Space of kink solutions in $SU(N)\times Z_2$

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We find $(N+1)/2$ distinct classes ("generations") of kink solutions in an $SU(N)\times Z_2$ field theory. The classes are labeled by an integer q. The members of one class of kinks will be globally stable while those of the other classes may be locally stable or unstable. The kink solutions in the q^{th} class have a continuous degeneracy given by the manifold $\Sigma_q = H/K_q$, where H is the unbroken symmetry group and $K_q \subseteq H$ is the group under which the kink solution remains invariant. The space Σ_q is found to contain two incontractable spheres for some values of q, indicating the possible existence of certain incontractable spherical structures in three dimensions. We explicitly construct the three classes of kinks in an $SU(5)$ model with a quartic potential and discuss the extension of these ideas to magnetic monopole solutions in the model.

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I. INTRODUCTION

It is relatively easy to determine if a field theory with spontaneous symmetry breaking admits topological defects. If the asymptotic field configuration is topologically nontrivial, the interior field configuration must have a topological defect. However, there can be a large class of asymptotic field configurations, all having the same topological characteristics. Which of the many different boundary conditions with a given topology should one use when trying to find a topological defect solution?

We will restrict our attention to the simplest kind of topological defects, namely kinks in one spatial dimension. However the field theories we will consider are rather general, having symmetry groups $SU(N)\times Z_2$ with N being an odd integer. The field content will be a scalar field Φ transforming in the adjoint representation of $SU(N)$, and the Z_2 takes Φ to $-\Phi$. The potential of the field theory is taken to be such that it gives a vacuum expectation value of Φ that breaks the symmetry spontaneously to $H=\{SU[(N+1)/2]\times SU[(N-1)/2]\times U(1)\}/C$, where $C=Z_{(N+1)/2}\times Z_{(N-1)/2}$ is the center of $SU[(N+1)/2]\times SU[(N-1)/2]$; other than having this property the potential is not restricted in any way. The vacuum manifold of the theory is disconnected because the Z_2 is broken down completely by the vacuum expectation value. Hence there are topological kinks in the theory.

Suppose we want to find the explicit solution for these kinks. Let $\Phi(x=-\infty)=\Phi_-$ and $\Phi(x=+\infty)=\Phi_+$. Then, to obtain a topological defect, the only constraint is that Φ_+ and Φ_- should lie in distinct topological sectors of the vacuum manifold. In fact, if Φ_+ is a choice, $U\Phi_+U^\dagger$ for $U\in SU(N)$ is also a valid choice. In [1] it was shown that the $SU(5)\times Z_2$ kink with $\Phi_+=-\Phi_-$ is unstable to small perturbations and that there exists a stable domain wall solution of lower energy corresponding to a different choice of Φ_+. These results were generalized to $SU(N)\times Z_2$ in [2] where the concept of different classes of kink solutions was introduced. Given a kink solution, the rest of the solutions from the same class can be constructed by applying global gauge transformations from the coset space H/I where H is the unbroked symmetry group and $I\subseteq H$ is the "internal" symmetry group that leaves the original kink solution invariant. One such class of solutions was constructed in [2], however, several questions of relevance were left unanswered. Will there exist a kink solution for any choice of Φ_+? Are the different solutions really distinct? How many distinct solutions can one obtain? Are these solutions stable? We will answer these questions in this paper.

In Sec. III we will show that not all choices of Φ_+ lead to kink solutions and we find that we must have $[\Phi_+,\Phi_-]=0$ in order for a solution to exist. This leads to a finite, discrete set of topological boundary conditions that can yield distinct kink solutions. Each boundary condition determines a class of continuously degenerate kink solutions in the model. Surprisingly, we also find that there are nontopological kink solutions for which the boundary conditions do not lie in distinct topological sectors. These solutions can also be classified and counted. We then find the manifold that describes the continuous degeneracy of every class. This manifold has non-trivial topological properties which suggests that certain closed domain walls are incontractable. In Sec. V we consider the specific example of an $SU(5)$ model with a quartic potential and construct the topological and non-topological kink solutions explicitly. In this case we also analyze the stability of the kink solutions in the three different classes. There is one globally stable class of solutions; another is locally stable for some parameters; the remaining classes are unstable for our choice of potential.

In Sec. VII we discuss the extension of our results on domain walls to $SU(5)$ magnetic monopoles. With fixed asymptotic field configurations, our findings suggest that there should exist three generations of fundamental $SU(5)$ magnetic monopole solutions. We summarize our results in Sec. VIII.

II. KINK BOUNDARY CONDITIONS

The Lagrangian of our (1+1 dimensional) model is

$$L=\mathrm{Tr}(\partial_\mu \Phi)^2-V(\Phi). \tag{1}$$

$V(\Phi)$ is a potential invariant under

$$G\equiv SU(N)\times Z_2, \tag{2}$$

N is taken to be odd, and the parameters in V are such that Φ has an expectation value that can chosen to be

$$\Phi_0=\eta\sqrt{\frac{2}{N(N^2-1)}}\begin{pmatrix} n\mathbf{1}_{n+1} & \mathbf{0} \\ \mathbf{0} & -(n+1)\mathbf{1}_n \end{pmatrix}, \tag{3}$$

where $\mathbf{1}_p$ is the $p\times p$ identity matrix and η is an energy scale determined by the minima of the potential V. Such an expectation value spontaneously breaks the symmetry down to

$$H=[SU(n+1)\times SU(n)\times U(1)]/C, \tag{4}$$

where we have defined

$$N\equiv 2n+1, \tag{5}$$

with $n\geqslant 1$ being an integer. The exact form of $V(\Phi)$ will not be important for most of our analysis. However, it does play a role in the stability of solutions and then we will choose it to be a quartic polynomial in Φ.

If $\Phi(x=-\infty)=\Phi_-$, then $\Phi(x=+\infty)=\Phi_+=-U\Phi_-U^\dagger$ for any $U\in SU(N)$ implies that the boundary conditions are topologically non-trivial. For example, if $U\in H$, the symmetry group that leaves Φ_- invariant, then $\Phi_+=-\Phi_-$. The first question we ask is: for a fixed Φ_-, for what choices of Φ_+ can we obtain kink solutions? As we shall now see, for a solution to exist, we must necessarily choose Φ_+ such that $[\Phi_+,\Phi_-]=0$.

In Appendix A we will prove the stronger result that if $\Phi_k(x)$ is a solution then $[\Phi_\pm,\Phi_k(x)]=0$. Here we will give a qualitative argument in support of this statement. Once the boundary condition at $x=-\infty$ is fixed, the various small excitations of the field Φ around Φ_- can be classified as massless or massive. The only components of Φ that can be non-trivial in the kink solution are the massive modes since the massless modes, also called the Nambu-Goldstone modes, if non-vanishing inside the kink, will not decay as we go further away from the kink. The massive modes are given precisely by the generators that commute with Φ_- while the Nambu-Goldstone modes are those that do not commute. Hence $[\Phi_-,\Phi_k(x)]=0$ and, in particular, $[\Phi_-,\Phi_+]=0$.

Therefore to construct a kink solution, one needs to fix Φ_- to a vacuum expectation value and consider all possible commuting vacuum expectation values for Φ_+. Φ_- can be chosen to be diagonal and by performing rotations that leave Φ_- invariant (i.e. lie in the unbroken group H at $x=-\infty$) Φ_+ can also be brought to diagonal form.

Now we can explicitly list all the possible boundary conditions (up to gauge rotations) that can lead to kink solutions. At $x=-\infty$, we fix $\Phi_-=\Phi_0$ given in Eq. (3). Then we can have

$$\Phi_+=\epsilon_T\eta\sqrt{\frac{2}{N(N^2-1)}}\mathrm{diag}(n\mathbf{1}_{n+1-q},-(n+1)\mathbf{1}_q,n\mathbf{1}_q,-(n+1)\mathbf{1}_{n-q}), \tag{6}$$

where we have introduced a parameter $\epsilon_T=\pm 1$ and another $q=0,\ldots,n$. The label ϵ_T is $+1$ when the boundary conditions are topologically trivial and is -1 when they are topologically non-trivial. q tells us how many diagonal entries of Φ_- have been permuted in Φ_+. The case $q=0$ is when $\Phi_+=\epsilon_T\Phi_-$. The case $q=n$ was considered in detail in Ref. [2].

III. KINK SOLUTIONS

We now find kink solutions for any allowed boundary conditions $\Phi_\pm$. As a starting point we take the following ansatz:

$$\Phi_k=F_+(x)\mathbf{M}_++F_-(x)\mathbf{M}_-+g(x)\mathbf{M}, \tag{7}$$

where

$$\mathbf{M}_+=\frac{\Phi_++\Phi_-}{2},\quad \mathbf{M}_-=\frac{\Phi_+-\Phi_-}{2}, \tag{8}$$

$g(\pm\infty)=0$ and M is yet to be found. Explicitly, for $\epsilon_T=-1$, we have

$$\mathbf{M}_+=\eta N\sqrt{\frac{1}{2N(N^2-1)}}\mathrm{diag}(0_{n+1-q},\mathbf{1}_q,-\mathbf{1}_q,\mathbf{0}_{n-q}), \tag{9}$$

$$\mathbf{M}_-=\eta\sqrt{\frac{1}{2N(N^2-1)}}\mathrm{diag}(-2n\mathbf{1}_{n+1-q},\mathbf{1}_q,\mathbf{1}_q,2(n+1)\mathbf{1}_{n-q}). \tag{10}$$

Note that the matrices $\mathbf{M}_\pm$ are orthogonal:

$$\mathrm{Tr}(\mathbf{M}_+\mathbf{M}_-)=0, \tag{11}$$

but are not normalized to 1/2. The boundary conditions for $F_\pm$ are:

$$F_-(-\infty)=-1,\quad F_-(+\infty)=+1,$$
$$F_+(-\infty)=+1,\quad F_+(+\infty)=+1. \tag{12}$$

The advantage of this form of the ansatz is that, for particular values of the parameters of a quartic potential in the $q=n$ topological ($\epsilon_T=-1$) case, one finds the explicit and simple solution $F_-(x)=\tanh(\sigma x)$, $F_+(x)=1$ and $g(x)=0$, where σ is the kink width which can be written in terms of the parameters [1,2]. Also, for $q=0$, $\epsilon_T=-1$, the solution is the embedded Z_2 kink i.e. $F_+(x)=g(x)=0$, $F_-(x)=\tanh(\sigma x)$.

Now we would like to find the unknown matrix $\mathbf{M}$ in the ansatz (7). This can be done by treating $g(x)\mathbf{M}$ as a small perturbation to

$$\Phi_k^{(0)} \equiv F_+(x)\mathbf{M}_+ + F_-(x)\mathbf{M}_- . \tag{13}$$

The perturbation is restricted to generators that are orthogonal to $\Phi_k^{(0)}$:

$$\mathrm{Tr}(\Phi_k^{(0)}\mathbf{M}) = 0. \tag{14}$$

We need to check if the energy density contains any terms that are linear in $g(x)$, otherwise we could always construct a stable kink solution with $g(x)=0$. The quadratic terms in the energy density clearly will not have such terms since $\mathrm{Tr}(\Phi_k^{(0)}\mathbf{M})=0$. The only terms that may be linear in $g(x)$ will be from terms in the potential such as $\mathrm{Tr}(\Phi^s)$ for even $s\geqslant 4$. (s has to be even since the potential is taken to have a Z_2 symmetry under $\Phi\rightarrow-\Phi$.) There will be no terms linear in $g(x)$ only if

$$\mathrm{Tr}((\Phi_k^{(0)})^{s-1}\mathbf{M}) = 0 \tag{15}$$

for every possible choice of $\mathbf{M}$ satisfying the conditions

$$\mathrm{Tr}(\mathbf{M})=0, \quad \mathrm{Tr}(\mathbf{M}_-\mathbf{M})=0, \quad \mathrm{Tr}(\mathbf{M}_+\mathbf{M})=0. \tag{16}$$

If $\mathbf{M}$ is off-diagonal, Eq. (15) is satisfied because the trace of the product of a diagonal and an off-diagonal matrix vanishes. ($\Phi_k^{(0)}$ is diagonal.) The non-trivial part is to check the condition for diagonal $\mathbf{M}$ and we shall now concentrate on this case.

Let us write $\mathbf{M}$ as

$$\mathbf{M} = \mathrm{diag}(\mathbf{U}_{n+1-q}, \mathbf{V}_q, \mathbf{W}_q, \mathbf{X}_{n-1}), \tag{17}$$

where $\mathbf{U}_{n+1-q}$, $\mathbf{V}_q$, $\mathbf{W}_q$ and $\mathbf{X}_{n-1}$ are diagonal matrices of order given by their subscripts. Implementation of the conditions in Eq. (16) leads to

$$\mathrm{Tr}\,\mathbf{U}_{n+1-q} = -\mathrm{Tr}\,\mathbf{V}_q = -\mathrm{Tr}\,\mathbf{W}_q = \mathrm{Tr}\,\mathbf{X}_{n-q}. \tag{18}$$

Note that if $q=0$ or if $q=n$, this condition enforces each matrix to be traceless.

Now, to check if Eq. (15) is satisfied, we insert the form of $\Phi_k^{(0)}$ from Eq. (13). From the boundary conditions in Eq. (12), it is clear that the functions $F_\pm(x)$ are linearly independent and so Eq. (15) can only be satisfied if:

$$\mathrm{Tr}(\mathbf{M}_+^\alpha \mathbf{M}_-^\beta \mathbf{M}) = 0 \tag{19}$$

for integers α, β such that $0\leqslant\alpha+\beta\leqslant s-1$. Explicit evaluation of this trace, together with the relations in Eq. (18) shows that the condition is satisfied by all $\mathbf{M}$ with $\mathrm{Tr}\,\mathbf{V}_q=0$. However, for $\mathbf{M}$ with $\mathrm{Tr}\,\mathbf{V}_q\neq 0$, the condition is not met if α is an even integer.

How many generators are there for which $\mathrm{Tr}\,\mathbf{V}_q\neq 0$ and that satisfy the conditions in Eq. (18)? There are a total number of $N-1$ diagonal $SU(N)$ generators. Of these, the number of generators satisfying the conditions in Eq. (18) together with $\mathrm{Tr}\,\mathbf{V}_q=0$ are

$$(n+1-q-1)+(q-1)+(q-1)+(n-q-1) = N-4.$$

Hence there are $(N-1)-(N-4)=3$ choices of $\mathbf{M}$ for which the condition in Eq. (18) plus $\mathrm{Tr}\,\mathbf{V}_q=0$ is not met. However this number includes the two possibilities $\mathbf{M}=\mathbf{M}_\pm$. Hence there is only one remaining possible choice of $\mathbf{M}$ and this is

$$\mathbf{M} = \mu\,\mathrm{diag}\big(q(n-q)\mathbf{1}_{n+1-q}, \quad -(n-q)(n+1-q)\mathbf{1}_q, -(n-q)(n+1-q)\mathbf{1}_q, q(n+1-q)\mathbf{1}_{n-q}\big) \tag{20}$$

with μ being a normalization factor in which we also include the energy scale η for convenience:

$$\mu = \eta[2q(n-q)(n+1-q)\{2n(n+1-q)-q\}]^{-1/2}. \tag{21}$$

Note that the matrix $\mathbf{M}$ is not normalizable if $q=0$ or if $q=n$. For these values of q, we can set $g(x)=0$ and $\Phi_k^{(0)}$ coincides with the ansatz Φ_k.

It is easy to see that Φ_k is a valid ansatz. Any perturbations that are orthogonal to Φ_k would have to satisfy Eq. (18) as well as be orthogonal to $\mathbf{M}$. Such perturbations necessarily have $\mathrm{Tr}\,\mathbf{V}_q=0$. Further, all traces of the kind in Eq. (15) are proportional to $\mathrm{Tr}\,\mathbf{V}_q$ and hence vanish. This justifies the ansatz in Eq. (7).

The functions $F_\pm(x)$ and $g(x)$ can be found by solving their equations of motion derived from the Lagrangian together with the specified boundary conditions. There is no guarantee that a solution will exist and so we find the solutions explicitly for $N=5$ with a quartic potential in Sec. V.

An interesting point to note is that the ansatz is valid even if $\Phi_\pm$ are not in distinct topological sectors i.e. even if $\epsilon_T=+1$. These imply the existence of non-topological kink solutions in the model. If we include a subscript NT to denote "non-topological" and T to denote "topological," we have

$$\Phi_{NTk} = F_+(x)\mathbf{M}_{NT+} + F_-(x)\mathbf{M}_{NT-} + g(x)\mathbf{M}_{NT}. \tag{22}$$

Since $\Phi_{NT+}=-\Phi_{T+}$, we find

$$\mathbf{M}_{NT+}=\mathbf{M}_{T-}, \quad \mathbf{M}_{NT-}=\mathbf{M}_{T+}, \quad \mathbf{M}_{NT}=\mathbf{M}_T. \tag{23}$$

Hence

$$\Phi_{NTk} = F_-(x)\mathbf{M}_{T+} + F_+(x)\mathbf{M}_{T-} + g(x)\mathbf{M}_T. \tag{24}$$

So to get F_- (F_+) for the non-topological kink we have to solve the topological F_+ (F_-) equation of motion with the boundary conditions for F_- (F_+). To obtain g for the non-topological kink, we need to interchange F_+ and F_- in the topological equation of motion. The boundary conditions for g are unchanged.

In Sec. V we will find the topological and the non-topological kinks explicitly for $N=5$. Generally the non-topological solutions, if they exist, will be unstable. However, the possibility that some of them may be locally stable for certain potentials cannot be excluded.

IV. KINK CLASSES

In Sec. II we showed that there is a discrete set of boundary conditions that lead to different topological kink solutions. The discrete set is labeled by the integer q which runs from 0 to n. Hence there are $n+1$ distinct classes of kink solutions in the $SU(N)\times Z_2$ model under consideration [2].

The explicit construction of the $n+1$ classes of kinks has already been described in Sec. III. Equation (7) describes the form of the solution for a fixed value of q. A solution of this form is one member of the class of kinks labeled by q. What are the other members of the class?

The members of a class of kinks is given by the set of boundary conditions that will lead to gauge equivalent kinks. In other words, there is a set of transformations belonging to the unbroken symmetry group, H_- in Eq. (4) defined by the vacuum expectation value Φ_-, that will leave Φ_- invariant but will rotate Φ_+ non-trivially. The kink solutions obtained by these global gauge transformations will appear different from the original kink at the level of field configurations but are degenerate and belong to the same class. If K_q is the subgroup of H_- that leaves the q-kink solution, Φ_k, invariant, then

$$\Sigma_q \equiv H_-/K_q$$

describes the class of q-kinks.

Another way to describe Σ_q is in terms of all perturbative modes that do not change the energy of the solution i.e. the zero modes on the solution background. This will include modes that give spatial translations and internal space rotations. The translations have not been included in Σ_q, while the internal space rotations have been included just as in the case of a ''moduli space.'' However, the internal zero modes may not vanish at $x=+\infty$ and hence are not required to be normalizable.

Now we will find Σ_q for various q.

When $q=0$, Φ_k is proportional to Φ_- and $K_q=H_-$ i.e. the symmetry group that leaves the kink invariant is the entire unbroken symmetry group. Therefore $\Sigma_0=1$ and there is only one element in the $q=0$ kink class.

When $0<q<n$, it is clear from Eq. (6) that the elements of H_- that leave Φ_+ invariant are $SU(n+1-q)$ in the first block, $SU(q)$ in the second block, $SU(q)$ in the third block, and $SU(n-q)$ in the fourth block. In addition, the diagonal generators of H_- commute with Φ_+ and these yield another three $U(1)$ factors. Hence the boundary condition at $x=+\infty$ is invariant under

$$\{SU(n+1-q)\times[SU(q)]^2\times SU(n-q)\times U(1)^3\}/Z_K, \tag{25}$$

where we have modded out the continuous group by its center, symbolically denoted by Z_K. [This is necessary since the center of $SU(n+1-q)$ for example, is also contained in the $U(1)$ factors.] From the form of **M** in Eq. (20), it is clear that the group in Eq. (25) is also the symmetry group that leaves **M** invariant. Hence it is also the symmetry group that leaves the entire kink solution Φ_k invariant and so

$$K_q=\{SU(n+1-q)\times[SU(q)]^2\times SU(n-q)U(1)^3\}/Z_K. \tag{26}$$

Therefore $\Sigma_q=H/K_q$ where H is given in Eq. (4) and K_q in Eq. (26).

When $q=n$, the analysis is modified a little bit since now $n-q=0$ and the last block in Φ_+ is absent. So now we have

$$K_n=\{[SU(n)]^2\times U(1)^2\}/Z_K. \tag{27}$$

Note that the above classification scheme holds for both topological ($\epsilon_T=-1$) and non-topological ($\epsilon_T=+1$) kink solutions.

The space Σ_q ($q\neq 0$) has interesting topological properties. For example, it has a non-trivial second homotopy group. This suggests that certain spherical configurations of domain walls (in three spatial dimensions) will be topologically non-trivial and may not be able to contract. We postpone a detailed investigation of the interpretation of the non-trivial topology of Σ_q and its consequences for future work.

V. KINK SOLUTIONS FOR $N=5$

In this section we will explicitly construct the kink solutions when $N=5$ and when the potential is quartic:

$$V(\Phi)=-m^2\mathrm{Tr}[\Phi^2]+h(\mathrm{Tr}[\Phi^2])^2+\lambda\,\mathrm{Tr}[\Phi^4]+V_0. \tag{28}$$

The desired symmetry breaking to

$$H=[SU(3)\times SU(2)\times U(1)]/[Z_3\times Z_2] \tag{29}$$

is achieved in the parameter range

$$\frac{h}{\lambda}>-\left.\frac{N^2+3}{N(N^2-1)}\right|_{N=5}=-\frac{7}{30}. \tag{30}$$

The vacuum expectation value, Φ_- is

$$\Phi_-=\eta\frac{1}{\sqrt{60}}(2,2,2,-3,-3) \tag{31}$$

with

$$\eta\equiv\frac{m}{\sqrt{\lambda'}} \tag{32}$$

and

$$\lambda'\equiv h+\left.\frac{N^2+3}{N(N^2-1)}\right|_{N=5}\lambda=h+\frac{7}{30}\lambda. \tag{33}$$

The $q=0$ topological kink ($\Phi_+=-\Phi_-$) has been found in Ref. [1] and is simply an embedded Z_2 kink for all parameters:

$$\Phi_k^{q=0}=\tanh\left(\frac{mx}{\sqrt{2}}\right)\Phi_-. \tag{34}$$

As discussed in Sec. IV, there is only one kink solution in this class.

To find the $q=1$ topological kink solution, we use the ansatz found in Sec. III

$$\Phi_k^{q=1}=F_+\mathbf{M}_+ + F_-\mathbf{M}_- + g\mathbf{M} \tag{35}$$

with

$$\mathbf{M}_+ = \eta\sqrt{\frac{5}{48}}\mathrm{diag}(0,0,1,-1,0), \tag{36}$$

$$\mathbf{M}_- = \eta\frac{1}{\sqrt{240}}\mathrm{diag}(-4,-4,1,1,6), \tag{37}$$

$$\mathbf{M}= \eta\frac{1}{2\sqrt{7}}\mathrm{diag}(1,1,-2,-2,2). \tag{38}$$

Inserting the ansatz in the Lagrangian we can derive the equations of motion for the functions $F_\pm$ and g. (These are given in Appendix B.) The boundary conditions on these functions are:

$$F_+(\pm\infty)=1, \quad F_-(\pm\infty)=\pm 1, \quad g(\pm\infty)=0. \tag{39}$$

If we assume that $|g''|\ll m^2|g|\ll 1$ and $|F_+''|\ll m^2|F_+|$, an approximate analytic solution can be obtained when $h=-3\lambda/70$. (The assumptions can later be checked for self-consistency.) The approximate solution is:

$$F_-\simeq \tanh\left(\frac{m}{\sqrt{2}}x\right), \tag{40}$$

$$g\simeq -\frac{\gamma_6 F_-(\alpha_1+\alpha_2 F_-^2)}{(\alpha_2\gamma_1-\alpha_1\gamma_3)+(\alpha_2\gamma_4+\alpha_5\gamma_6)F_-^2}, \tag{41}$$

$$F_+\simeq \alpha_2^{-1/2}[-\alpha_1-\alpha_5 g F_-]^{1/2}, \tag{42}$$

where the coefficients α_i and γ_i are given in Appendix B. This approximate solution can be extended to other near-by parameters and a comparison with the numerically obtained solutions shows that the approximation is reasonably good except at the turning points of F_+ and g. However, the qualitative features of the numerical solution are captured by the approximation. We show the numerical solution for $h=-3\lambda/70$ in Fig. 1. A numerical investigation for other values of h/λ shows that a solution always exists for the $q=1$ topological kink.

The class of $q=1$ kinks is described by the space

$$\Sigma_1=H/K_1, \tag{43}$$

where

$$K_1=[SU(2)\times U(1)^3]/Z_2. \tag{44}$$

The $q=2$ kink has been found in Ref. [1] (also see [2]). In the case when

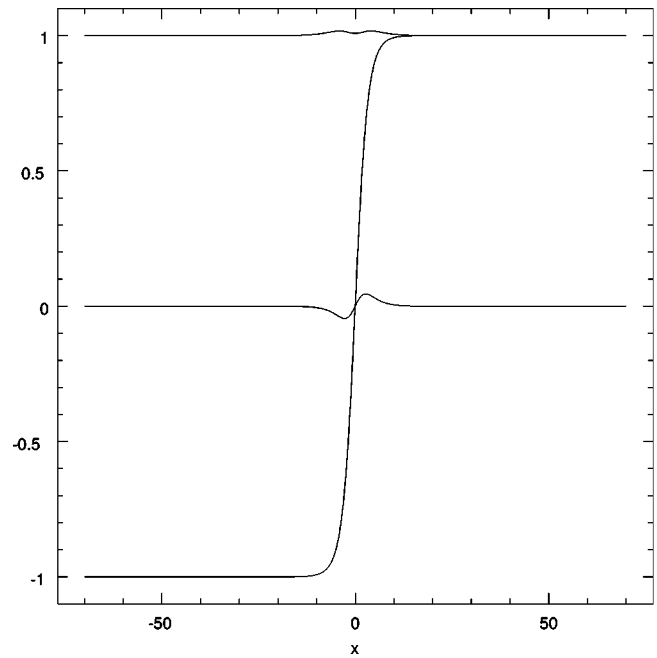

FIG. 1. The profile functions F_+ (nearly 1 throughout), F_- (shaped like a tanh function), and g (nearly zero) for the $q=1$ topological kink with parameters $h=-3/70$, $\lambda=1$ and $\eta=1$.

$$\frac{h}{\lambda}=-\frac{3}{20} \tag{45}$$

the solution can be written down simply as

$$\Phi_k^{q=2}=\frac{1-\tanh(\sigma x)}{2}\Phi_- + \frac{1+\tanh(\sigma x)}{2}\Phi_+ \tag{46}$$

with

$$\Phi_+ = -\eta\frac{1}{\sqrt{60}}(2,-3,-3,2,2). \tag{47}$$

[Φ_- is given by Eq. (31) and $\sigma=m/\sqrt{2}$.]

A more general ansatz, valid for all values of h/λ, is

$$\Phi_k^{q=2}=\frac{F_+(x)-F_-(x)}{2}\Phi_- + \frac{F_+(x)+F_-(x)}{2}\Phi_+, \tag{48}$$

where functions F_+ and F_- satisfy the same boundary conditions as in Eq. (39). The equations of motion for the $q=2$ kink along with a numerical solution were presented in [1].

The class of $q=2$ kinks is described by the space

$$\Sigma_2=H/K_2, \tag{49}$$

where

$$K_2=[SU(2)^2\times U(1)^2]/Z_2^2. \tag{50}$$

Now we will also construct the nontopological ($\epsilon_T=+1$) kinks in the model.

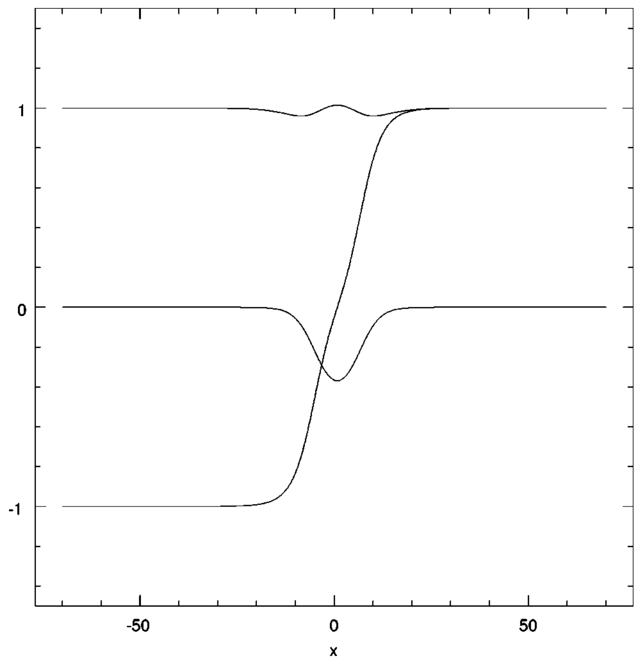

FIG. 2. The profile functions F_+ (shaped like a tanh function), F_- (nearly constant at 1), and g (asymptotically zero) for the $q=1$ non-topological kink with parameters $h=-14/70$, $\lambda=1$ and $\eta=1$.

The $q=0$ non-topological kink is simply the vacuum $\Phi_{NTk}=\Phi_+$ and there is only one member in this class.

As discussed at the end of Sec. III, to construct the $q=1$ nontopological kink we can use the same equations as for the topological case but we should switch the boundary conditions on F_+ and F_- [Eq. (39)]. The system of equations has been solved numerically for a few choices of parameters. For $h=-14\lambda/70$, the profile functions are shown in Fig. 2. For $h=-3\lambda/70$ we find that the $q=1$ non-topological kink breaks up into two $q=2$ topological kinks. Specifically the $q=1$ kink interpolating between $\Phi\propto(2,2,2,-3,-3)$ and $(2,2,-3,2,-3)$ breaks up into one $q=2$ kink interpolating between $(2,2,2,-3,-3)$ and $-(-3,-3,2,2,2)$ and another interpolating between $-(-3,-3,2,2,2)$ and $(2,2,-3,2,-3)$. This suggests that there is a repulsive force between different $q=2$ kinks for parameters close to $h=-3\lambda/70$ and so there will be no non-topological $q=1$ kink solution in a certain range of parameters. Numerically we have determined the critical parameter where the $q=1$ non-topological boundary conditions lead to two well-separated topological $q=2$ kinks instead of one bound object. Hence we find that there are no $q=1$ non-topological kink solutions for $h>-0.18\lambda$.

The $q=2$ non-topological kink can be found by solving the same equations of motion as for the topological $q=2$ kink after switching the boundary conditions on F_+ and F_- ($g=0$ in this case). Then, for the parameter $h=-3\lambda/20$, one has

$$\Phi_{NTk}^{q=2}=\frac{1-\tanh(\sigma x)}{2}\Phi_-+\frac{1+\tanh(\sigma x)}{2}\Phi_+, \tag{51}$$

where

$$\Phi_+=+\eta\frac{1}{\sqrt{60}}(2,-3,-3,2,2). \tag{52}$$

For general values of parameters the profile functions can be found by numerical relaxation.

VI. KINK STABILITY

To analyze the stability of the various kink solutions, we have to expand the energy density to second order in perturbations and then look for unstable modes. This would have to be done on a case by case basis for every different choice of potential. Here we will analyze the stability of the $SU(5)$ kinks constructed in the previous section.

The $q=0$ topological kink is known to be unstable [1]. To see this, note that the Nambu-Goldstone modes are massless at $x=\pm\infty$ and have a negative mass squared at the origin where $\Phi_k=0$. Furthermore, it can be checked that the mass squared for the Nambu-Goldstone modes is everywhere negative for any choice of parameters. We know that an everywhere negative potential in one dimension always admits a bound state. Therefore the $q=0$ topological kink is unstable towards the growth of the Nambu-Goldstone modes for all parameters.

The $q=1$ topological kink is perturbatively unstable. The unstable modes correspond to the four generators of $SU(5)$ which commute with $\Phi_k^{q=1}(0)\propto\mathbf{M}_+$ and do not commute with Φ_- and Φ_+. These modes are massless at $x=\pm\infty$ and have a non-zero mass at the origin. The corresponding potential is given by

$$U^{q=1}(x)=-m^2+\frac{7}{12}\left(h+\frac{2\lambda}{5}\right)\eta^2F_-^2+\frac{5}{12}\eta^2hF_+^2 +\left(h+\frac{\lambda}{2}\right)\eta^2g^2+\sqrt{\frac{7}{60}}\eta^2\lambda F_-g. \tag{53}$$

We have evaluated $U^{q=1}(x)$ numerically and found that it is everywhere negative for any choice of parameters.

As shown in Ref. [1], the $q=2$ topological kink is perturbatively stable, at least for a range of parameters around the choice in Eq. (45).

Next we discuss the perturbative stability of non-topological kinks.

The $q=0$ non-topological kink is simply the vacuum and is trivially stable.

We have seen that the $q=1$ non-topological kink solution may not exist for some parameter values. In other words, the $q=1$ configuration may split and become two $q=2$ topological kinks. When the $q=1$ non-topological kink does not split into two well-separated $q=2$ topological kinks, we find that it is locally stable. The potentially unstable modes are the two generators of $SU(5)$ that commute with $\Phi_k^{q=1NT}(0)\propto\mathbf{M}_-$ and do not commute with Φ_- and Φ_+. The corresponding potential has a particularly simple form:

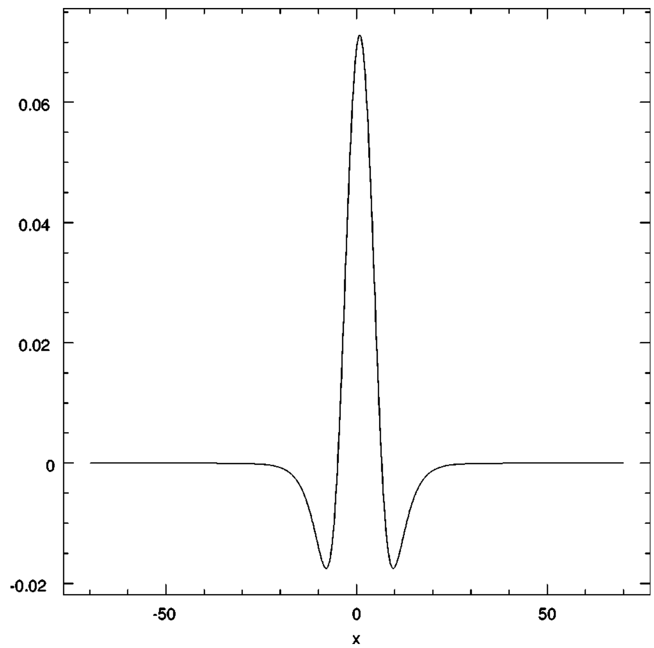

FIG. 3. $U_{NT}^{q=1}(x)$ versus x for $h=-14\lambda/70$ with $\lambda=1$ and $\eta=1$.

$$U_{NT}^{q=1}(x)=\frac{F''_+}{F_+}. \tag{54}$$

The plot of $U_{NT}^{q=1}(x)$ versus x for $h/\lambda=-14/70$ is shown in Fig. 3. We have checked that the value of the potential at $x=0$ remains positive for all parameters for which the $q=1NT$ kink solution exists.

The $q=2$ non-topological kink is perturbatively unstable for all parameter choices. The unstable modes are the eight Nambu-Goldstone modes for which the potential is given by the same expression as in Eq. (54). Numerically we find that $U_{NT}^{q=2}(x)<0$ for all x.

A general statement we can make is that the topological kinks in one of the classes will be globally stable. This just follows from the fact that the kinks are topological and so there must be a lowest energy kink. In the analysis done for the $SU(5)$ case in Sec. V, the $q=n$ kink is the least energetic while the $q=0$ kink has the largest number of unstable modes. This suggests that perhaps the $q=n$ topological kink is the globally stable kink for any choice of potential and not just the quartic potential considered in this section. Another argument in support of this conjecture is that the change in the values of the field components in going from $x=-\infty$ to $+\infty$ is the least for the $q=n$ kink. Only one component need vanish inside the core of the $q=n$ kink while a greater number of components vanish inside the core for $q\leqslant n-1$. The situation with the nontopological kinks is precisely the opposite. Here we know that the $q=0$ nontopological kink is the vacuum and hence is the least energy state.

VII. $SU(5)$ MAGNETIC MONOPOLES

A possible ansatz for a spherically symmetric $SU(5)$ fundamental magnetic monopole solution is [3,4]:

$$\Phi_M \equiv \sum_{a=1}^{3} P(r)\hat{r}^a T^a + M(r)T^4 + N(r)T^5, \tag{55}$$

where the subscript M denotes the monopole field configuration,

$$T^a=\frac{1}{2}\mathrm{diag}(0,0,\sigma^a,0), \quad T^4=\frac{1}{2\sqrt{3}}(1,1,0,0,-2),$$

$$T^5=\frac{1}{2\sqrt{15}}(2,2,-3,-3,2), \tag{56}$$

σ^a are the Pauli spin matrices, $r=\sqrt{x^2+y^2+z^2}$ is the spherical radial coordinate, and $\hat{r}^a$ denotes the unit radial vector. The ansatz for the gauge fields for the monopole can also be written down

$$W_i^a=\epsilon^a{}_{ij}\frac{\hat{r}^j}{er}[1-K(r)] \quad (a=1,2,3),$$

$$W_i^b=0 \quad (b\neq 1,2,3), \tag{57}$$

where e is the gauge coupling. $P(r)$, $M(r)$, $N(r)$ and $K(r)$ are profile functions.

In the Bogomol'nyi-Prasad-Sommerfield (BPS) case, when the $SU(5)$ potential vanishes, the exact, minimal energy solution is known [5]:

$$P(r)=\frac{1}{er}\left(\frac{Cr}{\tanh(Cr)}-1\right), \quad K(r)=\frac{Cr}{\sinh(Cr)}, \tag{58}$$

$$M(r)=\frac{2}{\sqrt{3}}\frac{C}{e}, \quad N(r)=\frac{1}{\sqrt{15}}\frac{C}{e}, \tag{59}$$

where C is a constant.

We can also write the monopole aysmptotic field configuration in more transparent form as $\Phi_M(r=\infty)=U_{34}^\dagger\Phi_+U_{34}$ where

$$U_{34}(\theta,\phi)=e^{-i\phi T^3}e^{-i\theta T^2}e^{+i\phi T^3},$$

θ, ϕ are spherical angular coordinates and the generators T^a are given in Eq. (56). Note that the winding of the monopole lies entirely in the (3,4) block of Φ. We are now — in contrast to the earlier sections — also choosing

$$\Phi_+=\eta\frac{1}{\sqrt{60}}(2,2,2,-3,-3). \tag{60}$$

Any other choice can be transformed to this choice by a global $SU(5)$ rotation.

The existence of the BPS solution does not preclude the existence of other higher energy magnetic monopole solutions even for fixed asymptotics since the boundary conditions at the origin can be chosen in different ways. (*Ansatzë* with other asymptotics can be found in [3].) One possible route to determining the different monopole boundary conditions at $r=0$ is to assume that the cores of magnetic monopoles are like the cores of domain walls. Then we would like to find the different spherical domain walls that have the asymptotics of the BPS solution. This will provide the spherical domain walls with monopole topology. If these spherical domain walls can shrink to zero size, the collapse will produce a monopole whose core is the same as that of the spherical domain wall that we started out with. In this way we might hope to determine the different possibilities for the boundary condition $\Phi_M(0)$.

We have three classes $q=0,1,2$ each of topological and non-topological walls. Let us consider each of these classes one by one.

The $q=0NT$ ($q=0$, non-topological) kink is trivial and we need not discuss it any further. The $q=0T$ ($q=0$, topological) kink has

$$\Phi_k^{q=0}(x)=\tanh(\sigma x)\Phi_+ .$$

Using the kink solution, we can write down a field configuration corresponding to a spherical $q=0T$ domain wall:

$$\Phi^{q=0T}(r,\theta,\phi)\approx\tanh(\sigma(r-R))\Phi_+,$$

where R, the radius of the spherical domain wall, is taken to be very large. Next we would like to introduce monopole topology as a boundary condition to get an object that is a monopole in which all the energy resides in a shell made of a domain wall. [We call this object a ''monopole wall'' (MW).] To do this we need to apply an $SU(5)$ rotation U_{34} on Φ. This will generally be ill-defined at the center ($r=0$) of the spherical domain wall since the field there will then become multivalued. However, we are ultimately interested in letting the radius of the spherical domain wall go to zero and hence we need only apply the gauge transformation on Φ for $r\geqslant R$. Therefore Φ for the monopole wall is

$$\Phi_{MW}^{q=0T}(r,\theta,\phi)\approx \tanh(\sigma(r-R))U_{34}^{\dagger}\Phi_+U_{34}, \quad r>R.$$

Note that the value of the field in the core of the wall is the same everywhere on the wall, that is, $\Phi_{MW}^{q=0}(R,\theta,\phi)=0$ regardless of the spherical angular coordinates. Therefore the monopole-wall can collapse to a point and the field will remain single-valued. The resulting monopole will have $\Phi_M(r=0)=0$. That is, the new boundary conditions on $M(r)$ and $N(r)$ suggested by this argument are: $M(0)=0=N(0)$.

Next consider the $q=1NT$ kink. Here $\Phi^{q=1NT}(0)\propto(4,4,-1,-1,-6)$ in the core of the domain wall. Once again we may construct the monopole-wall by applying the transformation U_{34}. Since

$$U_{34}^{\dagger}\Phi^{q=1NT}(0)U_{34}\propto\Phi(0),$$

the monopole-wall can collapse into a monopole. This suggests that we should be able to find a monopole solution with $\Phi_M(r=0)\propto(4,4,-1,-1,-6)$. This is precisely the monopole with boundary conditions given in Eq. (59).

The $q=1T$ kink has $\Phi^{q=1T}(0)\propto(0,0,1,-1,0)$ and this is not invariant under rotations by U_{34}. Therefore once we impose monopole boundary conditions on a spherical domain wall of this type, the field in the core of the domain wall will depend on the angular coordinates. Such a wall cannot simply collapse to zero radius since that would violate single-valuedness of the field. Hence we do not expect to find a monopole whose center has Φ proportional to $(0,0,1,-1,0)$.

The $q=2T$ kink has $\Phi^{q=2T}(0)\propto(0,1,1,-1,-1)$ and, as this is not invariant under U_{34}, a monopole with $\Phi_M(0)\propto(0,1,1,-1,-1)$ is not possible.

The $q=2NT$ kink as described in Sec. V has $\Phi^{q=2NT}(0)\propto(-4,1,1,1,1)$ and this is invariant under U_{34}. This suggests that a monopole with $\Phi_M(0)\propto(-4,1,1,1,1)$ is possible. However, this monopole-wall does not quite fit the form of the monopole solution given in Eq. (55). We find that if we choose $M(0)=-\sqrt{5}N(0)$, the center of the monopole has $\Phi_M(0)\propto(1,1,1,1,-4)$ and not $(-4,1,1,1,1)$. A global $SU(5)$ rotation on the monopole solution could be used to make $\Phi_M(0)\propto(-4,1,1,1,1)$, however this would then rotate the asymptotic field to $\Phi_M(z=\infty)\propto(-3,2,2,-3,2)$, once again providing a mismatch between the monopole-wall and the monopole ansatz in Eq. (55). In spite of this mismatch, the monopole-wall has the same topologically non-trivial asymptotic field configuration as the BPS solution and can also contract to a point without any conflict with single-valuedness. Hence we think that a monopole solution with $\Phi_M(0)\propto(-4,1,1,1,1)$ should exist.

The above discussion, suggesting that there could be several monopole solutions corresponding to different boundary conditions on the scalar field at $r=0$, clearly applies to global monopoles. In the case of gauge monopoles, the only non-trivial gauge fields are the three fields associated with the $SU(2)$ group of the embedded monopole, as in the BPS case above. These fields still satisfy the form in Eq. (57) and the only quantity that will depend on the ''monopole generation'' is the profile function $K(r)$.

This completes an analysis of all the cases. Three of the five non-trivial cases led to the possibility of a monopole solution. This suggests the existence of three classes of fundamental monopoles in $SU(5)$ with the same asymptotics as the BPS monopole.

VIII. CONCLUSIONS

We have shown that the kink solutions in $SU(N)\times Z_2$ occur in $(N+1)/2$ classes. All the kink solutions, regardless of class, have the same topological charge. Borrowing the terminology of the standard model where particles come in ''generations'' (or ''families''), we dub the kink classes ''kink generations.'' We have determined the continuous degeneracy associated with every kink generation. The degeneracy is described by certain manifolds which themselves have interesting topological properties. In particular, the manifolds have non-trivial second homotopy, suggesting that certain configurations of closed domain walls in three spatial dimensions may be incontractable.

We have also examined the stability of the various classes of kinks in an $SU(5)$ model with quartic potential. Our analysis shows that two classes of solutions are perturbatively stable (for some parameters) while the other non-trivial kinks are unstable.

The generation structure of domain walls suggests a generation structure for the magnetic monopoles in the gauged version of the model — a possibility that seems worth exploring further in the context of the dual standard model [6]. We have found that spherical domain walls of the $q=0T,1NT,2NT$ classes can collapse into monopoles that al. have the same asymptotic field configurations. Hence monopole solutions with $\Phi_M(0)=0$ and $\Phi_M(0)\propto(-4,1,1,1,1)$ should be possible to construct in addition to the known case where $\Phi_M(0)\propto(4,4,-1,-1,-6)$. If all these different boundary conditions lead to magnetic monopole solutions and there are none others,[1] it would indicate that there are exactly three generations of $SU(5)$ magnetic monopole solutions. To confirm this statement would require an explicit construction of the $SU(5)$ monopole solutions with the various possible boundary conditions.

We anticipate that a survey of the space of $SU(N)$ magnetic monopole solutions will show novel features, similar to those we have discovered in the case of kinks.

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APPENDIX A: PROOF THAT SOLUTIONS REQUIRE $[\Phi_+,\Phi_-]=0$

Let $\Phi_k(x)$ be a kink solution. We can expand the solution in an orthonormal set of $SU(N)$ generators T^a $[\mathrm{Tr}(T_aT_b)=\delta_{ab}/2]$:

$$\Phi_k(x)=\sum_a \phi_a(x)T^a. \tag{A1}$$

Here an alternate expansion will be more convenient:

$$\Phi_k(x)=\sum_a \psi_a(x)R^a, \tag{A2}$$

where

$$R^1\equiv\frac{1}{\eta}\Phi_-\equiv R_-,\quad R^2\equiv\frac{1}{\eta}\Phi_+\equiv R_+, \tag{A3}$$

where η is a normalization factor so that $\mathrm{Tr}(R_\pm^2)=1/2$ and the remaining R^a complete the set of generators. Depending on the boundary conditions, it may well turn out that $\mathrm{Tr}(R_+R_-)\neq 0$ and so these generators are not orthogonal. However, we shall choose the other generators, i.e., R^a with $a\neq 1,2$, to satisfy the orthogonality conditions $\mathrm{Tr}(R_+R^a)=0=\mathrm{Tr}(R_-R^a)$ and also normalize them to satisfy $\mathrm{Tr}(R_aR^a)=1/2$. We define new structure constants r_{abc} by

$$[R^a,R^b]=ir_{abc}R^c. \tag{A4}$$

Next we need to state certain properties of the functions $\psi_a(x)$. Due to the boundary conditions $\Phi(x\to\pm\infty)\to\Phi_\pm$, we have

$$\psi_1(-\infty)=\eta,\quad \psi_a(-\infty)=0\ \ (a\neq 1), \tag{A5}$$

$$\psi_2(+\infty)=\eta,\quad \psi_a(+\infty)=0\ \ (a\neq 2). \tag{A6}$$

(Just as for the generators, $\psi_-\equiv\psi_1$ and $\psi_+\equiv\psi_2$.) These boundary conditions ensure that there is no non-trivial solution of the kind $\psi_a(x)=\mathrm{const}$.

Let us now perturb the kink solution $\Phi_k(x)$. For this, consider the field configuration

$$\Phi_1(x)=U(x)\Phi_kU^\dagger(x), \tag{A7}$$

where $U(x)\in SU(N)$. Note that $V(\Phi_1)=V(\Phi_k)$ since the potential is invariant under $SU(N)$ local gauge transformations. Then the energy of the configuration Φ_1 is:

$$E[\Phi_1]=E[\Phi_k]+2\,\mathrm{Tr}(\partial_x\Phi_k[U^\dagger\partial_xU,\Phi_k]) +\mathrm{Tr}([U^\dagger\partial_xU,\Phi_k]^2). \tag{A8}$$

If we now consider infinitesimal rotations, the second term is linear in these while the last term is quadratic. If Φ_k is to be a solution, the linear variation must vanish. Therefore,

$$\mathrm{Tr}(\partial_x\Phi_k[U^\dagger\partial_xU,\Phi_k])=0 \tag{A9}$$

for all $U(x)$ infinitesimally close to unity and for all x.

The condition in Eq. (A9) can also be rewritten as:

$$\mathrm{Tr}([\Phi_k,\partial_x\Phi_k]U^\dagger\partial_xU)=0, \tag{A10}$$

which should hold for any $U(x)\in SU(N)$. (For infinitesimal rotations this condition is

$$\mathrm{Tr}([\Phi_k,\partial_x\Phi_k]T^a)=0,\ \ \forall x,a\ , \tag{A11}$$

where T^a form a complete set of $SU(N)$ generators.) Hence the solution must necessarily satisfy

$$[\Phi_k,\partial_x\Phi_k]=0 \tag{A12}$$

for all x.

Next use the expansion of Φ_k of Eq. (A2) in Eq. (A12) and that gives us:

$$\sum_{b>a} r_{abc}[\psi_a(x)\psi_b'(x)-\psi_b(x)\psi_a'(x)]=0,\ \ \forall c,x. \tag{A13}$$

If the functions

$$F_{ab}\equiv\psi_a(x)\psi_b'(x)-\psi_b(x)\psi_a'(x)$$

[1] While the only $SU(5)$ monopole solution known to us is the BPS solution, an exhaustive list of spherically symmetric ansatze consistent with monopole topology is given in [3]. Some of these could possibly lead to other monopole solutions with different asymptotic field configurations.

are linearly independent, Eq. (A13) implies that $r_{abc}=0$ whenever $F_{ab}\neq 0$. It is easy to see that $F_{ab}\neq 0$ provided both ψ_a and ψ_b are non-trivial and linearly independent. Hence the (assumed) linear independence of F_{ab} implies that $r_{abc}=0$ whenever ψ_a and ψ_b are non-trivial and linearly independent. It is sufficient to assume that all the ψ_a are linearly independent since if two components are linearly dependent, the basis of generators, R^a, can be redefined so that only linearly independent functions occur in the expansion in Eq. (A2). This shows that if F_{ab} are linearly independent then $[R^a,R^b]=0$ if ψ_a and ψ_b are non-trivial. Therefore the solution Φ_k can be expanded in a Cartan basis and in particular $[\Phi_+,\Phi_-]=0$.

Without assuming the linear independence of the functions F_{ab}, we can still show the desired result $[\Phi_\pm,\Phi_k]=0$ by examining the condition in Eq. (A13) as $x\to+\infty$. In this spatial region, the only non-vanishing function is $\psi_+(x)\to\eta$. The term $\psi_+\psi_a'$ is small because all derivatives vanish at infinity. The terms $\psi_a\psi_b'$ with $\psi_a\neq\psi_+$ are also small since both ψ_a and ψ_b' tend to zero at $x=+\infty$. In this region, where the field is nearly at its vacuum value, we can examine the behavior of the fields by perturbing the potential around the vacuum. This tells us that ψ_a $(a\neq +)$ falls off exponentially as $x\to\infty$. Therefore,

$$\psi_+\psi_a'\gg\psi_a\psi_b'$$

for all $a\neq +$. So the condition in Eq. (A13) in the large, positive x region yields

$$\sum_{b\neq 2} r_{2bc}\psi_b'(x)=0. \tag{A14}$$

An integration over the interval $(x,+\infty)$ then gives

$$\sum_{b\neq 2} r_{2bc}\psi_b(x)=0, \tag{A15}$$

where we have used the boundary conditions $\psi_b(+\infty)=0$ except for $b=2$ (which does not appear in the sum). As discussed above, it is sufficient to consider the case when the set of functions $\psi_b(x)$ are linearly independent. Therefore, if ψ_b is non-trivial, we get

$$r_{2bc}=0, \quad \forall b,c. \tag{A16}$$

Similarly, by considering the region with $x\to-\infty$,

$$r_{1bc}=0, \quad \forall b,c. \tag{A17}$$

This shows that $[R_+,R^a]=0=[R_-,R^a]$ if $\psi_a\neq 0$ for any choice of a and hence $[\Phi_\pm,\Phi_k(x)]=0$. In particular, we can only get a kink solution if $[R_+,R_-]=0$ which is equivalent to $[\Phi_+,\Phi_-]=0$.

APPENDIX B: EQUATIONS OF MOTION FOR THE $q=1$ KINK IN $SU(5)$

The equations of motion for the topological $q=1$ kink functions $F_\pm$ and g are:

$$-F_+''+\alpha_1F_++\alpha_2F_+^3+\alpha_3F_+F_-^2+\alpha_4g^2F_++\alpha_5gF_-F_+=0, \tag{B1}$$

$$-F_-''+\beta_1F_-+\beta_2F_-^3+\beta_3F_+^2F_-+\beta_4g^2F_-+\beta_5g(3F_-^2-F_+^2)+\beta_6g^3=0, \tag{B2}$$

$$-g''+\gamma_1g+\gamma_2g^3+\gamma_3gF_+^2+\gamma_4gF_-^2+\gamma_5g^2F_-+\gamma_6F_-(F_-^2-F_+^2)=0, \tag{B3}$$

where

$$\begin{aligned}
\alpha_1&=\beta_1=\gamma_1=-m^2,\\
\alpha_2&=\eta^2\frac{5}{12}\left(h+\frac{1}{2}\lambda\right),\\
\alpha_3&=\eta^2\frac{7}{12}\left(h+\frac{3}{70}\lambda\right),\\
\alpha_4&=\eta^2\left(h+\frac{6}{7}\lambda\right),\\
\alpha_5&=-\eta^2\lambda\sqrt{\frac{3}{35}},\\
\beta_2&=\eta^2\frac{7}{12}\left(h+\frac{181}{490}\lambda\right),\\
\beta_3&=\eta^2\frac{5}{12}\left(h+\frac{3}{70}\lambda\right),\\
\beta_4&=\eta^2\left(h+\frac{138}{245}\lambda\right),\\
\beta_5&=\eta^2\lambda\frac{5}{14}\sqrt{\frac{3}{35}},\\
\beta_6&=\eta^2\lambda\frac{12}{49}\sqrt{\frac{3}{35}},\\
\gamma_2&=\eta^2\left(h+\frac{25}{98}\lambda\right),\\
\gamma_3&=\eta^2\frac{5}{12}\left(h+\frac{6}{7}\lambda\right),\\
\gamma_4&=\eta^2\frac{7}{12}\left(h+\frac{138}{245}\lambda\right),\\
\gamma_5&=\eta^2\lambda\frac{3}{7}\sqrt{\frac{3}{35}},\\
\gamma_6&=\eta^2\lambda\frac{5}{24}\sqrt{\frac{3}{35}}.
\end{aligned} \tag{B4}$$

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