{aligned} D_{[Dd]\mu\nu}^{(0)}(x-y) &= e^2 \int \frac{d^{d+1} k d^{D-d-1} \bar{k}}{(2\pi)^D} \exp(ik(x-y)) \left(\delta_{\mu\nu} \right. \\ &\quad \left. - (1-\xi) \frac{k_\mu k_\nu}{\bar{k}^2 + k^2} \right) \frac{1}{\bar{k}^2 + k^2} \end{aligned} \quad (6)$$

with $\mu, \nu = D, 1, \dots, d$ (the notation for momenta we use here is self-explanatory).

As it will be shown in the next sections, after integrating over \bar{k} momenta, the effective action can be rewritten in the following general form:

$$\begin{aligned} S_{[Dd]eff} &= \int d^{d+1} x \left[\frac{1}{4e^2} F_{\mu\nu} I(-\partial^2) F^{\mu\nu} + A_\mu j^\mu \right. \\ &\quad \left. + \text{gauge term} \right], \end{aligned} \quad (7)$$

where ∂^2 is the Laplacian in $d+1$ dimensions and $I(-\partial^2)$ is a nonlocal (i.e., integral) operator.

The following properties of the action (7) are noticeable.

(a) The interacting term $A_\mu j^\mu$ is conformally invariant for all D and d . This point will be important for the dynamics of chiral symmetry breaking in reduced QED.

(b) When $d = D - 2$, the kinetic term in expression (7) is finite. However, when $d < D - 2$, there are ultraviolet divergences in it. The reason for that is simple. Because of a delta function in the fermion current (4), integrating over $\bar{x} = (x_{d+1}, \dots, x_{D-1})$ encounters a classical self-energy of a pointlike particle in $D - d - 1$ dimensions. It is finite in the

one-dimensional case ($d = D - 2$) and divergent otherwise. Therefore, when $d < D - 2$, one should regularize the delta function, i.e., introduce a finite thickness for a D -brane. This is an additional source of the breakdown of conformal symmetry. Notice that in the reduced QED_{3+1} with $d = 2$, the kinetic term is both finite and conformally invariant.

(c) Effective action (7) describes fermion fields and a projection of the gauge field on a brane. Since gauge bosons can escape from the brane to the bulk, the unitarity does not fulfill in the brane dynamics. In the next section, we will discuss explicit manifestations of this feature of reduced QED.

III. REDUCED QED_{3+1} WITH A 2-BRANE

In this section we will study spontaneous chiral symmetry breaking in the reduced QED_{3+1} with a 2-brane, i.e., with $D = 4$, $d = 2$ and $\bar{k} = k_3$. Integrating over k_3 in expression (6), we obtain the bare gauge field propagator of an effective $(2+1)$ -dimensional theory on a 2-brane:

$$\begin{aligned} D_{[42]\mu\nu}^{(0)}(x-y) &= \frac{e^2}{2} \int \frac{d^3 k}{(2\pi)^3} \exp(ik(x-y)) \left(\delta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{\sqrt{k^2}}, \end{aligned} \quad (8)$$

where $\mu, \nu = 4, 1, 2$ and, for convenience, we made the substitution $\xi \rightarrow 2\xi - 1$, i.e., $(1-\xi) \rightarrow 2(1-\xi)$. Introducing a $2+1$ vector field $A_\mu(x)$, the effective action (5) can be rewritten in the following form [compare with Eq. (7)]:

$$\begin{aligned} S_{[42]eff} &= \int d^3 x \left[\frac{1}{2e^2} F_{\mu\nu} \frac{1}{\sqrt{-\partial^2}} F^{\mu\nu} + A_\mu j^\mu \right. \\ &\quad \left. + \frac{1}{e^2 \xi} \partial_\mu A^\mu \frac{1}{\sqrt{-\partial^2}} \partial_\nu A^\nu \right]. \end{aligned} \quad (9)$$

One should add the kinetic term of fermions on a 2-brane to this action:

$$S_{kin} = \int d^3 x \bar{\psi} (i \gamma^\mu \partial_\mu) \psi. \quad (10)$$

As is well known, there are two (two-dimensional) inequivalent representations of the Clifford algebra in $2+1$ dimensions. Following Refs. [5–7], we will consider four component fermion fields which contain these two inequivalent representations. In this case, there exists a fermion mass term preserving parity. If there are N_f fermion flavors, the symmetry of the action is $U(2N_f)$ [6,7]. The dynamical generation of a fermion mass leads to spontaneous breakdown of this symmetry down to $U(N_f) \times U(N_f)$.

A remarkable feature of the action (9) is that it is conformal invariant. Since the initial QED theory is renormalizable, one should expect that this feature plays an important role in the dynamics. Our aim is to describe spontaneous chiral symmetry breaking in the theory with this effective

action. Since there is no dimensional parameters in the action (9), a fermion dynamical mass m_d can be induced only through the mechanism of dynamical transmutation. In our case, it means that one should introduce an ultraviolet cutoff Λ , thus breaking the conformal symmetry. Then the dynamical mass, if it arises at all, will be proportional to Λ . We will be especially interested in the near-critical regime of the dynamics, when $m_d \ll \Lambda$.

The dynamical chiral symmetry breaking in this model is a highly nontrivial problem, and our strategy for solving it (at least approximately) will be to find a framework in which the improved ladder (rainbow) approximation would be reliable. We recall that while in the ladder (rainbow) approximation there is only a Schwinger-Dyson (SD) equation for the fermion propagator (with both the vertex and the gauge field propagator being bare), in the improved ladder (rainbow) approximation there are two SD equations (with a bare vertex), both for the fermion and for the gauge field propagators.

The SD equation for the fermion propagator in Minkowski space in the improved ladder approximation has the following form:

$$G^{-1}(p) = G^{(0)-1}(p) + i \int \frac{d^3 q}{(2\pi)^3} \gamma^\mu G(q) \gamma^\nu D_{[42]\mu\nu}(p-q), \quad (11)$$

where $G^{(0)}(p)$ is the bare fermion propagator and

$$D_{[42]\mu\nu} = \left(g_{\mu\nu} - (1 - \xi(k^2)) \frac{k_\mu k_\nu}{k^2} \right) D(k^2) \quad (12)$$

is the full gauge field propagator for which there is its own SD equation (with a bare vertex in this approximation). Here we use a general nonlocal gauge with $\xi(k^2)$ being a function of k^2 [a need for considering such gauges will soon become clear: see Eq. (21) below]. The bare gauge field propagator is now [compare with Eq. (8)]

$$D_{[42]\mu\nu}^{(0)} = \left(g_{\mu\nu} - (1 - \xi(k^2)) \frac{k_\mu k_\nu}{k^2} \right) \frac{e^2}{2\sqrt{-k^2}} \quad (13)$$

and the full gauge field propagator is related to the vacuum polarization tensor $\Pi_{\mu\nu}(k)$:

$$D_{[42]\mu\nu}^{-1}(k) = D_{[42]\mu\nu}^{(0)-1}(k) + \Pi_{\mu\nu}(k),$$

$$\Pi_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k^2). \quad (14)$$

The structure of the propagator $G(p^2)$ is $G(p^2) = (A(p^2)\hat{p} - B(p^2))^{-1}$, $\hat{p} \equiv \gamma^\mu p_\mu$, and from Eqs. (11) and (12) we obtain the following equations for $A(p^2)$ and $B(p^2)$ in the Euclidean space ($p_0 = ip_4$):

$$A(p^2) = 1 + \frac{1}{p^2} \int \frac{d^3 q}{(2\pi)^3} \frac{A(q^2)}{q^2 A^2(q^2) + B^2(q^2)} D[(p-q)^2] \times \left(pq + (1 - \xi[(p-q)^2]) \left(pq - \frac{2(p^2 q^2 - (pq)^2)}{(p-q)^2} \right) \right), \quad (15)$$

$$B(p^2) = \int \frac{d^3 q}{(2\pi)^3} \frac{B(q^2)}{q^2 A^2(q^2) + B^2(q^2)} D[(p-q)^2] (2 + \xi[(p-q)^2]). \quad (16)$$

Notice that the function $D(k^2)$ is expressed through the vacuum polarization function $\Pi(k^2)$ as

$$D(k^2) = \frac{1}{\frac{2k}{e^2} + \Pi(k^2)}. \quad (17)$$

The full gauge field propagator (12) satisfies its own SD equation and therefore is in principle a complicated functional of the functions $A(p^2)$ and $B(p^2)$. Fortunately, in the present case the situation can be considerably simplified. First of all, as we will see below, one can choose a gauge in which the function $A(p^2)$ is identically equal to 1, and we will use such a gauge. Second, it will be shown that, in the near-critical regime (when $m_d \ll \Lambda$), the fermion dynamical mass, defined as $m_d = B(m_d^2)$, is mainly induced in the kinematic region with $m_d^2 \ll k^2$. In that region, fermions can be treated as massless, and, if $A(p^2) = 1$, the polarization function is given by the one loop expression with the fermion propagators of free, $(2+1)$ -dimensional, massless fermions.

For completeness and convenience, however, we will use the one loop expression for $\Pi(k^2)$ taking free fermions with the mass $m = m_d$. On a 2-brane, i.e., in $2+1$ dimensions, it is

$$\Pi(k^2) = \frac{N_f}{4\pi} \left[2m_d + \frac{k^2 - 4m_d^2}{k} \arctan \frac{k}{2m_d} \right]. \quad (18)$$

Notice that

$$\Pi(k^2) \rightarrow \frac{N_f k}{8} \quad (19)$$

for $k \gg m_d$, and

$$\Pi(k^2) \rightarrow \frac{N_f k^2}{6\pi m_d} \quad (20)$$

for $k \ll m_d$.

When can the improved ladder approximation be reliable? The simplest case would be of course the dynamics with a small coupling constant $\alpha = e^2/4\pi$ (notice that α is a bare coupling constant here). In that case, even the ladder (rain-

bow) approximation would be good enough. Unfortunately, as it will be shown below, for small α there is no solution with spontaneous chiral symmetry breaking in the reduced QED₃₊₁ with a 2-brane. Therefore one should try something else.

Our initial observation is that the structure of SD equations (15) and (16) is similar to that in usual, nonreduced, QED₂₊₁. It had been recognized long ago that the $1/N_f$ expansion can be useful in that theory [6–9]. Though being very nontrivial, the $1/N_f$ expansion is helpful in putting under control of nonperturbative dynamics. The crucial point is the selection of a “right” gauge in the leading order in $1/N_f$, in which the improved ladder approximation would be reliable [10]. In particular, appropriate Ward identities have to be satisfied in that gauge. In other gauges, the results can be found by gauge-transforming Green’s functions from the “right” gauge to those gauges. Such a transformation in general changes the initial improved ladder approximation to another one, though the gauge invariant quantities remain of course the same. We will adopt this strategy for the present problem and, first of all, check the Ward identity for the vertex. Since in this approximation, by definition, the vertex is bare, the function $A(p^2)$ has to be equal one. It is known [11], that for the full photon propagator (12), and in arbitrary d space dimensions, this function is identically equal to 1 if one uses a nonlocal (in general) gauge with the following gauge function $\xi(k^2)$:

$$\xi(z) = d - \frac{d(d-1)}{z^d D(z)} \int_0^z dt t^{d-1} D(t). \quad (21)$$

We will see that, in the near-critical regime, the momentum region mostly responsible for the mass generation is $k \gg m_d$. As it follows from Eq. (19), in that region $\Pi(k^2) = N_f k/8$, i.e., the function $D(k^2)$ (17) is proportional to k^{-1} . For such a function $D(k^2)$ and $d=2$, one gets $\xi(k^2) = 2/3$ (the so-called Nash gauge [9]), and the gap equation takes the form

$$B(p^2) = 4\pi^2 \lambda \int \frac{d^3 q}{(2\pi)^3} \frac{B(q^2)}{q^2 + B^2(q^2)} \frac{1}{\sqrt{(p-q)^2}}, \quad (22)$$

$$\lambda = \frac{e^2}{3\pi^2 \left(1 + \frac{N_f e^2}{16}\right)}.$$

The validity of the Ward identity is a necessary but not of course sufficient condition for the reliability of the improved ladder approximation. The crucial point for that is a justification of the use of a bare vertex. This approximation for the vertex can be justified in the leading order of the $1/N_f$ expansion [6–10].

Integrating over angles in Eq. (22), we obtain

$$B(p^2) = \lambda \int_0^{\Lambda^2} dq^2 \frac{\sqrt{q^2} B(q^2)}{q^2 + B^2(q^2)} \frac{\sqrt{2}}{\sqrt{p^2 + q^2 + |p^2 - q^2|}}. \quad (23)$$

Here the ultraviolet cutoff Λ was introduced.

Notice that in the momentum region $q^2 \gg m_d^2 \equiv B^2(m_d^2)$, the term $B^2(q^2)$ in the denominator of the integrand of expression (23) is irrelevant. The only role of this term is to provide a cutoff in the infrared region. Therefore one can drop this term, introducing an explicit infrared cutoff in the integral. Then we obtain the following equation ($x = p^2, y = q^2$):

$$B(x) = \lambda \int_{m_d^2}^{\Lambda^2} \frac{dy}{y^{1/2}} B(y) \left[\frac{\theta(x-y)}{\sqrt{x}} + \frac{\theta(y-x)}{\sqrt{y}} \right]. \quad (24)$$

The transition from Eq. (23) to Eq. (24) corresponds to the so called bifurcation approximation (or method). For the problem of dynamical symmetry breaking, this method was introduced in Ref. [12] and since then has been widely used in the literature (for a review see Ref. [13]). This method is especially appropriate for the near-critical dynamics: the closer the dynamics is to a critical (bifurcation) point, the smaller the dynamical mass m_d , and therefore the term $B^2(q^2)$ in the denominator of the integrand (23), become.

It is easy to check that the integral equation (24) is equivalent to the differential equation

$$x^2 B'' + \frac{3x}{2} B' + \frac{\lambda}{2} B = 0 \quad (25)$$

with the following two boundary conditions:

$$B'(m_d^2) = 0, \quad (26)$$

$$(2xB' + B)|_{x=\Lambda^2} = 0. \quad (27)$$

A solution of Eq. (25) which satisfies the infrared boundary condition (26) is

$$B(x) = \frac{m_d^{3/2}}{x^{1/4} \sinh \delta} \sinh \left(\frac{\omega}{4} \log \frac{x}{m_d^2} + \delta \right), \quad (28)$$

where $\omega = \sqrt{1-8\lambda}$ and $\delta = \frac{1}{2} \log[(1+\omega)/(1-\omega)]$, and here we also used the normalization condition $B(m_d^2) = m_d$. The ultraviolet boundary condition (27) yields the following equation for the dynamical mass:

$$\tanh \left(\frac{\omega}{2} \log \frac{\Lambda}{m_d} + \delta \right) = -\omega. \quad (29)$$

Obviously, there is no solution $m_d \ll \Lambda$ for $\lambda < \lambda_{cr} = 1/8$. For supercritical values of λ ($\lambda > \lambda_{cr}$), Eq. (29) takes the form

$$\tan \left(\frac{\nu}{2} \log \frac{\Lambda}{m_d} + \arctan \nu \right) = -\nu, \quad (30)$$

where $\nu = \sqrt{8\lambda - 1}$. Therefore for small ν the mass is

$$m_d \approx \Lambda \exp \left[-\frac{2\pi}{\nu} + 4 \right]. \quad (31)$$

The critical line in the plane (N_f, e^2) is given by

$$e_{cr}^2 = \frac{16}{N_{max} - N_f}, \quad (32)$$

where $N_{max} = 128/3\pi^2$. Spontaneous chiral symmetry breaking takes place for $e > e_{cr}$, and the value N_{max} defines the upper limit for the number of fermion flavors N_f for which spontaneous chiral symmetry breaking is possible. When $N_f \rightarrow N_{max}$, the critical value $e_{cr}^2 \rightarrow \infty$.

Let us now discuss self-consistency of the assumption that the region of momenta $q \gg m_d$ is mostly responsible for the generation of the mass in the near-critical regime ($m_d \ll \Lambda$). The point is that in this regime $\ln \Lambda/m_d \sim 2\pi/\nu$ is large. On the other hand, the behavior of the integrand on the right-hand side of Eq. (23) is smooth as $q^2 \rightarrow 0$. The smooth behavior of the integrand in the infrared region implies that the region $0 \leq q \leq m_d$ is too small to generate the large logarithm $\ln \Lambda/m_d$. It [and therefore the essential singularity in expression (31)] is generated in the large region $m_d \leq q \leq \Lambda$. A variation of the kernel in the infrared region can at most change the overall coefficient in that expression. This heuristic argument is supported by numerical studies of integral equation (23).

The critical line (32) implies that it is a strong coupling dynamics, with $e^2 > e_{cr}^2$, that provides spontaneous chiral symmetry breaking on a brane. Indeed, the lowest e_{cr}^2 corresponds to $N_f = 1$ and it is $e_{cr}^2 \approx 4.81$, i.e., $\alpha_{cr} \equiv e_{cr}^2/4\pi \approx 0.38$.

This strong coupling dynamics is provided by (essentially) conformal invariant interactions in the most important region of momenta $m_d \leq q \leq \Lambda$. Indeed, up to the irrelevant $B^2(q^2)$ term in the denominator, the kernel of integral equation (22) transforms as $K(p^2, q^2) \rightarrow K(s^2 p^2, s^2 q^2) = s^{-3} K(p^2, q^2)$ under the scale transformation $p, q \rightarrow sp, sq$. This, together with the transformation of the measure $d^3 q \rightarrow s^3 d^3 q$, implies that the interactions are indeed (essentially) conformal invariant in that region. [In the integral equation (24), the conformal symmetry is broken only by the dimensional boundary parameters Λ and m_d in the integral.] This reflects the presence of long range, Coulomb like, interactions which provide the essential singularity in expression (31).

The critical line (32) corresponds to the so called conformal phase transition (CPT) introduced in Ref. [14]. There are the following characteristic features of the CPT.

(a) Unlike the conventional Ginzburg-Landau (GL) phase transition, a parameter governing the phase transition in the CPT is connected with a marginal (i.e., renormalizable) operator [in the GL phase transition, such a parameter is connected with a relevant (i.e., superrenormalizable) operator; it is usually a mass term].

(b) Though the CPT is a continuous phase transition, there is an abrupt change of the spectrum of light excitations at a critical point (line). This is unlike the GL phase transition where the spectrum is continuous at a critical point (line).

In the present model, the parameter governing the phase transition is the coupling constant e . It is connected with the marginal operator $j_\mu A^\mu$. The spectrum of the light excitations is discontinuous at the critical line (32). Indeed, in the

subcritical region, with massless fermions, there is a Coulomb, conformally invariant, phase describing interactions of massless fermions and gauge bosons. In the supercritical region, with massive fermions, there are a lot of bound states, including $2N_f^2$ Nambu-Goldstone bosons corresponding to spontaneous breakdown of the $U(2N_f)$ to $U(N_f) \times U(N_f)$. Therefore these two criteria of the CPT are indeed realized in this model.

Notice that though QED_{3+1} is a renormalizable theory, there are new, nonperturbative, divergences in the supercritical phase [see Eq. (31)]. These divergences are connected *not* with introducing a 2-brane of vanishing thickness in the model but with the strong coupling dynamics. As is well known, such divergences occur in the strong coupling phase of QED_{3+1} in the absence of any brane [15–17]. They lead to breakdown of the conformal symmetry (nonperturbative scale anomaly).

It is instructive to compare the reduced QED_{3+1} with a 2-brane with the conventional QED_{2+1} . The SD equations in these two models are similar. The difference is in the form of the gauge field propagator. Instead of expression (17), one has [7,8]

$$D(k^2) = \frac{1}{\frac{k^2}{e_3^2} + \Pi(k^2)}, \quad (33)$$

where e_3 is the (dimensional) coupling constant in QED_{2+1} and $\Pi(k^2)$ is the (same) polarization function (18). The appearance of the term k^2/e_3^2 , instead $2k/e^2$, makes quite a difference. On the one hand, it provides a dynamical ultraviolet cutoff $\sim e_3^2$ in the SD equation and, on the other hand, since this term is suppressed in the region $k \ll e_3^2$, it does not contribute to the fermion dynamical mass. This implies reducing screening of Coulomb like interactions as compared to the reduced QED_{3+1} with a 2-brane. Indeed, the dynamical mass in QED_{2+1} is [8–10]

$$m_{3d} \sim e_3^2 \exp\left(-\frac{2\pi}{\nu_3}\right), \quad (34)$$

where $\nu_3 = \sqrt{8\lambda_3 - 1}$ with $\lambda_3 = 16/3\pi^2 N_f$. The parameter ν_3 coincides with ν in Eq. (31) only in the limit $e^2 \rightarrow \infty$, i.e., in the limit of maximally strong interactions in the reduced QED_{3+1} with a 2-brane.¹

Therefore we conclude that there are important similarities and important differences between the dynamics in QED_{2+1} and reduced QED_{3+1} with a 2-brane. Both dynamics are intimately connected with long range Coulomb like interactions. Both dynamics provide a realization of the con-

¹The critical value of N_f is $N_f^{cr} = 128/3\pi^2 \approx 4.32$ in QED_{2+1} . Since this result was obtained in the framework of the $1/N_f$ expansion, there may be some concern about its reliability [18]. Although it would be too strong to say that this issue has been finally resolved, different studies indicate that $1/N_f$ corrections are small for N_f around 4 [9,10].

formal phase transition. In particular, like in the reduced QED₃₊₁, there is an abrupt change of the spectrum of light excitations at the critical point $N_f = N_f^{cr}$ in QED₂₊₁ [19]. On the other hand, since QED₂₊₁ is superrenormalizable (and therefore asymptotically free) theory, there is no (nonperturbative) ultraviolet divergence in the dynamical mass. Also, its dynamics is more effective in generating a fermion mass in that it corresponds to the dynamics in the reduced QED₃₊₁ when the coupling constant e of the latter goes to ∞ .

This point is intimately connected with the violation of the unitarity in the brane theory. Indeed, because of the first term in the denominator of expression (17), there is an imaginary part for *all* time like momenta k in the gauge field propagator (12), *independently* of the value of the mass m_d . This feature reflects the process of escaping of a gauge boson from the brane to the bulk. This ‘‘instability’’ of brane gauge bosons leads to an effective reduction of interactions on the brane. Only in the limit $e \rightarrow \infty$ the gauge bosons are localized on the brane, i.e., become ‘‘stable.’’

IV. REDUCED QED₃₊₁ WITH A 1-BRANE

In this section we will consider the dynamics in the reduced QED₃₊₁ with a 1-brane, i.e., with $D=4$ and $d=1$. As it was pointed out in Sec. II, there are (classical) ultraviolet divergences in the theory with a 1-brane of vanishing thickness in this case. Because of that, one needs to introduce a finite thickness for the brane, which will play a role of a regularization parameter.

To get the reduction $3+1 \rightarrow 1+1$, we perform integration in Eq. (2) with the sources taken as

$$\begin{aligned} J^a(X) &= 0 \quad \text{for } a=2,3, \\ J^a(X) &= j^a(x_4, x_1) f(x_2) f(x_3) \quad \text{for} \\ & a=0,1, \end{aligned} \quad (35)$$

where the regularization function

$$f(x) = \sqrt{\frac{a}{\pi}} \exp(-ax^2), \quad f(x) \rightarrow \delta(x), \quad a \rightarrow \infty. \quad (36)$$

Integrating over x_2, x_3 and y_2, y_3 in Eq. (2), we obtain the reduced (1+1)-dimensional action (5) with the bare gauge field propagator

$$\begin{aligned} D_{[41]\mu\nu}^{(0)}(x-y) &= e^2 \int \frac{d^2 k dk_2 dk_3}{(2\pi)^4} \exp\left(ik(x-y) - \frac{k_2^2 + k_3^2}{2a}\right) \\ & \times \left(\delta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2 + k_2^2 + k_3^2} \right) \frac{1}{k^2 + k_2^2 + k_3^2} \\ & = \int \frac{d^2 k}{(2\pi)^2} \exp(ik(x-y)) D_{[41]\mu\nu}^0(k), \\ & \mu, \nu = 4, 1, \end{aligned} \quad (37)$$

where $k = (k_4, k_1)$, $k^2 = k_4^2 + k_1^2$. It is

$$D_{[41]\mu\nu}^{(0)}(k) = \left[\delta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \cdot k^2 \frac{d}{dk^2} \right] D^{(0)}(k^2), \quad (38)$$

where

$$D^{(0)}(k^2) = -\frac{e^2}{4\pi} \exp\left(\frac{k^2}{2a}\right) \text{Ei}\left(-\frac{k^2}{2a}\right) \quad (39)$$

and $\text{Ei}(-x)$ is the integral exponential function.

By introducing a 1+1 gauge field A_μ , we obtain an effective (1+1)-dimensional action:

$$\begin{aligned} S_{[41]eff} &= \int d^2 x \frac{1}{4e^2} F_{\mu\nu} \frac{1}{-\partial^2 D^{(0)}(-\partial^2)} F^{\mu\nu} + A_\mu j^\mu + \frac{1}{2e^2} \\ & \times \partial_\mu A^\mu \frac{1}{-\partial^2 [D^{(0)}(-\partial^2) + (1-\xi)\partial^2 D^{(0)' }(-\partial^2)]} \\ & \times \partial_\nu A^\nu, \end{aligned} \quad (40)$$

where $D^{(0)' }(-\partial^2) = D^{(0)' }(x)|_{x=-\partial^2}$. If there were no need for a regularization, this effective action would be conformal invariant. The finite thickness of the 1-brane breaks the conformal symmetry. We will return to this point below.

One should add the kinetic term of fermions on a 1-brane to the action (40):

$$S_{kin} = \int d^2 x \bar{\psi} (i \gamma^\mu \partial_\mu) \psi. \quad (41)$$

We will consider N_f two component (i.e., vectorlike) fermions. The chiral group is $U(N_f)_L \times U(N_f)_R$. Our aim is to find whether a fermion dynamical mass is generated in the theory with effective action (40). Naively, one might expect that in this case the chiral symmetry $U(N_f)_L \times U(N_f)_R$ would be spontaneously broken down to its vector subgroup $U(N_f)_V$. However, this is not the case in 1+1 dimensions. Because of the Mermin-Wagner-Coleman (MWC) theorem [20], there cannot be spontaneous breakdown of continuous symmetries in 1+1 dimensions. The MWC theorem is based on the fact that gapless Nambu-Goldstone bosons cannot exist in 1+1 dimensions. It however does not prevent a generation of a fermion mass. In this case, the so-called Berezinski-Kosterlitz-Thouless phase would realize [21,22]. We will return to this point at the end of this section.

As in the previous section, we will change the initial gauge in such a way that the full gauge propagator takes the form

$$\begin{aligned} D_{[41]\mu\nu}(k) &= \left[\delta_{\mu\nu} - (1-\xi(k^2)) \frac{k_\mu k_\nu}{k^2} \right] D(k^2) \equiv \left[\delta_{\mu\nu} - (1 \right. \\ & \left. - \xi(k^2)) \frac{k_\mu k_\nu}{k^2} \right] \frac{1}{D_0^{-1}(k^2) + \Pi(k^2)}. \end{aligned} \quad (42)$$

Our aim is to find such a function $\xi(k^2)$ that the fermion wave function $A(k^2)$ would be identically equal to 1 in the improved ladder approximation. Fortunately, as Eq. (21) shows, for $d=1$ the choice of $\xi=1$, i.e., the Feynman gauge, provides $A(k^2)=1$ for *any* gauge field propagator. Then the SD equation for the fermion mass function takes the following form in the improved ladder approximation:

$$B(p^2) = \frac{2}{(2\pi)^2} \int \frac{d^2 k B(k^2)}{k^2 + B^2(k^2)} D((p-k)^2). \quad (43)$$

As was shown in the previous section, in the reduced QED₃₊₁ with a 2-brane, there is no generation of the fermion mass for a small coupling constant α , and we used the $1/N_f$ expansion in order to justify the use of the improved ladder approximation. As we will see below, the situation in the reduced QED₃₊₁ with a 1-brane is different: there may exist a solution with a nonzero m_d even for an arbitrary small α . For such a α , one can expect that even the ladder approximation is justifiable.

Using the same arguments as in the reduced QED₃₊₁ with a 2-brane, one can show that in the present case the momentum region yielding a dominant contribution to SD equation (43) is $m_d \ll k \ll \Lambda$ [we will see below that the parameter $a^{1/2}$ (the inverse thickness of the 1-brane) plays the role of an ultraviolet cutoff Λ]. Therefore, like in the previous section, for the vacuum polarization function one can use the one-loop expression with the propagators for free massive fermions with the mass m_d . In 1+1 dimensions, this expression is

$$\Pi(k^2) = \frac{N_f}{\pi} \left[1 - \frac{2m_d^2}{\sqrt{k^2(k^2+4m_d^2)}} \ln \frac{\sqrt{k^2+4m_d^2} + \sqrt{k^2}}{\sqrt{k^2+4m_d^2} - \sqrt{k^2}} \right]. \quad (44)$$

The asymptotics of this expression are

$$\Pi(k^2) \rightarrow \frac{N_f k^2}{6\pi m_d^2}, \quad \text{for } k^2 \ll m_d^2, \quad (45)$$

$$\Pi(k^2) \rightarrow \frac{N_f}{\pi}, \quad \text{for } k^2 \gg m_d^2. \quad (46)$$

From Eqs. (39) and (46) we find that for momenta $k^2 \gg 2a$ the photon propagator (42) rapidly decreases:

$$D(k^2) \sim \frac{1}{\frac{4\pi k^2}{2e^2 a} + \frac{N_f}{\pi}} \sim \frac{2ae^2}{4\pi k^2}. \quad (47)$$

Therefore the whole integrand in SD equation (43) rapidly decreases for $k^2 \gg 2a$. Because of that, one can neglect the region of those large momenta and put the cutoff $\Lambda_a^2 = 2a$ or, more precisely, $\Lambda_a^2 = 2a \exp(-\gamma)$, with γ the Euler constant, in the SD equation.

This also implies that one can keep the leading order term in expansion of $D^{(0)}$ (39) in k^2/a :

$$D_0(k^2) \simeq \frac{e^2}{4\pi} \log \frac{2a \exp(-\gamma)}{k^2} \equiv \frac{e^2}{4\pi} \log \frac{\Lambda_a^2}{k^2}. \quad (48)$$

From here and Eqs. (42), (45), and (46) we find that

$$D^{-1}(k^2) \simeq \frac{4\pi}{e^2 \log \frac{\Lambda_a^2}{k^2}}, \quad \text{at } k^2 \leq m_d^2,$$

$$D^{-1}(k^2) \simeq \frac{4\pi}{e^2 \log \frac{\Lambda_a^2}{k^2}} + \frac{N_f}{\pi}, \quad \text{at } k^2 \geq m_d^2. \quad (49)$$

Now we proceed at solving the SD equation. In order to get a hint of the character of the solution, we will first consider the so-called a constant mass approximation, taking the external momentum being equal to zero and replacing the running mass function in the integrand by its value $m_d = B(0)$. Then we get the equation

$$1 = \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 + m_d^2} \frac{2}{\frac{4\pi}{e^2 \log \frac{\Lambda_a^2}{k^2}} + \frac{N_f}{\pi}}. \quad (50)$$

The main contribution comes from the range of momenta $k^2 \gg m_d^2$. Therefore one can omit the term m_d^2 in denominator and put instead the parameter m_d^2 as the lower limit in the integral. Then the integral can be easily evaluated and one gets the following algebraic transcendental equation:

$$1 = \frac{1}{2N_f} \left\{ \log \frac{\Lambda_a^2}{m_d^2} - \frac{4\pi^2}{e^2 N_f} \log \left[1 + \frac{e^2 N_f}{4\pi^2} \log \frac{\Lambda_a^2}{m_d^2} \right] \right\}. \quad (51)$$

Introducing the variable $y = (e^2 N_f / 4\pi^2) \log(\Lambda_a^2 / m_d^2)$, it can be rewritten as

$$\frac{e^2 N_f^2}{2\pi^2} = y - \log(1+y). \quad (52)$$

The function on the right-hand side of this equation is monotonically increasing, starting from zero value at $y=0$ and going to ∞ as $y \rightarrow \infty$. Therefore this equation always has a solution. However, the character of the solution depends on the value of the parameter $e^2 N_f^2 / 2\pi^2$. Indeed, one gets

$$m_d^2 \simeq \Lambda_a^2 \exp\left(-\frac{4\pi}{e}\right), \quad \text{for } \frac{e^2 N_f^2}{2\pi^2} \ll 1, \quad (53)$$

$$m_d^2 \approx \Lambda_a^2 \left(\frac{e^2 N_f^2}{2\pi^2} \right)^{-4\pi^2/e^2 N_f} \times \exp(-2N_f), \quad \text{for } \frac{e^2 N_f^2}{2\pi^2} \gg 1. \quad (54)$$

Notice that solution (53) corresponds to a weak coupling regime. It does not depend on N_f and therefore comes from the range of momenta in the integral equation where one can neglect the vacuum polarization, i.e., one can use the ladder (rainbow) approximation in this case.

A closer look at Eq. (51) reveals that the solution (53) emerges from the range of momenta where a double logarithmic contribution dominates in the gap equation (50). On the other hand, the solution at large N_f [Eq. (54)] comes from the region of momenta generating a one logarithmic contribution.

We will turn now at studying SD equation (43) for the running mass function. In order to integrate over the angle variable there, we will use the following conventional approximation for the vector boson propagator [23] (for a review see Ref. [13]):

$$D((p-k)^2) \approx D(p^2)\theta(p^2-k^2) + D(k^2)\theta(k^2-p^2). \quad (55)$$

This approximation is justifiable: the measure of the only ‘‘dangerous’’ (for this approximation) region, with $|p^2| \approx |k^2|$, is small and the dependence of the propagator $D((p-k)^2)$ on the angular variable is rather smooth.

Neglecting then the term $B^2(k^2)$ in the denominator and instead putting the infrared cutoff $m_d^2 \equiv B^2(m_d^2)$ in the integral (the bifurcation approximation discussed in the previous section), one gets a simple integral equation. It is easy to show that it is equivalent to a differential equation with two [infrared (IR) and ultraviolet (UV)] boundary conditions:

$$B''(x) - \frac{D''(x)}{D'(x)} B'(x) - \frac{D'(x)}{2\pi x} B(x) = 0, \quad x = p^2, \quad (56)$$

$$B'(x)|_{x=m_d^2} = 0,$$

$$[D(x)B'(x) - D'(x)B(x)]|_{x=\Lambda_a^2} = 0. \quad (57)$$

It is convenient to introduce the variable $z = (e^2 N_f / 4\pi^2) \log(\Lambda_a^2/x)$ in terms of which Eq. (56) becomes

$$B''(z) - \frac{D''(z)}{D'(z)} B'(z) - \frac{2\pi}{e^2 N_f} D'(z) B(z) = 0,$$

$$D(z) = \frac{\pi}{N_f} \frac{z}{z+1}. \quad (58)$$

Together with the boundary conditions, it can be rewritten as

$$B''(z) + \frac{2}{z+1} B'(z) + \frac{2\pi^2}{e^2 N_f^2 (z+1)^2} B(z) = 0, \quad (59)$$

$$B'(z)|_{z=(e^2 N_f / 4\pi^2) \log(\Lambda_a^2/m_d^2)} = 0,$$

$$[zB'(z) - B(z)]|_{z=0} = 0. \quad (60)$$

The solution $B(z)$ satisfying the UV boundary condition and the normalization $B(x=m_d^2) = m_d$ is given by

$$B(z) = m_d \left(\frac{z_0+1}{z+1} \right)^{1/2} \frac{\sinh\left[\frac{\omega}{2} \log(z+1)\right]}{\sinh\left[\frac{\omega}{2} \log(z_0+1)\right]},$$

$$\omega = \sqrt{1 - \frac{8\pi^2}{e^2 N_f^2}}, \quad (61)$$

where $z_0 \equiv z(x=m_d^2)$. The IR boundary condition leads to the equation for the dynamical mass:

$$\tanh\left[\frac{\omega}{2} \log(z_0+1)\right] = \omega. \quad (62)$$

For real ω ($e^2 N_f / 8\pi^2 > 1$) it can be easily solved:

$$m_d^2 \approx \Lambda_a^2 \exp(-N_f \Sigma(\omega)),$$

$$\Sigma(\omega) = \left[\left(\frac{1+\omega}{1-\omega} \right)^{1/\omega} - 1 \right] \frac{1-\omega^2}{2}. \quad (63)$$

For large $e^2 N_f$ this solution becomes

$$m_d^2 \approx \Lambda_a^2 \left(\frac{e^2 N_f^2}{2\pi^2} \right)^{-8\pi^2/e^2 N_f} \exp(-2N_f), \quad (64)$$

and coincides, up to minor difference in preexponential factor, with expression (54) obtained in the constant mass approximation.

The line $\alpha \equiv e^2/4\pi = 2\pi/N_f^2$ divides the region in plane (α, N_f) in two parts with different dependence of a dynamical mass on α and N_f . Indeed, at $\alpha < 2\pi/N_f^2$, when $\omega = i\nu$, $\nu = \sqrt{8\pi^2/e^2 N_f^2 - 1}$, Eq. (62) gives

$$m_d^2 \approx \Lambda_a^2 \exp\left(-\frac{\pi^2 \sqrt{2}}{e}\right) = \Lambda_a^2 \exp\left(-\pi \sqrt{\frac{\pi}{2\alpha}}\right). \quad (65)$$

The ratio of powers of two exponents in Eqs. (65) and (53) is $\pi/2\sqrt{2} \approx 1.11$ that shows that the constant mass approximation in this case is also rather reliable. It is peculiar that the expression (65) for a dynamical mass coincides with the expression for a dynamical mass generated by a magnetic field in quenched QED₃₊₁ [see Eq. (111) in Ref. [24]]. In fact, in the ladder (rainbow) approximation, used in the weak coupling regime, the present SD equation essentially coincides

with the SD equation in that paper (see especially Appendix C there). The origin of this similarity is in the dimensional reduction $3+1 \rightarrow 1+1$ in the dynamics of spontaneous chiral symmetry breaking in a magnetic field [24].

The existence of the two types solutions, corresponding to the weak ($\bar{e}^2 \equiv e^2 N_f \ll 1$) and the strong ($\bar{e}^2 \gg 1$) coupling regimes, is intriguing. While the strong coupling solution essentially coincides with that in the $(1+1)$ -dimensional Thirring model (see below), the weak coupling one yields a new type solution, characteristic for a 1-brane physics in a $(3+1)$ -dimensional bulk. These two solutions are generated by very different dynamics: while in the strong coupling regime the gauge field propagator is dominated by the 1-brane vacuum polarization operator, in the weak coupling one the propagator is dominated by the bare term coming from the bulk. In particular, while the polarization operator is generated by the conformal invariant interaction $j_\mu A^\mu$, the bare term breaks the conformal symmetry as result of a finite thickness $1/a^{1/2}$ of a 1-brane. We will argue below that this point can be important in the connection with the MWC theorem.

As we already stated above, there cannot be spontaneous breakdown of a continuous symmetry in $1+1$ dimensions (the MWC theorem) [20]. This happens because strong fluctuations of would be NG modes lead to vanishing order parameter connected with such a breakdown. Let us recall how this theorem is realized in the case of the $(1+1)$ -dimensional Thirring model with the color group $U(N_c)$ and the chiral symmetry $U(N_f)_L \times U(N_f)_R$. It is relevant for our case since the dynamics of the strong coupling solution found above essentially coincides with the dynamics of the Thirring model with the color group $U(1)$. Indeed, since for this solution the gauge field propagator is dominated by the 1-brane vacuum polarization function, which is essentially constant in this case [see Eq. (46)], the interaction is of a current \times current form, as in the Thirring model.

First of all, using the Fierz identities, it is easy to show that, in $1+1$ dimensions, the Thirring model is equivalent to the Gross-Neveu (GN) model [25]. The interaction term of the latter is

$$S_{int}^{GN} = \int d^2x \frac{G}{2} [(\bar{\psi} \lambda^s \psi)^2 + (\bar{\psi} \lambda^s i \gamma^5 \psi)^2], \quad (66)$$

where λ^s are flavor matrices, $s=0,1,\dots,N_f^2-1$, and the summation over s and color indices of the fermion fields is assumed. The λ^s matrices are normalized according to $\text{tr}(\lambda^s \lambda^k) = 2 \delta^{sk}$.

Let us first consider the case of the $U(1)_L \times U(1)_R$ chiral group. In this case the model is soluble [26]. There is a nonzero dynamical mass for fermions for all $N_c \geq 2$. However, there is *no* NG boson in the model. Instead of that, there is a Berezinski-Kosterlitz-Thouless (BKT) gapless mode. This mode is described by the exponent field $U(x) = \exp(i\theta(x))$, where θ satisfies the constraint $0 \leq \theta(x) < 2\pi$. More precisely, the BKT mode is described by a usual Lagrangian density of a massless free field, $f/2(\partial_\mu \theta \partial^\mu \theta)$ with $f \approx N_c/4\pi$. However, the corresponding observables are

described not by Green's functions of the field θ but by Green's functions of the field $U(x)$ and its derivatives, including the derivative $\partial_\mu \theta = iU \partial_\mu U^\dagger$. The point is that while the propagator and other Green's functions of the $\theta(x)$ field do not exist in $1+1$ dimensions (they are divergent for all x), Green's functions of the $U(x)$ field and its derivatives are well defined. Moreover, the corresponding field theory is conformal invariant and the parameter f defines anomalous dimensions of its Green's functions.

The case of the GN model with one color is special. For $N_c=1$ and the chiral group $U(1)_L \times U(1)_R$, fermions are massless and, moreover, the bosonization of the model leads exactly to the Lagrangian of the free massless BKT mode [27]. Therefore in this particular case, the *whole* dynamics is conformal invariant.

Though the dynamics with $N_f \geq 2$ is more involved, some of the basic points described above survive. In this case one should distinguish the $U(1)_L \times U(1)_R$ and the $SU(N_f)_L \times SU(N_f)_R$ sectors. The dynamics in the first one is essentially the same as in the model with $N_f=1$ and one should expect that while for $N_c \geq 2$ fermions are massive, they become massless for $N_c=1$. In the second sector, because of a strong self-interaction between N_f^2-1 would be NG bosons, all of them acquire a (same) mass, thus leading to a Wigner realization of the dynamics with the exact $SU(N_f)_L \times SU(N_f)_R$ symmetry [28].

In our case the number of colors $N_c=1$. Does it necessarily imply that the dynamical mass of fermions will disappear in the exact solution in the reduced QED_{3+1} with a 1-brane? We do not think that the situation is so simple. First of all, even the status of the Goldstone theorem is not completely clear in this case: some of the assumptions the theorem is based on are violated in the brane world. Indeed, in the initial bulk theory, the $(D-1)+1$ Lorentz invariance is broken because of the presence of a d-brane. On the other hand, while on a d-brane the $d+1$ Lorentz symmetry is preserved, the corresponding effective theory in nonlocal. Second, as was emphasized above, in the $(1+1)$ -dimensional Thirring (or Gross-Neveu) model, it is important that the conformal symmetry is exact in the sector with the BKT field $U(x)$. On the other hand, in the reduced QED_{3+1} with a 1-brane, the conformal symmetry is necessarily broken by a finite thickness of the brane. The latter is especially important for the weak coupling solution (65) in which the gauge field propagator is dominated by the bare term (48) which explicitly breaks the conformal symmetry. The dynamics described by that solution is very different from that of the Thirring model.

It remains a challenge to clarify these various issues in the brane dynamics.

V. REDUCED QED_{2+1} WITH A 1-BRANE

In this section we will study spontaneous chiral symmetry breaking in the reduced QED_{2+1} with a 1-brane, i.e., with $D=3$ and $d=1$. Recall that the gauge coupling constant is dimensional in $2+1$ dimensions: its dimension is $[e_3] = m^{1/2}$, and we will see that the parameter $e_3^2 N_f$ plays the role of an ultraviolet cutoff, which is a typical feature for QED_{2+1} [6,7]. Notice also that as it follows from the discus-

sion in Sec. II, there is no need for introducing a finite thickness for a 1-brane in a $(2+1)$ -dimensional bulk.

With trivial modifications, the effective action can be derived as in the case of the reduced QED $_{3+1}$ with a 2-brane. It is

$$S_{[32]eff} = \int d^2x \left[\frac{1}{2e_3^2} F_{\mu\nu} \frac{1}{\sqrt{-\partial^2}} F^{\mu\nu} + \bar{\psi} (i\gamma^\mu \partial_\mu) \psi + A_{\mu j}{}^\mu + \frac{1}{e_3^2 \xi} \partial_\mu A^\mu \frac{1}{\sqrt{-\partial^2}} \partial_\nu A^\nu \right]. \quad (67)$$

As in the previous section, we will consider N_f two component fermion fields [see Eq. (41)]. The chiral group is $U(N_f)_L \times U(N_f)_R$.

The full photon propagator in a nonlocal gauge is given by

$$D_{\mu\nu}(k) = \left[\delta_{\mu\nu} - (1 - \xi(k^2)) \frac{k_\mu k_\nu}{k^2} \right] \frac{1}{\frac{2k}{e_3^2} + \Pi(k^2)}, \quad (68)$$

where the vacuum polarization function $\Pi(k^2)$ is given in Eq. (44). As was shown in Sec. IV, the convenient choice of the gauge for the study of fermion dynamics on a 1-brane is $\xi=1$. Then the SD equation for the fermion mass function takes the form of Eq. (43) with the function $D(k)$ given now by

$$D(k) = \frac{1}{\frac{2k}{e_3^2} + \Pi(k^2)}. \quad (69)$$

Like in the case of the reduced QED $_{3+1}$ with a 1-brane, the bare term $2k/e_3^2$ breaks the conformal symmetry. However, this bare term is very different from that one in Eq. (50). Its strong dependence on momentum implies that it is important both in the infrared and ultraviolet regions. Taking into account this term and the asymptotics of the vacuum polarization function $\Pi(k^2)$ (45) and (46), one concludes that the dominant, logarithmic, contribution to the SD equation should come from the range of momenta $12\pi m_d^2/e_3^2 N_f < k < e_3^2 N_f 2\pi$.

An analysis of this SD equation is done in the Appendix. It is shown there that a solution with a nonzero dynamical mass exists for all values of N_f and e_3^2 . It is also shown that the dynamical mass satisfies the following constraint:

$$\frac{N_f e_3^2}{\pi} \exp(-2N_f) \lesssim m_d \lesssim \frac{N_f e_3^2}{2\sqrt{6}\pi} \exp\left(-\frac{N_f}{7}\right). \quad (70)$$

In the case when $N_f \gg 1$, the dynamical mass is

$$m_d \approx \frac{N_f e_3^2}{2\sqrt{6}\pi} \exp\left[-\left(N_f + \frac{1}{8} + \gamma - 3 \log 7\right)\right]. \quad (71)$$

It is interesting that, unlike the previous model with $D=4$ and $d=1$, in this model the constant mass approximation is unreliable. In particular, it is not difficult to show (see Appendix A) that it would yield the following expression for the dynamical mass in the case $N_f \gg 1$:

$$m_d \approx \frac{N_f e_3^2}{2\sqrt{6}\pi} \exp\left(-\frac{N_f}{7}\right), \quad (72)$$

which is very different from expression (71). The reason for that is that, unlike the previous model, the bare term in the propagator (69) (now strongly depending on momentum) does not decouple even in the dynamical regime with $N_f \gg 1$.

We would like also to add that all remarks made in Sec. IV concerning the status of the problem of the fermion mass generation on a 1-brane, in particular, its connection with the Mermin-Wagner-Coleman theorem, are also relevant for the present case.

VI. CONCLUSION

The dynamics of chiral symmetry breaking in reduced QED is rich and quite nontrivial. Its characteristic features are intimately connected with the structure of the gauge field propagator. It includes two terms: the vacuum polarization function, completely defined by the brane dynamics, and the ‘‘bare’’ term coming from the bulk. The vacuum polarization function is connected with the conformal invariant term $j_\mu A^\mu$. Therefore, since in $1+1$ and $2+1$ dimensions there are no divergences in the polarization function, it is conformal invariant for massless fermions. This feature essentially survives in the near-critical regime of chiral symmetry breaking: in this regime, a fermion dynamical mass m_d is small and the dominant region is that with momenta $k \gg m_d$.

On the other hand, in many cases, the bare term breaks the conformal symmetry: either because of a finite thickness of a brane or because an initial bulk theory (as QED $_{2+1}$) is not conformal invariant. The interplay between those two dynamical sources provides rich nonperturbative dynamics.

In this paper, the improved rainbow approximation (with a bare vertex) was used. It would be interesting to study the dynamics beyond this approximation, though it is not straightforward at all. The point is that, besides the bare spin structure γ_μ , the vertex can have other ones. For example, in the case of a 2-brane, there are in principle 11 other structures. The crucial point in the present analysis is decoupling of the Schwinger-Dyson equations. Therefore the role of the gauge where the function $A(p^2)=1$ is very important. As it is discussed in Sec. III, for the bare vertex, one can find such a gauge for any vector boson propagator. However, beyond the approximation with the bare vertex, new structures in the vertex can appear. In order to find these structures, one either should consider the equation for the vertex (that is quite complicated) or try to construct an ansatz for the vertex consistent with such general constraints as Ward identities, the absence of kinematic singularities, the correct perturbative limit, etc. This last approach was successful in $(3+1)$ -dimensional QED [30]. However, studies of this

problem in $(2+1)$ -dimensional QED (which is similar to the dynamics on a 2-brane) have revealed that it is a hard (and still unresolved) problem [31]. We hope to turn to this problem elsewhere.

At last, we would like to indicate that this analysis can be useful for studying dynamical chiral symmetry breaking in higher dimensional brane theories [32,33]. In this connection, it is noticeable that in Ref. [33] the consequences of the existence of a ultraviolet stable fixed point in higher dimensional gauge theories were considered.

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APPENDIX: ANALYSIS OF THE GAP EQUATION FOR THE REDUCED QED₂₊₁ WITH A 1-BRANE

In this appendix we analyze the SD equation for the case of the QED₂₊₁ with a 1-brane. The equation has the form

$$B(p^2) = \frac{2}{(2\pi)^2} \int \frac{d^2k B(k^2)}{k^2 + B^2(k^2)} D((p-k)^2), \quad (\text{A1})$$

where

$$D((p-k)^2) = \frac{1}{\frac{2\sqrt{(p-k)^2}}{e_3^2} + \Pi((p-k)^2)},$$

and the vacuum polarization function $\Pi(k^2)$ is given in Eq. (44).

We will first obtain the constraint (70) for the dynamical mass. We begin by deriving the lower limit for m_d . As was already indicated in Sec. V, the dominant contribution to SD equation (A1) comes from the range of momenta $\mu < k < \Lambda$ with infrared and ultraviolet cutoffs given by $\mu = 12\pi m_d^2/N_f e_3^2$, $\Lambda = e_3^2 N_f / 2\pi$. Since the kernel of this integral equation is positive (corresponding to an attractive interaction), we obviously obtain a lower limit for m_d if integrate only over this range of momenta and, furthermore, replace $D((p-k)^2)$ in the kernel by its minimal value in this interval. Taking into account Eq. (55), one finds that the minimal value is $\pi/2N_f$. Then the gap equation becomes simple:

$$B(p^2) = \frac{2}{(2\pi)^2} \int_{\mu}^{\Lambda} \frac{d^2k B(k^2)}{k^2 + B^2(k^2)} \frac{\pi}{2N_f}. \quad (\text{A2})$$

It has the following solution:

$$B(p^2) = m_d \simeq \frac{e_3^2 N_f}{\pi} \exp(-2N_f). \quad (\text{A3})$$

Since the initial interaction is stronger, the true m_d is larger than the value (A3).

Let us find an estimate from above for the dynamical mass. To do this, we consider the integral equation at $p^2 = 0$. It is

$$B(0) = \frac{2}{(2\pi)^2} \int \frac{d^2k B(k^2)}{k^2 + m_d^2} D(k^2, m_d^2), \quad (\text{A4})$$

and we explicitly indicated the dependence of the interaction kernel $D(k^2, m_d^2)$ on the dynamical mass m_d^2 . Equation (A4) is equivalent to

$$1 = \frac{2}{(2\pi)^2} \int \frac{d^2k f(k^2)}{k^2 + m_d^2} D(k^2, m_d^2), \quad (\text{A5})$$

where $f(k^2) = B(k^2)/B(0)$. It follows from the gap equation that $B'(p^2) < 0$, i.e., $B(p^2)$ is a decreasing function of p^2 . Therefore, $f(k^2) < 1$ in Eq. (A5) for $k^2 > 0$. In the case of the constant mass approximation [where $B(k^2)$ is a constant $B(k^2) = M$] the square mass M^2 satisfies the following gap equation:

$$1 = \frac{2}{(2\pi)^2} \int \frac{d^2k}{k^2 + M^2} D(k^2, M^2). \quad (\text{A6})$$

By using asymptotics (45) and (46) for the vacuum polarization function, the integration region in Eq. (A6) is divided into two regions $k \lesssim M\sqrt{6}$ and $k \gtrsim M\sqrt{6}^2$ that gives us the following gap equation:

$$1 = \frac{1}{\pi} \left[\int_0^{M\sqrt{6}} \frac{dk}{k^2 + M^2} \frac{1}{\frac{2}{e_3^2} + \frac{N_f}{6\pi} \frac{k}{M^2}} + \int_{M\sqrt{6}k^2 + M^2}^{\infty} \frac{dk}{k^2 + M^2} \frac{k}{\frac{2k}{e_3^2} + \frac{N_f}{\pi}} \right]. \quad (\text{A7})$$

We can further neglect k^2 term in comparison to M^2 in the first integral in Eq. (A7), while in the second one we can neglect M^2 in comparison to k^2 . Evaluating the integrals, we come to the following expression:

$$1 = \frac{6}{N_f} \log \left(1 + \frac{N_f e_3^2}{2\sqrt{6}\pi M} \right) + \frac{1}{N_f} \log \left(1 + \frac{N_f e_3^2}{2\sqrt{6}\pi M} \right). \quad (\text{A8})$$

²The value $M\sqrt{6}$ here was determined from matching small and large k asymptotics of the vacuum polarization function.

The corresponding solution for a small dynamical mass ($M \ll e_3^2$) is

$$M \simeq \frac{N_f e_3^2}{2\sqrt{6}\pi} \exp\left(-\frac{N_f}{7}\right). \quad (\text{A9})$$

It is obviously valid for $N_f \gg 1$.

Let us prove that $m_d^2 < M^2$, where m_d^2 is the solution of the gap equation with the running mass function, by assuming the opposite and then showing that it leads to a contradiction.

So let us assume that $m_d^2 > M^2$ and consider the integral

$$\int \frac{d^2 k f(k^2)}{k^2 + m_d^2} D(k^2, m_d^2). \quad (\text{A10})$$

Since $f(k^2) < 1$, we have

$$\int \frac{d^2 k f(k^2)}{k^2 + m_d^2} D(k^2, m_d^2) < \int \frac{d^2 k}{k^2 + m_d^2} D(k^2, m_d^2). \quad (\text{A11})$$

By calculating

$$I(m_d^2) = \int \frac{d^2 k}{k^2 + m_d^2} D(k^2, m_d^2),$$

one can show that $I'(m_d^2) < 0$, i.e., $I(m_d^2)$ is a decreasing function of m_d^2 . Since we assumed that $m_d^2 > M^2$, we have

$$\int \frac{d^2 k}{k^2 + m_d^2} D(k^2, m_d^2) < 1$$

and, consequently, we obtain that

$$\int \frac{d^2 k f(k^2)}{k^2 + m_d^2} D(k^2, m_d^2) < 1 \quad (\text{A12})$$

for all $m_d^2 > M^2$. Then, since we cannot satisfy equation (A5) with $m_d^2 > M^2$, the assumption that $m_d^2 > M^2$ leads to a contradiction. Therefore, we get the inequality $m_d^2 < M^2$ with M given in Eq. (A9). This and the lower limit we obtained earlier lead us to constraint (70).

Can one get an explicit solution of the integral equation (A1) in a reliable approximation? The answer is affirmative.

To solve Eq. (A1), we use approximation (55) in order to be able to perform integration over angles. Since we already know that the main (logarithmic) contribution comes from the range of momenta $12\pi m_d^2/N_f e_3^2 < k < e_3^2 N_f/2\pi$, we put infrared and ultraviolet cutoffs in the integral equation at $\mu = 12\pi m_d^2/N_f e_3^2$ and $\Lambda = e_3^2 N_f/2\pi$, respectively. Then the integral equation (A1) takes the form

$$B(p) = \frac{1}{\pi} \left[D(p) \int_{\mu}^p \frac{dk B(k)}{k^2 + m_d^2} + \int_p^{\Lambda} \frac{dk B(k) D(k)}{k^2 + m_d^2} \right]. \quad (\text{A13})$$

Furthermore, we approximate the function $D(p)$ on the interval $\mu < p < \Lambda$ as

$$D(p) = \theta(p_m - p) \frac{6\pi m_d^2}{N_f p^2} + \theta(p - p_m) \frac{1}{\frac{2p}{e_3^2} + \frac{N_f}{\pi}}, \quad (\text{A14})$$

where the parameter p_m is determined from the condition of continuity of $D(p)$ at the point $p = p_m$ [$p_m = 6\pi m_d^2/N_f e_3^2 + \sqrt{(6\pi m_d^2/N_f e_3^2)^2 + 6m_d^2} \approx m_d \sqrt{6}$].

It is convenient to represent the mass function as

$$B(p) \equiv B_i(p) \theta(p_m - p) + B_u(p) \theta(p - p_m). \quad (\text{A15})$$

For ‘‘infrared’’ B_i and ‘‘ultraviolet’’ B_u (with respect to the parameter p_m) parts of the mass function, we get the following equations:

$$B_i(p) = \frac{6m_d^2}{N_f p^2} \int_{\mu}^p \frac{dk B_i(k)}{k^2 + m_d^2} + \int_p^{p_m} \frac{dk B_i(k)}{k^2 + m_d^2} \cdot \frac{6m_d^2}{N_f k^2} + \frac{1}{\pi} \int_{p_m}^{\Lambda} \frac{dk B_u(k)}{k^2 + m_d^2} \cdot \frac{1}{\frac{2k}{e_3^2} + \frac{N_f}{\pi}}, \quad (\text{A16})$$

$$B_u(p) = \frac{1}{\pi} \left[\frac{1}{\frac{2p}{e_3^2} + \frac{N_f}{\pi}} \int_{p_m}^p \frac{dk B_u(k)}{k^2 + m_d^2} + \int_p^{\Lambda} \frac{dk B_u(k)}{k^2 + m_d^2} \cdot \frac{1}{\frac{2k}{e_3^2} + \frac{N_f}{\pi}} \right] + \frac{1}{\pi} \frac{1}{\frac{2p}{e_3^2} + \frac{N_f}{\pi}} \int_{\mu}^{p_m} \frac{dk B_i(k)}{k^2 + m_d^2}. \quad (\text{A17})$$

Taking the derivatives on the both sides of these equations, we get

$$B_i'(p) = \frac{1}{\pi} \left[-\frac{12\pi m_d^2}{N_f p^3} \int_{\mu}^p \frac{dk B_i(k)}{k^2 + m_d^2} \right], \quad (\text{A18})$$

$$B_u'(p) = -\frac{1}{\pi} \frac{2}{e_3^2} \frac{1}{\left(\frac{2p}{e_3^2} + \frac{N_f}{\pi}\right)^2} \left[\int_{\mu}^{p_m} \frac{dk B_i(k)}{k^2 + m_d^2} + \int_{p_m}^p \frac{dk B_u(k)}{k^2 + m_d^2} \right]. \quad (\text{A19})$$

Differentiating the last equations one more time we obtain

$$B_i''(p) + \frac{3}{p} B_i'(p) + \frac{12m^2}{N_f} \frac{B(p)}{p^2(p^2 + m_d^2)} = 0, \quad (\text{A20})$$

$$B_u''(p) + \frac{2}{p + \frac{e_3^2 N_f}{2\pi}} B_u'(p) + \frac{e_3^2}{2\pi} \frac{p B_u(p)}{\left(p + \frac{e_3^2 N_f}{2\pi}\right)^2 (p^2 + m_d^2)} = 0. \quad (\text{A21})$$

We have also the following IR and UV boundary conditions:

$$B_i'(p)|_{p=\mu} = 0, \quad [(p + \Lambda) B_u(p)]'|_{p=\Lambda} = 0, \quad (\text{A22})$$

where the prime denotes the derivative with respect to p . Furthermore, the mass function is continuous at the point p_m , therefore, $B_i(p_m) = B_u(p_m)$ and the first derivatives satisfy

$$B_u'(p_m) = \frac{6\pi m_d^2}{N_f e_3^2 p_m} B_i'(p_m) \quad (\text{A23})$$

[the condition of continuity and Eq. (A23) follow from Eqs. (A16), (A17) and Eqs. (A18), (A19), respectively].

The general solution of Eq. (A20) is given in terms of hypergeometric functions:

$$B(p^2) = C_1 \left(\frac{m_d^2}{p^2}\right)^{(1-\omega)/2} F\left(\frac{-1+\omega}{2}, \frac{1+\omega}{2}, 1+\omega; -\frac{p^2}{m_d^2}\right) + C_2 \left(\frac{m_d^2}{p^2}\right)^{(1+\omega)/2} F\left(-\frac{1+\omega}{2}, \frac{1-\omega}{2}, 1-\omega; -\frac{p^2}{m_d^2}\right), \quad (\text{A24})$$

where $\omega = \sqrt{1 - 12/N_f}$. The IR boundary condition gives a relation between the constants C_1 and C_2

$$C_1 (1-\omega) \left(\frac{\mu}{m_d}\right)^\omega F\left[\frac{1+\omega}{2}, \frac{1+\omega}{2}; 1+\omega; -\left(\frac{\mu}{m_d}\right)^2\right] + C_2 \left(1 + \omega\right) \left(\frac{\mu}{m_d}\right)^{-\omega} F\left[\frac{1-\omega}{2}, \frac{1-\omega}{2}; 1-\omega; -\left(\frac{\mu}{m_d}\right)^2\right] = 0, \quad (\text{A25})$$

where we used the formula for differentiating the hypergeometric function [29]

$$\frac{d^n}{dz^n} [z^{a+n-1} F(a, b; c; z)] = (a)_n z^{a-1} F(a+n, b; c; z). \quad (\text{A26})$$

Since for $B_u(p)$ the corresponding momenta are larger than $m_d (p \geq p_m > m_d)$, we approximate $p^2 + m_d^2$ by p^2 in Eq. (A21). This gives us

$$B_u''(p) + \frac{2}{p + \frac{e_3^2 N_f}{2\pi}} B_u'(p) + \frac{e_3^2}{2\pi} \frac{B_u(p)}{p \left(p + \frac{e_3^2 N_f}{2\pi}\right)^2} = 0. \quad (\text{A27})$$

Introducing the variable $z = -2\pi p / N_f e_3^2$ and making the substitution

$$B_u(z) = f(t), \quad \frac{z}{z-1} = t, \quad (\text{A28})$$

Eq. (A27) reduces to the hypergeometric differential equation

$$t(1-t)f''(t) + \frac{1}{N_f} f'(t) = 0. \quad (\text{A29})$$

A solution regular at zero is

$$f_1(t) = t F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 2; t\right), \quad \nu = \sqrt{1 + \frac{4}{N_f}}, \quad (\text{A30})$$

and the second independent solution is [29]

$$f_2(t) = (1-t) F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 2; 1-t\right). \quad (\text{A31})$$

Thus the general solution for the mass function is

$$B_u(z) = C_3 \frac{z}{1-z} F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 2; \frac{z}{z-1}\right) + C_4 \frac{1}{1-z} F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 2; \frac{1}{1-z}\right). \quad (\text{A32})$$

The UV boundary condition (A22), which can be rewritten as

$$\frac{d}{dz} [(1-z)B(z)]|_{z=-1} = 0, \quad (\text{A33})$$

allows us to fix the ratio C_3/C_4

$$\frac{C_3}{C_4} = \frac{F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 2; \frac{1}{2}\right) - F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 1; \frac{1}{2}\right)}{F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 1; -\frac{1}{2}\right) + F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 2; -\frac{1}{2}\right)}, \quad (\text{A34})$$

where the formula for differentiating the hypergeometric function

$$\frac{d^n}{dz^n} [z^{c-1} F(a, b; c; z)] = (c-n)_n z^{c-1-n} F(a, b; c-n; z) \quad (\text{A35})$$

has been used. For $N_f \gg 1$ we have $C_3/C_4 \approx 1/8N_f$.

Finally, matching the solutions $B_i(p)$ and $B_u(p)$ at the point $p = p_m \approx m\sqrt{6}$, we obtain two other equations for the constants C_1, C_2, C_3, C_4 :

$$C_1 6^{(\omega-1)/2} F\left(\frac{-1+\omega}{2}, \frac{1+\omega}{2}; 1+\omega; -6\right) + C_2 6^{-(1+\omega)/2} F\left(-\frac{1+\omega}{2}, \frac{1-\omega}{2}; 1-\omega; -6\right) = \left[C_3 \frac{z}{1-z} F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 2; \frac{z}{z-1}\right) + C_4 \frac{1}{1-z} F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 2; \frac{1}{1-z}\right) \right] \Big|_{z=-\mu/p_m}, \quad (\text{A36})$$

$$C_1 (1-\omega) 6^{(\omega-1)/2} F\left(\frac{1+\omega}{2}, \frac{1+\omega}{2}; 1+\omega; -6\right) + C_2 (1+\omega) 6^{-(1+\omega)/2} F\left(\frac{1-\omega}{2}, \frac{1-\omega}{2}; 1-\omega; -6\right) = \frac{1}{3(1-z)^2} \left[C_3 F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 1; \frac{z}{z-1}\right) + C_4 F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; 1; \frac{1}{1-z}\right) \right] \Big|_{z=-\mu/p_m}. \quad (\text{A37})$$

The determinant of the set of homogeneous equations (A25), (A34), (A36), and (A37) gives an equation for determining the dynamical mass. Since we look for a solution with $\mu/p_m \ll 1$, we can simplify the equations for C_i by using the corresponding formulas for hypergeometrical functions [29]. Finally, we obtain the following equation for the dynamical mass:

$$\sqrt{1-\omega^2} A(\omega) \frac{\sinh\left[\omega \left(\log \frac{p_m}{\mu} + \delta_1(\omega)\right)\right]}{\sinh\left[\omega \left(\log \frac{p_m}{\mu} + \delta_2(\omega)\right)\right]} = \frac{1}{3} \frac{C_3}{C_4} \Gamma\left(\frac{3-\nu}{2}\right) \Gamma\left(\frac{3+\nu}{2}\right) - \frac{1}{3N_f} \log\left(\frac{p_m \exp(h_0'')}{\mu}\right), \quad (\text{A38})$$

where

$$A(\omega) = \left[\frac{F\left(\frac{1+\omega}{2}, \frac{1+\omega}{2}; 1+\omega; -6\right) F\left(\frac{1-\omega}{2}, \frac{1-\omega}{2}; 1-\omega; -6\right)}{F\left(\frac{-1+\omega}{2}, \frac{1+\omega}{2}; 1+\omega; -6\right) F\left(-\frac{1+\omega}{2}, \frac{1-\omega}{2}; 1-\omega; -6\right)} \right]^{1/2}, \quad (\text{A39})$$

$$\delta_1(\omega) = \frac{1}{2\omega} \log \frac{F\left(\frac{1+\omega}{2}, \frac{1+\omega}{2}; 1+\omega; -6\right)}{F\left(\frac{1-\omega}{2}, \frac{1-\omega}{2}; 1-\omega; -6\right)}, \quad (\text{A40})$$

$$\delta_2(\omega) = \frac{1}{2\omega} \log \frac{(1+\omega) F\left(\frac{-1+\omega}{2}, \frac{1+\omega}{2}; 1+\omega; -6\right)}{(1-\omega) F\left(-\frac{1+\omega}{2}, \frac{1-\omega}{2}; 1-\omega; -6\right)}, \quad (\text{A41})$$

and the constant h_0'' is

$$h_0'' = 2\psi(1) - \psi\left(\frac{1+\nu}{2}\right) - \psi\left(\frac{1-\nu}{2}\right). \quad (\text{A42})$$

In the limit of large $N_f \gg 1$ the last equation can be solved explicitly and we find

$$m_d = \frac{N_f e^{\frac{2}{3}}}{2\sqrt{6}\pi} \exp\left[-\left(N_f + \frac{1}{8} + \gamma - 3 \log 7\right)\right], \quad (\text{A43})$$

and we used that $F(1,1;2; -6) = \log 7/6$ (note also that $\frac{1}{8} + \gamma - 3 \log 7 \approx -5.14$). It is obvious from comparison with Eqs. (A3) and (A9) that our solution (A43) satisfies the estimates from below and above, which we obtained earlier.

Notice that up to the preexponential factor the dependence of this solution on N_f coincides with the corresponding dependence of the (strongly coupling) solution (54) in the case of the reduced QED₃₊₁ with a 1-brane. The cause of that is the fact that, when $N_f \gg 1$, in both cases the gauge field propagators are dominated by the 1-brane vacuum polariza-

tion function, which is the same. All the information about extra dimensions (like the number of dimensions, geometry, etc.) is contained in the preexponential factor.

The reason why the gauge field propagator is dominated by the 1-brane vacuum polarization function in the reduced QED₂₊₁ is rather subtle. An analysis of the gap equation (A1) for the running mass function shows that its nontrivial solution is formed by momenta on the interval $(m_d\sqrt{6}, \Lambda)$: this equation with the low ultraviolet cutoff $m_d\sqrt{6}$ does not have a nontrivial solution. In the limit $N_f \gg 1$, the vacuum polarization dominates on the interval $(m_d\sqrt{6}, \Lambda)$, and the equation reduces to a simple Gross-Neveu like equation whose solution is $m_d \sim \exp(-N_f)$.

Notice that this is not true for the constant mass approximation, where a nontrivial mass is generated even for the low ultraviolet cutoff $m_d\sqrt{6}$. Therefore, the constant mass approximation in this case gives a different result for the dynamical mass (A9) than the correct solution for the running mass function (A43). This is unlike the case of the reduced QED₃₊₁ with a 1-brane, where the constant mass approximation is reliable.

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- [1] V. A. Rubakov, hep-ph/0104152.
- [2] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Redwood City, CA, 1991).
- [3] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
- [4] G. Dvali, G. Gabadadze, and M. Shifman, Phys. Lett. B **497**, 271 (2001).
- [5] R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981); S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
- [6] R. D. Pisarski, Phys. Rev. D **29**, 2423 (1984).
- [7] T. W. Appelquist, M. Bowick, D. Karabali, and L. C. Wijewardhana, Phys. Rev. D **33**, 3704 (1986); **33**, 3774 (1986).
- [8] T. W. Appelquist, D. Nash, and L. C. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).
- [9] D. Nash, Phys. Rev. Lett. **62**, 3024 (1989).
- [10] K.-I. Kondo and H. Nakatani, Mod. Phys. Lett. A **5**, 407 (1990); I. J. Aitchison, N. E. Mavromatos, and D. McNeill, Phys. Lett. B **402**, 154 (1997); K. I. Kondo and T. Murakami, *ibid.* **410**, 257 (1997); V. P. Gusynin, A. H. Hams, and M. Reenders, Phys. Rev. D **53**, 2227 (1996); **63**, 045025 (2001).
- [11] T. Kugo and M. G. Mitchard, Phys. Lett. B **282**, 162 (1992); K.-I. Kondo, T. Ebihara, T. Iizuka, and E. Tanaka, Nucl. Phys. **B434**, 85 (1995).
- [12] D. Atkinson, J. Math. Phys. **28**, 2494 (1987).
- [13] V. A. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific, Singapore, 1993).
- [14] V. A. Miransky and K. Yamawaki, Phys. Rev. D **55**, 5051 (1997).
- [15] P. I. Fomin and V. A. Miransky, Phys. Lett. **64B**, 166 (1976); P. I. Fomin, V. P. Gusynin, and V. A. Miransky, *ibid.* **78B**, 136 (1978); P. I. Fomin, V. P. Gusynin, V. A. Miransky, and Yu. A. Sitenko, Riv. Nuovo Cimento **6**, 1 (1983).
- [16] V. A. Miransky, Phys. Lett. **91B**, 421 (1980); V. A. Miransky, Nuovo Cimento A **90**, 149 (1985); V. A. Miransky and V. P. Gusynin, Prog. Theor. Phys. **81**, 426 (1989).
- [17] W. A. Bardeen, C. N. Leung, and S. T. Love, Phys. Rev. Lett. **56**, 1230 (1986); C. N. Leung, S. T. Love, and W. A. Bardeen, Nucl. Phys. **B273**, 649 (1986).
- [18] M. R. Pennington and D. Walsh, Phys. Lett. B **253**, 246 (1991); D. C. Curtis, M. R. Pennington, and D. Walsh, *ibid.* **295**, 313 (1992).
- [19] T. W. Appelquist, J. Terning, and L. C. Wijewardhana, Phys. Rev. Lett. **77**, 1214 (1996); V. P. Gusynin, V. A. Miransky, and A. V. Shpagin, Phys. Rev. D **58**, 085023 (1998).
- [20] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); S. Coleman, Commun. Math. Phys. **31**, 259 (1973).
- [21] V. L. Berezinski, Sov. Phys. JETP **32**, 493 (1970); J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973).
- [22] E. Witten, Nucl. Phys. **B145**, 110 (1978).
- [23] K. Higashijima, Phys. Rev. D **29**, 1228 (1984); V. A. Miransky, Sov. J. Nucl. Phys. **38**, 280 (1983).
- [24] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Nucl. Phys. **B462**, 249 (1996).
- [25] D. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
- [26] N. Andrei and J. H. Lowenstein, Phys. Rev. Lett. **43**, 1698 (1979); N. Andrei and J. H. Lowenstein, Phys. Lett. **90B**, 106 (1980).
- [27] S. Coleman, Phys. Rev. D **11**, 2088 (1975).
- [28] A. Polyakov and P. B. Wiegmann, Phys. Lett. **131B**, 121 (1983); P. B. Wiegmann, *ibid.* **142B**, 173 (1984).
- [29] H. Bateman and A. Erdelyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1.
- [30] D. C. Curtis and M. R. Pennington, Phys. Rev. D **42**, 4165 (1990).
- [31] C. J. Burden and P. C. Tjiang, Phys. Rev. D **58**, 085019 (1998); A. Bashir, A. Kizilersü, and M. R. Pennington, *ibid.* **62**, 085002 (2000).

- [32] B. A. Dobrescu, Phys. Lett. B **461**, 99 (1999); A. B. Kobakhidze, Yad. Fiz. **64**, 1010 (2001) [Phys. At. Nucl. **64**, 941 (2001)]; H.-C. Cheng, B. A. Dobrescu, and C. T. Hill, Nucl. Phys. **B573**, 597 (2000); N. Arkani-Hamed, H.-C. Cheng, B. A. Dobrescu, and L. J. Hall, Phys. Rev. D **62**, 096006 (2000); H. Abe, H. Miguchi, and T. Muta, Mod. Phys. Lett. A **15**, 445 (2000); H. Abe, T. Inagaki, and T. Muta, hep-ph/0104002; N. Rius and V. Sanz, Phys. Rev. D **64**, 075006 (2001).
- [33] M. Hashimoto, M. Tanabashi, and K. Yamawaki, Phys. Rev. D **64**, 056003 (2001).