Effect of weak interactions on the ultrarelativistic Bose-Einstein condensation temperature

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We calculate the ultrarelativistic Bose-Einstein condensation temperature of a complex scalar field with weak $\lambda(\Phi^\dagger\Phi)^2$ interaction. We show that at high temperature and finite density we can use dimensional reduction to produce an effective three-dimensional theory which then requires nonperturbative analysis. For simplicity and ease of implementation we illustrate this process with the linear delta expansion.

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I. INTRODUCTION

The inclusion of finite densities of conserved charges in thermal field theory poses well known problems in the study of phase transitions. On one hand, perturbative analyses of charged scalar fields [1–3] give information about the phase structure, but cannot probe the critical point. Standard perturbative calculations are plagued with infrared divergences and after dealing with these, the asymptotic expansion breaks down. On the other hand, finite charges cannot be easily represented in lattice Monte Carlo simulations. Encoding a non-zero charge in the grand canonical ensemble renders the action complex and therefore useless as a statistical weighting [4]. These obstacles invoke a demand for alternative methods.

Several authors have recently considered the effect of repulsive interactions on the condensation temperature of a dilute Bose gas, a nonrelativistic problem. It was pointed out in Ref. [5] that the leading correction could be isolated in the static Matsubara mode and a mean field calculation could be performed on this mode alone. Since then the static mode has been studied using the $1/N$ expansion [6,7], with the linear delta expansion [8], and now numerically [9–11].

In this paper we consider the effect of interactions on the transition temperature for scalars in the ultrarelativistic limit. By this we mean that we are at the high temperature limit, but that the typical self-energy corrections Σ, the chemical potential μ, or the cube root of the charge density, ρ, can be of the same order $T \gg \Sigma, \mu, \rho^{1/3}$. This limit is appropriate for the study of high temperature symmetry breaking where Bose-Einstein condensation and spontaneous symmetry breaking have interesting similarities. The relevant finite temperature and density 4D Feynman diagrams are difficult to handle and only Jones and Parkin [12] include the setting-sun diagram for the self-energy. To avoid these problems, we use dimensional reduction [13–16] and this is the main focus of our paper. In our regime, we will show that dimensional reduction for a relativistic model is much more complicated than in the nonrelativistic limit of our model which studied in [5–11], but we will show it is still manageable. As in the nonrelativistic case, we take advantage of the good infrared behavior of the nonstatic Matsubara modes. This allows us to integrate these modes perturbatively in an attempt to simplify our calculation.

Following dimensional reduction, the problem of relativistic Bose-Einstein condensation has been reduced to the study of a phase transition in an effective three-dimensional theory at zero temperature and zero density. Standard nonperturbative methods can then be employed to study this model but we do not investigate the many alternatives here, merely choosing one for exemplary purposes. Here we use the linear delta expansion for its simplicity and resemblance to standard perturbation theory.

By way of contrast, the only other study of a relativistic finite density system using dimensional reduction we know of is a study of QCD [17], rather than the Higgs sector studied here. The resulting 3D action is complex, making subsequent numerical analysis of [17] more complicated.

II. DIMENSIONAL REDUCTION

We begin by considering a relativistic system of bosons described by a complex scalar field theory. We encode a conserved charge by working in the grand canonical ensemble with chemical potential μ. The partition function is then

$$Z = \int [d\Psi^\dagger][d\Psi] \exp\{-S\} \tag{1}$$

where field integrations are periodic over imaginary time $\beta = 1/T$ and the action is given by

$$S = \int_0^\beta d\tau \int d^3x \{[(\partial_\tau + \mu)\Psi^\dagger][(\partial_\tau - \mu)\Psi] + \nabla\Psi^\dagger\nabla\Psi + m^2\Psi^\dagger\Psi + \lambda(\Psi^\dagger\Psi)^2\}. \tag{2}$$

The charge density is obtained from Z as follows:

$$\rho = \frac{T}{V}\frac{1}{Z}\frac{\partial Z}{\partial \mu} \tag{3}$$

where we let the volume V tend to infinity.

The periodicity of the fields is made explicit by a mode expansion and the nonstatic modes are integrated

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perturbatively.[1] This not only gives an overall factor to Z but also renormalizes the parameters of the static mode, Φ. The result is an effective three-dimensional theory whose dependence on the temperature and chemical potential is contained within its mass and coupling:

$$Z=\exp\{\beta V F(\mu)\}\int [d\Phi^\dagger][d\Phi]\exp\{-S_{3D}\}. \tag{4}$$

Writing $F=F^{(0)}+\lambda F^{(1)}+\mathcal{O}(\lambda^2)$, the quadratic part of the action can be integrated to give

$$F^{(0)}(\mu)=T\sum_{n\neq 0}\int\frac{d^3p}{(2\pi)^3}\ln\Delta(\omega_n,p) \tag{5}$$

where $\Delta(\omega_n,p)=[(\omega_n-i\mu)^2+p^2+m^2]^{-1}$ and, since we are dealing with bosons, $\omega_n=2\pi nT$. The leading perturbative correction is given by the figure-of-eight diagram:

$$F^{(1)}(\mu)=-2T^2\left[\sum_{n\neq 0}\int\frac{d^3p}{(2\pi)^3}\Delta(\omega_n,p)\right]^2. \tag{6}$$

We use dimensional regularization in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, making the replacement

$$\int\frac{d^3p}{(2\pi)^3}\to\int_{\text{p}}=\left(\frac{e^\gamma M^2}{4\pi}\right)^\epsilon\int\frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \tag{7}$$

and subtracting only the terms which are divergent as $\epsilon\to 0$. M is an arbitrary renormalization scale and γ is the Euler-Mascheroni constant.

The ultrarelativistic limit is defined by $\rho\gg m^3$ or, equivalently, $T\gg m$. In order to avoid confusion over the different expansion parameters we will set $m=0$. Since we shall handle the infrared region nonperturbatively, this does not create additional problems. We keep corrections of $\mathcal{O}(\mu^2/T^2)$ as we will find these are in fact $\mathcal{O}(\lambda)$ for the critical theory. Ignoring any μ-independent terms, the factor F is then given by (see [2] for details)

$$F(\mu)=\frac{\mu^2T^2}{6}\left[1-\frac{\mu^2}{4\pi^2T^2}+\frac{\lambda}{4\pi^2}+\mathcal{O}(\lambda^2)\right], \tag{8}$$

where by $\mathcal{O}(\lambda^2)$ we mean $\mathcal{O}(\mu^4/T^4,\lambda\mu^2/T^2,\lambda^2)$.

The nonstatic modes also give corrections to the static mode action. We write the effective 3D action as

$$S_{3D}=\int d^3x[\nabla\Phi^\dagger\nabla\Phi+r\Phi^\dagger\Phi+u(\Phi^\dagger\Phi)^2], \tag{9}$$

where we neglect the higher dimensional operators $(\Phi^\dagger\Phi)^n$ for $n\geqslant 3$. This is first on the grounds that their coefficients are high order in λ and further that all nonrenormalizable interactions are suppressed by factors of $1/T^2$ (for further discussion of this point see [16]). It can be checked that in the calculation which follows, these neglected terms contribute at higher order in the λ expansion than is considered.

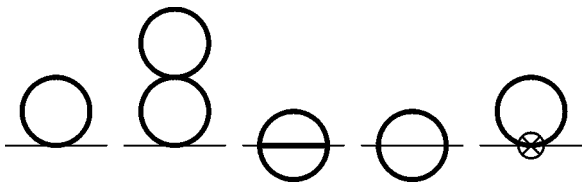

FIG. 1. Contributions to the mass of the static mode. Bold lines correspond to nonstatic internal lines; thin lines correspond to the static mode. The last diagram contains the one-loop vertex counterterm.

The fields should be scaled by $\sqrt{T}$ in order to remove an overall factor of $1/T$ from the action and to the order we work they undergo no field renormalization from the nonstatic mode integration.

Referring to Fig. 1, the 3D mass is given by

$$r=-\mu^2+\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\mathcal{O}(\lambda^3), \tag{10}$$

where

$$\Sigma_1=4\lambda T\sum_{n\neq 0}\int_{\text{p}}\Delta(\omega_n,p) \tag{11}$$

$$=\frac{\lambda T^2}{3}\left[1-\frac{3\mu^2}{2\pi^2T^2}+\mathcal{O}(\lambda^2)\right]; \tag{12}$$

$$\Sigma_2=-16\lambda^2T^2\sum_{n_1\neq 0}\int_{\text{p}_1}\Delta(\omega_{n_1},p_1)\sum_{n_2\neq 0}\int_{\text{p}_2}\Delta^2(\omega_{n_2},p_2) \tag{13}$$

$$=-\frac{\lambda^2T^2}{6\pi^2}\left[\frac{1}{2\epsilon}+2\ln\frac{Me^\gamma}{4\pi T}+1-\gamma+\frac{\zeta'(-1)}{\zeta(-1)}+\mathcal{O}(\lambda)\right]. \tag{14}$$

Though Σ_4 involves a static internal line it should be included since it will not arise from the effective static 3D theory.[2] This contribution is in fact well behaved in the infrared and to leading order we find (see Appendix)

[1]Alternatively, one can calculate static quantities in the 3D effective theory and in the full 4D theory. By matching these results one can then relate the coefficients of the 3D effective theory to those of the 4D one [16]. However there is a lack of results in the literature for our model at high densities.

[2]Note that at this order, integrating out the heavy modes actually leaves one with a nonlocal action with terms such as $\int d^3x d^3x'|\Phi(x)|^2B(x,x')|\Phi(x')|^2$ where B is an $O(\lambda^2)$ bubble diagram. It is when approximating the 3D theory by the purely local one (9) that contributions, such as Σ_4, coming purely from nonlocal terms in the exact nonlocal effective theory must not be forgotten. For instance, the same type of non-local B term should also lead to a diagram similar to Σ_2 but with a light petal on the top and one has to check that this is of lower order than required. Such problems suggest that the matching of Green functions approach to dimensional reduction [16] might be simpler.

$$\Sigma_3+\Sigma_4=0+\mathcal{O}(\lambda^3T^2). \tag{15}$$

Finally we include the diagram with the one-loop vertex counterterm which is given by $(5\lambda^2T/8\pi^2\epsilon)(\Phi^\dagger\Phi)^2$. This is

$$\Sigma_5=4\frac{5\lambda^2T}{8\pi^2}\frac{1}{\epsilon}\sum_{n\neq 0}\int_{\rm p}\Delta(\omega_n,p) \tag{16}$$

$$=\frac{5\lambda^2T^2}{12\pi^2}\left[\frac{1}{2\epsilon}+\ln\frac{Me^\gamma}{4\pi T}+1-\gamma+\frac{\zeta'(-1)}{\zeta(-1)}+\mathcal{O}(\lambda)\right]. \tag{17}$$

We may now sum all the contributions to Eq. (10) giving

$$r=-\mu^2+\frac{\lambda T^2}{3}\left[1-\frac{3\mu^2}{2\pi^2T^2}+\frac{3\lambda}{4\pi^2}\left(\frac{1}{2\epsilon}+\frac{1}{3}\ln\frac{Me^\gamma}{4\pi T}+1-\gamma+\frac{\zeta'(-1)}{\zeta(-1)}\right)+\mathcal{O}(\lambda^2)\right]. \tag{18}$$

We shall find the coupling to be given with sufficient accuracy by

$$u=\lambda T+\mathcal{O}(\lambda^2). \tag{19}$$

The coupling is now dimensionful due to the scaling of the static fields by $\sqrt{T}$.

Use of Eqs. (3) and (4) with Eqs. (8), (9), (18) and (19) gives

$$\rho=\frac{\partial F}{\partial\mu}+2\mu T\left[1+\frac{\lambda}{2\pi}+\mathcal{O}(\lambda^2)\right]\langle\Phi^\dagger\Phi\rangle \tag{20}$$

$$=\frac{\mu T^2}{3}\left[1-\frac{\mu^2}{2\pi^2T^2}+\frac{\lambda}{4\pi^2}+\mathcal{O}(\lambda^2)\right]+2\mu T\left[1+\frac{\lambda}{2\pi^2}+\mathcal{O}(\lambda^2)\right]\langle\Phi^\dagger\Phi\rangle, \tag{21}$$

where $\langle\Phi^\dagger\Phi\rangle$ denotes the Green function evaluated in the effective 3D theory. This we cannot calculate perturbatively since the expansion will break down when probing large length scales greater than $1/u$.

III. LINEAR DELTA EXPANSION

To evaluate the quantity $\langle\Phi^\dagger\Phi\rangle$ we need a nonperturbative method as the 3D sector retains all the infrared divergences of the 4D theory. The effective 3D theory is studied at zero temperature and density so the problem is greatly simplified. Any standard non-perturbative method can be used at this point.

We will use the linear delta expansion (LDE), though the method is also known by several other names (see [20] for a brief summary). LDE has been used successfully in many situations, including studies of scalar theories at nonzero density such as [8,18,19,12,20]. In toy models, where exact results are achievable, LDE is known to produce convergent results and to do so much faster than alternatives, for instance see [21,22] and references therein. In full quantum field theory (QFT), LDE has also often proved to be better than other methods [23].

We begin by defining S_δ which interpolates between our 3D action (from which we now drop the subscript) and some soluble action S_0 as δ varies from 1 to 0:

$$S\to S_\delta=\delta S+(1-\delta)S_0. \tag{22}$$

We are free to choose

$$S_0=\int d^3x[\nabla\Phi^\dagger\nabla\Phi+\Omega^2\Phi^\dagger\Phi] \tag{23}$$

such that

$$S_\delta=\int d^3x[\nabla\Phi^\dagger\nabla\Phi+(\Omega^2-\delta\Omega^2+\delta r)\Phi^\dagger\Phi+\delta u(\Phi^\dagger\Phi)^2]. \tag{24}$$

Any physical quantity $\mathcal{P}$ is evaluated as a power series in δ to some finite order. This quantity will generally have some dependence on Ω which we fix by some specified criterion which we take to be the principle of minimal sensitivity (PMS) [24]:

$$\left.\frac{d\mathcal{P}}{d\Omega}\right|_{\delta=1,\Omega=\bar{\Omega}}=0. \tag{25}$$

This variational procedure allows for the emergence of nonperturbative behavior.

The Green function we require can now be written as

$$\langle\Phi^\dagger\Phi\rangle=\int\frac{d^3p}{(2\pi)^3}G_\delta(p) \tag{26}$$

where

$$G_\delta(p)=[p^2+\Omega^2-\delta\Omega^2+\delta r+\Sigma_\delta(p)]^{-1}. \tag{27}$$

The criterion defining the transition temperature is that the correlation length in the original theory be infinite. This can be expressed as $G_\delta^{-1}(0)|_{\delta=1}=0$ which is satisfied by imposing

$$\delta r+\Sigma_\delta(0)=0. \tag{28}$$

Use of this relation in Eq. (26) gives

$$\langle\Phi^\dagger\Phi\rangle=\int\frac{d^3p}{(2\pi)^3}[p^2+\Omega^2-\delta\Omega^2+\Sigma_\delta(p)-\Sigma_\delta(0)]^{-1} \tag{29}$$

and expanding to second order in δ we have

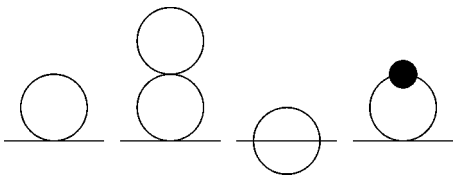

FIG. 2. Contributions to the self-energy for the effective 3D theory. The filled dot corresponds to a $\delta(r-\Omega^2)$ mass insertion.

$$\langle\Phi^\dagger\Phi\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+\Omega^2}\left[1+\frac{\delta\Omega^2}{p^2+\Omega^2}+\frac{\delta^2\Omega^4}{(p^2+\Omega^2)^2}-\frac{\Sigma_\delta(p)-\Sigma_\delta(0)}{p^2+\Omega^2}\right]+\mathcal{O}(\delta^3). \tag{30}$$

The only momentum dependent contribution to the self-energy results from the setting sun diagram (see Fig. 2):

$$\Sigma_3^{3D}(p)=-8\delta^2u^2\int\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{(k^2+\Omega^2)}\frac{1}{(q^2+\Omega^2)}\times\frac{1}{[(k+q+p)^2+\Omega^2]}. \tag{31}$$

We continue our use of dimensional regularization in the $\overline{\text{MS}}$ scheme, using the same scale M as in our 4D heavy mode calculations. The required integrals (see, for example, [8] and [25]) are

$$\int_{\text{p}}\frac{1}{p^2+\Omega^2}=-\frac{\Omega}{4\pi}\left[1+2\epsilon\left(\ln\frac{M}{2\Omega}+1\right)+\mathcal{O}(\epsilon^2)\right], \tag{32}$$

$$\Sigma_3^{3D}(0)=\frac{8\delta^2u^2}{(4\pi)^2}\left[\frac{1}{4\epsilon}+\ln\frac{M}{3\Omega}+\frac{1}{2}+\mathcal{O}(\epsilon)\right], \tag{33}$$

$$\int_{\text{p}}\frac{1}{p^2+\Omega^2}\Sigma_3^{3D}(p)=-\frac{2\delta^2u^2}{(4\pi)^3\Omega}\left[\frac{1}{2\epsilon}+3\ln\frac{M}{2\Omega}+1-2\ln 2+\mathcal{O}(\epsilon)\right]. \tag{34}$$

Further integrals may be derived from Eq. (32) by successive differentiations with respect to Ω.

Summing the contributions to the Green function, we find

$$\langle\Phi^\dagger\Phi\rangle=-\frac{\Omega}{4\pi}+\frac{\delta\Omega}{2(4\pi)}+\frac{\delta^2\Omega}{8(4\pi)}-\frac{4\delta^2u^2}{(4\pi)^3\Omega}\ln\frac{4}{3}. \tag{35}$$

The divergences cancel and there is no need to invoke any counterterms. We now apply the PMS condition to $\langle\Phi^\dagger\Phi\rangle$ and arrive at

$$\overline{\Omega}=\pm\frac{u}{\pi}\left(\frac{2}{3}\ln\frac{4}{3}\right)^{1/2}. \tag{36}$$

The question of which solution to choose can be answered by comparison with the solution in the large-N case for $N\to 2$. Referring to [6] and [7] we find that the positive solution is appropriate, giving

$$\langle\Phi^\dagger\Phi\rangle=-\frac{u\overline{f}}{(4\pi)^2}, \tag{37}$$

$$\overline{f}=\left(6\ln\frac{4}{3}\right)^{1/2}\approx 1.314. \tag{38}$$

Inserting this into Eq. (21) we have for the critical density

$$\rho=\frac{\mu T^2}{3}\left[1-\frac{\mu^2}{2\pi^2T^2}+\frac{\lambda}{(4\pi)^2}(4-6\overline{f})+\mathcal{O}(\lambda^2)\right]. \tag{39}$$

The chemical potential is an unwanted free variable in this expression and we must use relation (28) to constrain μ at the transition temperature. Along with the setting sun contribution to the self-energy, we also have to order δ^2 (see Fig. 2)

$$\Sigma_1^{3D}=4\delta u\int_{\text{p}}\frac{1}{p^2+\Omega^2}, \tag{40}$$

$$\Sigma_2^{3D}=-16\delta^2u^2\int_{\text{p}}\frac{1}{(p^2+\Omega^2)^2}\int_{\text{q}}\frac{1}{q^2+\Omega^2}, \tag{41}$$

$$\Sigma_4^{3D}=4\delta^2u(\Omega^2-r)\int_{\text{p}}\frac{1}{(p^2+\Omega^2)^2}. \tag{42}$$

Evaluating and summing these contributions gives

$$\Sigma_\delta(p=0)=-\frac{\delta u\Omega}{\pi}-\frac{8\delta^2u^2}{(4\pi)^2}\left[\frac{1}{4\epsilon}+\ln\frac{M}{3\Omega}-\frac{1}{2}\right]+\frac{\delta^2u\Omega}{2\pi}-\frac{\delta^2ur}{2\pi\Omega}+\mathcal{O}(\delta^3), \tag{43}$$

which upon substituting into Eq. (28) gives

$$\delta r=\frac{\delta u\Omega}{\pi}-\frac{\delta^2u\Omega}{2\pi}+\frac{\delta^2u^2}{2\pi^2}\left[\frac{1}{4\epsilon}+\ln\frac{M}{3\Omega}+\frac{1}{2}\right]+\mathcal{O}(\delta^3). \tag{44}$$

We apply the PMS condition to r, giving $\overline{\Omega}=u/\pi$, and insert this value into Eq. (44):

$$r=\frac{u^2}{2\pi^2}\left[\frac{1}{4\epsilon}+\ln\frac{M}{u}+c_r\right], \tag{45}$$

$$c_r = \ln\frac{\pi}{3} + \frac{3}{2} \approx 1.546. \tag{46}$$

The dependence of r on the scale M is exact because of super-renormalizability and agrees with that noted elsewhere (e.g., [11]) even though it appears here in the context of a particular nonperturbative calculation. Comparing with Eq. (18) we have

$$-\mu^2 + \frac{\lambda T^2}{3}\left[1 - \frac{3\mu^2}{2\pi^2 T^2} + \frac{3\lambda}{4\pi^2}\left(\frac{1}{2\epsilon} + \frac{1}{3}\ln\frac{M e^{\gamma}}{4\pi T} + 1 - \gamma + \frac{\zeta'(-1)}{\zeta(-1)}\right) + \mathcal{O}(\lambda^2)\right] = \frac{\lambda^2 T^2}{2\pi^2}\left[\frac{1}{4\epsilon} + \ln\frac{M}{\lambda T} + c_r\right]. \tag{47}$$

The divergences cancel and we can obtain μ in terms of the critical temperature:

$$\mu = \frac{\sqrt{\lambda}T}{\sqrt{3}}\left[1 + \frac{\lambda}{8\pi^2}\left(\alpha_{\rm mu}\ln\frac{T}{M} + \eta_{\rm mu}\ln\lambda + a_{\rm mu}\right) + \mathcal{O}(\lambda^2)\right], \tag{48}$$

$$\alpha_{\rm mu} = 5, \quad \eta_{\rm mu} = 6,$$

$$a_{\rm mu} = \ln\left(\frac{e^{\gamma}}{4\pi}\right) + 1 + 3\left(-\gamma + \frac{\zeta(1,-1)}{\zeta(-1)}\right) - 6c_r \approx 3.270 - 6c_r \approx -6.007. \tag{49}$$

It is now clear that for the critical theory, $\mu^2/T^2 \sim \mathcal{O}(\lambda)$. Substituting into the equation for the charge density gives

$$\rho = \frac{\sqrt{\lambda}T^3}{3\sqrt{3}}\left[1 + \frac{\lambda}{8\pi^2}\left(\alpha_{\rm rho}\ln\left(\frac{T}{M}\right) + \eta_{\rm rho}\ln(\lambda) + a_{\rm rho}\right) + \mathcal{O}(\lambda^2)\right], \tag{50}$$

$$\alpha_{\rm rho} = 5, \quad \eta_{\rm rho} = 6, \tag{51}$$

$$a_{\rm rho} = a_{\rm mu} + \frac{2}{3} - 3\overline{f} \approx 3.9236 - 6c_r - 3\overline{f} \approx -21.759.$$

Equation (50) relates the critical density to the critical temperature for the Bose-Einstein condensation of a complex scalar field in the ultrarelativistic limit. Though Eq. (50) looks like an expansion in λ we stress that the result is nonperturbative because of the severe IR problems in calculating $\langle\Phi^\dagger\Phi\rangle$ and self-energies at the critical point in the three-dimensional theory. We should expect ρ to have some renormalization scale dependence since we have included a one-loop vertex counterterm in the calculation.

IV. DISCUSSION

The leading term in our expressions for the critical chemical potential (48) or equivalently for the critical density (50), are the usual leading high temperature results, $\mu_0 = \sqrt{\lambda/3}T$ and $\rho_0 = \sqrt{\lambda/27}T^3$.

The $\lambda^2\ln(T/M)$ term is *exactly* that expected from the running of the 4D coupling λ in the leading term using perturbation theory where one finds

$$\lambda(M_2) = \lambda(M_1) + \frac{5\lambda^2}{8\pi^2}\ln\left(\frac{M_2^2}{M_1^2}\right). \tag{52}$$

The perturbative result is appropriate as the leading behavior comes only from the heavy modes, and these are dealt with perturbatively in this calculation. Thus we find that once the implicit scale dependence of λ is accounted for, our results for μ and ρ are actually independent of the scale M as all exact physics results should be.

The $\lambda^2\ln(\lambda)$ term comes directly from our expression (45) for r and in particular comes from dependence on the scale M which is exact for for the super-renormalizable theory. In the context of a Green function matching approach, as discussed for instance in [16], this term might be described as running the 3D mass r from the scale $M \sim T$, used when dealing with the heavy modes, down to $u = \lambda T$ appropriate for the 3D static theory. Overall then our expressions for the critical chemical potential and density agree with our expectations from other calculations.

In terms of actual numbers, for $\lambda = 1/8$ the fractional corrections to this leading term are of the order of a few percent, e.g., in units of the scale M, for $T = 10.0$, the critical chemical potential is $\mu = \mu_0 \times (1 - 0.010)$, $\mu_0 = 2.04$ and the critical density is $\rho = \rho_0 \times (1 - 0.058)$, $\rho_0 = 68.0$.

A good test of our central result, the dimensional reduction, is to compare our formulas for the critical chemical potential with that extracted from the results of Jones and Parkin [12]. They also use the linear delta expansion but they apply it directly to the full four-dimensional theory anywhere in the symmetric phase. They can in principle study any temperature and density, though they limit their analysis to high temperatures. We use dimensional reduction so our method is always limited to high temperatures. However, this brings several benefits to us as further fields, including fermions, can be added with great ease and our nonperturbative effort is much less. The effective three-dimensional theory derived here can in principle be studied using any nonperturbative technique, including Monte Carlo since the effective action is real. While we have only looked at the critical point, we have also given the critical density which is the actual measured quantity. Finally, our results are completely analytic, while Jones and Parkin can only find numerical solutions to their equations, though the numerics are relatively straightforward.

Turning to the details of the results, we find that the Jones-Parkin method gives the same qualitative behavior as our results for $\lambda = 1/8$, $T/M = 0.5 \ldots 10.0$. However, their results for these parameters [28] are best fitted with slightly

different coefficients,[3] namely $\alpha_{\mathrm{rho,JP}}/\alpha_{\mathrm{rho}}=1.2(1)$, and choosing $\eta_{\mathrm{rho,JP}}=\eta_{\mathrm{rho}}$ we find $a_{\mathrm{rho,JP}}+\eta_{\mathrm{rho,JP}}\ln\lambda=-17.8(2)$ compared with our value -18.5. In our calculation the α_{mu} value was set by perturbation theory and cannot be exact. Since Jones and Parkin use an entirely nonperturbative method, even though both methods ought to be valid in the parameter range considered, there are likely to be small differences between the two results. These ought to be $O(\lambda)$ fractional corrections so the results seem to be consistent as far as they go.

Finally, we can illustrate the power of our work by noting that any nonperturbative calculation can be used with our formulae[4] for dimensional reduction in the presence of a large density. For instance, since this work was completed, two lattice Monte Carlo studies of the 3D action relevant here have appeared [9–11]. While these numerical results were produced for the nonrelativistic 4D Bose-Einstein condensation problem, we can just as easily apply them to our relativistic case. The two studies are completely consistent but for definiteness we use the values in [11] which gives $c_{\mathrm{r,lat}}=0.671(1)$ and $\overline{f}_{\mathrm{lat}}=0.57(1)$. In this case, the α and η terms of both μ and ρ expressions are fixed by dimensional reduction so the only difference is in the a parameters, and we find that the lattice data leads to $a_{\mathrm{mu,\,lat}}=-0.756$ and $a_{\mathrm{rho,lat}}=-13.2$ compared to our values of -6.01 and -21.8 respectively. While these appear to be large differences, in physical quantities such as the critical density ρ the constant coming from a is in fact overwhelmed by the contribution from the $\ln\lambda$ term for $\lambda\ll 1$ which is where the dimensional reduction process is valid. The difference is more of a comment on the efficacy of different nonperturbative approximations in 3D calculations (see [10] for a good comparison) than particularly important to our results.

In conclusion, we have shown that by organizing the modes into those which can be perturbatively integrated and those which cannot, we minimize the nonperturbative effort needed to study Bose-Einstein condensation at relativistic temperatures and densities. The method is economical and provides a reliable estimate of the critical temperature and critical density.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we consider how the sunset diagram can be decomposed into contributions from different modes.

We begin by considering the $\mu=0$ case and state the results (see, for example, [26,27])

$$\sum_{n_1,n_2=-\infty}^{\infty}\int_{\mathrm{p}_1,\mathrm{p}_2}\frac{1}{(\omega_{n_1}^2+p_1^2+m^2)}\frac{1}{(\omega_{n_2}^2+p_2^2+m^2)}\times\frac{1}{[(\omega_{n_1}+\omega_{n_2})^2+(p_1+p_2)^2+m^2]}=\frac{1}{(4\pi)^2}\left[\frac{1}{4\epsilon}+\ln\frac{M}{3m}+\frac{1}{2}\right]+\mathcal{O}\left(\frac{m}{T}\right)\tag{A1}$$

and

$$\int_{\mathrm{p}_1,\mathrm{p}_2}\frac{1}{(p_1^2+m^2)}\frac{1}{(p_2^2+m^2)}\frac{1}{[(p_1+p_2)^2+m^2]}=\frac{1}{(4\pi)^2}\left[\frac{1}{4\epsilon}+\ln\frac{M}{3m}+\frac{1}{2}\right].\tag{A2}$$

We shall split Eq. (A1) into purely nonstatic internal lines ($n_1\neq 0;n_2\neq 0;n_1\neq n_2$), one static internal line ($n_1\neq 0;n_2=0;n_1=n_2$ and $n_1\neq 0;n_2=0$ and $n_1=0;n_2\neq 0$), and purely static internal lines ($n_1=0;n_2=0$). This covers all the possibilities and we may write

$$\sum_{n_1,n_2=-\infty}^{\infty}=\sum_{\substack{n_1,n_2\neq 0\\ n_1\neq n_2}}+6\,\delta_{n_1}\sum_{n_2>0}+\delta_{n_1}\delta_{n_2}.\tag{A3}$$

Denoting these contributions to Eq. (A1) as $I_{\mathrm{nonstatic}}$, I_{mixed}, and I_{static}, we immediately see that I_{static} is given by Eq. (A2).

The contribution with one static line can be calculated by taking the static line to be massless. This does not cause infrared divergences and is appropriate for the critical theory. We may also set $m=0$ in the other propagators since $T\gg m$. We thus have

$$I_{\mathrm{mixed}}=6\sum_{n>0}\int_{\mathrm{p}_1,\mathrm{p}_2}\frac{1}{p_1^2}\frac{1}{p_2^2+\omega_n^2}\frac{1}{(p_1+p_2)^2+\omega_n^2}+\mathcal{O}\left(\frac{m}{T}\right)\tag{A4}$$

$$=6\sum_{n>0}\left(\frac{e^{\gamma}M^2}{4\pi}\right)^{\epsilon}\frac{\Gamma^3(1/2+\epsilon)}{(4\pi)^{3/2-\epsilon}\Gamma(1+2\epsilon)}\times\int_{\mathrm{p}_1}\frac{1}{p_1^2}\frac{1}{(p_1^2+\omega_n^2)^{1/2+\epsilon}}+\mathcal{O}\left(\frac{m}{T}\right)\tag{A5}$$

$$=6\sum_{n>0}-\left(\frac{e^{\gamma}M^2}{4\pi\omega_n^2}\right)^{2\epsilon}\frac{\Gamma^2(1/2+\epsilon)}{2\epsilon(\epsilon-1/2)(4\pi)^{3-2\epsilon}}+\mathcal{O}\left(\frac{m}{T}\right)\tag{A6}$$

[3] A shift from MS to $\overline{\mathrm{MS}}$ scales is needed.

[4] Note that the normalization of fields and the definition of u or λ often differ from those used here by simple constant factors.

$$= -\left(\frac{e^{\gamma}M^2}{16\pi^3T^2}\right)^{2\epsilon}\frac{3\zeta(4\epsilon)\Gamma^2(1/2+\epsilon)}{\epsilon(\epsilon-1/2)(4\pi)^{3-2\epsilon}}+\mathcal{O}\left(\frac{m}{T}\right) \tag{A7}$$

$$= -\frac{3}{(4\pi)^2}\left[\frac{1}{4\epsilon}+\ln\frac{M}{2T}+\frac{1}{2}\right]+\mathcal{O}\left(\frac{m}{T}\right). \tag{A8}$$

Finally, the purely nonstatic contribution can be found by subtracting the other contributions from Eq. (A1):

$$I_{\text{nonstatic}}=\frac{3}{(4\pi)^2}\left[\frac{1}{4\epsilon}+\ln\frac{M}{2T}+\frac{1}{2}\right]+\mathcal{O}\left(\frac{m}{T}\right). \tag{A9}$$

Turning to the case where $\mu\neq 0$, $I_{\text{nonstatic}}$ is unchanged at leading order in the high temperature expansion since $\mu \ll T$ and we may clearly take the $\mu\to 0$ limit without causing infrared divergences. I_{mixed} is also unchanged upon choosing the static line to be critical. Up to corrections of $\mathcal{O}(\mu^2/T^2)$ and $\mathcal{O}(m/T)$, the non-static and mixed contributions to the sunset diagram sum to zero.

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