Energy and angular momentum flow into a black hole in a binary

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As a black hole in a binary spirals in gradually from large separation, energy and angular momentum flow not only to infinity but also into or out of the hole. In addition, the hole's horizon area increases slowly during this process. In this paper, the changes in the black hole's mass, spin, and horizon area during inspiral are calculated for a hole in a circular binary with a companion body of possibly comparable mass. When the binary is composed of equal mass black holes that have spins aligned with the orbital angular momentum and are rapidly rotating (with spins 99.8% of their maximal values), it is found that the fractional increase in the surface area of each hole's horizon is 1% by the time the binary spirals down to a separation b of $6M$ (where M is the binary's total mass), and 7% down to $b=2M$. The flow of energy and angular momentum into the black holes' horizons changes the number of gravitational-wave cycles in the LIGO band by no more than a tenth of a cycle by the time the binary reaches $b=2M$. The results obtained in this paper are relevant for the detection and analysis of gravitational waves from binary systems containing a black hole.

I. INTRODUCTION

Binary black holes are expected to be among the primary sources of gravitational waves for interferometric detectors [1]. Since we do not have exact solutions of Einstein's equations that represent binary black holes in sufficient generality, we must study these systems perturbatively and/or numerically. One regime in which the evolution of binary black holes is well understood is the early inspiral phase. In this phase, the holes' separation is still much larger than the binary's total mass, and post-Newtonian expansions can be used to analyze the system. Eventually radiation reaction drives the holes together and the post-Newtonian approximation fails. The binary's subsequent evolution must be studied numerically.

While the flow of energy and angular momentum to infinity during inspiral has been calculated to high post-Newtonian order, to date the flow into or out of the black holes' horizons has not been computed except in the extreme-mass-ratio limit; and in that limit, it has been done to very high post-Newtonian order [2] (for numerical work, see e.g. [3]). Absorption (or emission) of energy and angular momentum by the holes' horizons, while much smaller than emission to infinity, might still be important because extraction of weak gravitational signals from noisy detector output using matched filtering requires knowledge of the orbital evolution to very high accuracy, and black hole absorption/emission might affect the evolution at that level. Two purposes of this paper are first, to calculate black hole absorption/emission of energy and angular momentum to leading order in a circular binary with holes of possibly comparable mass, and second, to investigate whether it is relevant for detection and analysis of gravitational waves.

A third purpose of this paper is to provide some information on the interface between the inspiral and merger phases of binary evolution. Numerical simulations of binary black holes typically begin computing at this interface and need initial data representing holes that have spiraled in from infinity, i.e. initial data tied to the inspiral phase and to the post-Newtonian expansions used to describe it. One approach to obtaining such initial data is given in [4]. Since initial data of this sort are not yet being used, one needs to relate the masses, spins, and horizon areas of the black holes present in currently used initial data to the corresponding quantities when the holes were infinitely separated. For this purpose, it is necessary to know how these quantities change during inspiral. In this paper, I calculate the leading-order changes in the holes' masses, spins, and horizon areas during inspiral for a circular binary.

Recently, Price and Whelan [5] have emphasized the role of angular momentum absorption/emission by rapidly rotating black holes at the end of inspiral, when the holes are beginning to merge. Here I focus on the earlier stages of inspiral, when the holes are widely (or moderately) separated and their gravitational effects on each other can be described using black hole perturbation theory.

The results obtained in this paper are actually valid for a black hole in a binary with any companion body (e.g. a neutron star) that is well separated from the hole. The formulas for the changes in black hole quantities presented here depend only on the companion body's mass and not on its internal structure. These formulas therefore remain valid when the companion is not a black hole.

II. FRAMEWORK

I follow the field-theory-in-flat-spacetime notation used in the literature on post-Newtonian expansions (e.g. [6,7]) and denote 3-vectors by boldface letters. A centered dot between 3-vectors denotes the usual inner product in flat 3-space; a hatted 3-vector represents the unit vector in that direction.

Consider a black hole binary undergoing circular motion with the separation b between the holes much larger than their total mass $M=M_1+M_2$, where M_B denotes the mass of the Bth hole. Define $\mu=M_1M_2/M$ and $\eta=\mu/M$. Label the holes BH1 and BH2, and denote their spins by $\mathbf{S}_B$ and horizon areas by A_B for $B=1,2$. Let $S_B=(\mathbf{S}_B\cdot\mathbf{S}_B)^{1/2}$ be the spin magnitudes, and define the parameters χ_B by S_B

$=\chi_B M_B^2$ ($B=1,2$). Throughout this paper I assume $\chi_B \leqslant 0.998$ ($B=1,2$); this restriction is based on the analysis in [8].

Define each black hole's horizon radius $r_{HB}=M_B[1+(1-\chi_B^2)^{1/2}]$, angular velocity $\Omega_{HB}=\chi_B(2r_{HB})^{-1}$, and surface gravity $\kappa_B=(1-\chi_B^2)^{1/2}(2r_{HB})^{-1}$ ($B=1,2$). Introduce the following Newtonian quantities for the binary: the orbital angular momentum $\mathbf{L}_N$, the orbital angular velocity $\Omega_N=(M/b^3)^{1/2}$, and the relative velocity $v=(M/b)^{1/2}$. By assumption, $v\ll 1$.

Since the black holes are widely separated, each hole has a surrounding region that satisfies the following properties: (i) it is far enough from the hole that gravity is weak there; (ii) it does not extend so far that the companion hole's tidal field varies appreciably in the region [9]. We can place in this region an inertial coordinate system in which the hole is (momentarily) at rest. This region and its local coordinates are referred to as the black hole's local asymptotic rest frame (LARF) [9]. Label the two regions around the holes LARF1 and LARF2.

Usually mass and angular momentum are only defined globally in general relativity, using fields at infinity, since precise local definitions are not available. However, for a black hole well separated from its companion, one can define the hole's mass and angular momentum using fields measured in the hole's LARF; these definitions are inherently ambiguous [9–12]. (For further discussion of the ambiguities, see Sec. VI.) I refer to these definitions when discussing a black hole's mass and angular momentum in this paper. I calculate the rates of change of these quantities as measured in the LARF—that is, with respect to time t measured by an inertial observer in the LARF. When integrated over the duration of inspiral, these rates of change should give results exceeding the ambiguities in the definitions of mass and angular momentum, in order to be relevant to the analysis of initial data at the interface between inspiral and merger. This issue will be discussed further in Sec. VI.

I also consider slices of constant time t that begin in the LARF and extend into the black hole, intersecting the horizon in 2-surfaces that correspond to constant ingoing-time slices of a Kerr black hole's horizon. (Alternatively, one can consider slices that intersect a "stretched horizon" as discussed in [13] and references therein.) The rate of area increase of these 2-surfaces can be calculated using the results of Hawking and Hartle [14] combined with black hole perturbation theory [15,16]. The quantities dM_B/dt and dS_B/dt can then be obtained from dA_B/dt using the first law of black hole mechanics $dM=(\kappa/8\pi)dA+\Omega_H dJ$ and the relation $\omega dJ=mdM$ for black hole perturbation modes of angular frequency ω and azimuthal angular number m [16,17,13]. (Here J refers to the black hole's angular momentum.)

Throughout this paper, I focus on BH1 and the changes in its parameters. The corresponding formulas for BH2 are simply obtained by exchanging the subscripts $1\leftrightarrow2$ in the final results [e.g. Eqs. (11)]. In Sec. III, I consider the special situation in which BH2 is held stationary with respect to BH1. The results from this artificial scenario are used in Sec. IV to analyze a circular binary with black hole spins aligned

or anti-aligned with $\mathbf{L}_N$. The more general case of spins not fully aligned or anti-aligned with $\mathbf{L}_N$ is treated in Sec. V.

III. STATIONARY COMPANION

In this section, I calculate the tidal distortion BH1 suffers when BH2 is held stationary. This involves solving for the Weyl tensor component ψ_0, which contains complete information about the gravitational perturbation on BH1, using the Teukolsky formalism [15]. With ψ_0 in hand, the rates of change of BH1 parameters can be calculated using the results of Hawking and Hartle [14] and Teukolsky and Press [16].

The first step in this process is to calculate BH2's tidal field as seen in LARF1. I will consider only the lowest-order Newtonian tidal field, which is approximately constant throughout LARF1. To calculate this field and its effect on BH1, consider first a fictitious Euclidean 3-space containing a single stationary body of mass M_2 at coordinate location (b,θ_0,ϕ_0) in a spherical polar coordinate system. The Newtonian gravitational potential at the field point (r,θ,ϕ) is given in these coordinates by

$$\Phi(r,\theta,\phi)=-4\pi\frac{M_2}{b}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(2l+1)^{-1}\left(\frac{r}{b}\right)^l Y_{lm}^*(\theta_0,\phi_0)$$
$$\times Y_{lm}(\theta,\phi) \tag{1}$$

for $r<b$.

We are interested in the gravitational field only in a small neighborhood of the origin satisfying $r\ll b$. In particular, we would like to evaluate the body's tidal field at the origin, so only the $l=2$ part $\Phi^{(2)}$ of Φ is relevant. The (electric-type) tidal field is given by $\mathcal{E}_{ij}=\Phi_{,ij}^{(2)}$ in Cartesian coordinates. Taking these derivatives in spherical coordinates and evaluating in the usual spherical orthonormal basis yields the tidal field components $\mathcal{E}_{\hat\theta\hat\theta},\mathcal{E}_{\hat\theta\hat\phi},\mathcal{E}_{\hat\phi\hat\phi}$ near the origin $r=0$. The particular combination of relevance to us (see below) is in this way determined to be

$$\mathcal{E}_{\hat\phi\hat\phi}-\mathcal{E}_{\hat\theta\hat\theta}-2i\mathcal{E}_{\hat\theta\hat\phi}=8\pi\sqrt{\frac{6M_2}{5b^3}}\sum_{m=-2}^{2}{}_2Y_{2m}(\theta,\phi)$$
$$\times Y_{2m}^*(\theta_0,\phi_0). \tag{2}$$

Here the functions ${}_2Y_{2m}(\theta,\phi)$ are spin-weighted spherical harmonics [18].

Return now to the black hole binary. The region near BH1, including LARF1, can be described as a perturbed Kerr black hole, and so can be covered by Boyer-Lindquist coordinates (t,r,θ,ϕ). We would like to solve the Teukolsky equation [15] in this region for the Weyl tensor component $\psi_0(r,\theta,\phi)$. If we were considering a single perturbed Kerr black hole as the entire spacetime, the asymptotic form of ψ_0 as $r/M_1\to\infty$ would be the combination $\mathcal{E}_{\hat\phi\hat\phi}-\mathcal{E}_{\hat\theta\hat\theta}-2i\mathcal{E}_{\hat\theta\hat\phi}$ of the external tidal field [13], since ψ_0 vanishes for an unperturbed black hole. In our binary system, ψ_0 acquires this asymptotic form for $M_1\ll r\ll b$, i.e. in LARF1, with the tidal field $\mathcal{E}_{ij}$ being that of BH2. To lowest order, this tidal field is exactly the Newtonian field of a body of mass M_2 at sepa-

ration b, which was calculated above; in particular, the angular dependence of ψ_0 in LARF1 is given by Eq. (2), but with θ and ϕ now representing Boyer-Lindquist coordinates, and θ_0 and ϕ_0 now representing BH2's angular coordinates as seen in LARF1. Therefore, to solve for the perturbation ψ_0 on BH1, we impose the LARF1 boundary condition

$$\psi_0 \rightarrow \frac{8\pi\sqrt{6}M_2}{5b^3} \sum_{m=-2}^{2} {}_2Y_{2m}(\theta,\phi) Y_{2m}^*(\theta_0,\phi_0) \qquad (3)$$

for $M_1 \ll r \ll b$.

It now remains to solve the Teukolsky equation for ψ_0 with the boundary condition (3) and an appropriate no-outgoing-wave boundary condition at the black hole horizon [15]. We express ψ_0 as a sum of modes

$$\psi_0 = \sum_{m=-2}^{2} {}_2Y_{2m}(\theta,\phi) R_m(r) \qquad (4)$$

and solve the radial Teukolsky equation for $R_m(r)$ subject to the no-outgoing-wave boundary condition at the horizon. This yields the radial functions (Eq. (5.7) in Ch. VI of [19])

$$R_m(r) = C_m x^{\gamma_m - 2}(1+x)^{-\gamma_m - 2} F(-4,1,-1+2\gamma_m,-x) \qquad (5)$$

for $m \neq 0$. Here

$$\gamma_m = \frac{im\chi_1}{2(1-\chi_1^2)^{1/2}}, \quad x = \frac{r-r_{H1}}{2M_1(1-\chi_1^2)^{1/2}}, \qquad (6)$$

and F is a hypergeometric function. The $m=0$ mode can be treated separately; since a full treatment reveals that this mode does not contribute to the rates of change of black hole parameters, I ignore it here. The constants C_m are determined by imposing the LARF1 boundary condition (3); we obtain

$$C_m = \frac{8\pi M_2}{5b^3\sqrt{6}} \gamma_m(\gamma_m+1)(4\gamma_m^2-1) Y_{2m}^*(\theta_0,\phi_0). \qquad (7)$$

The leading-order tidal distortion of BH1 due to the presence of a stationary companion of mass M_2 has now been determined. This information allows us to calculate the rates of change of BH1 quantities using the results of Hawking and Hartle [14]. In fact, given the modal decomposition (4), we can easily obtain the relevant rates using explicit formulas provided by Teukolsky and Press [16]. The results are $dM_1/dt = 0$ and

$$\frac{dA_1}{dt} = \frac{64\pi M_1^5 M_2^2 \chi_1^2 \sin^2\theta_0}{5b^6(1-\chi_1^2)^{1/2}} \left(1 - \frac{3}{4}\chi_1^2 + \frac{15}{4}\chi_1^2\sin^2\theta_0\right),$$

$$\frac{dS_1}{dt} = -\frac{(1-\chi_1^2)^{1/2}}{8\pi\chi_1} \frac{dA_1}{dt}$$

$$= -\frac{8M_1^5 M_2^2}{5b^6} \chi_1 \sin^2\theta_0 \left(1 - \frac{3}{4}\chi_1^2 + \frac{15}{4}\chi_1^2\sin^2\theta_0\right). \qquad (8)$$

Here θ_0 is BH2's θ-coordinate—that is, its polar angle with respect to $\mathbf{S}_1$—as measured in LARF1.

Since the effects of only the leading-order tidal field were taken into account above, the expressions (8) are actually valid for any companion body of mass M_2, not just a black hole. The rates (8) of area increase and spin-down have already been derived by Teukolsky [19] in the extreme-mass-ratio limit, i.e. for $M_2 \ll M_1$. The derivation I have presented above establishes the validity of the expressions (8) for comparable-mass black holes as well. Hartle and collaborators [14,20,21,13] have shown that the spin-down of a black hole by an external tidal field is analogous to the Newtonian tidal friction process in a planet-moon system.

The results (8) will be used in the next sections to obtain the corresponding formulas for a binary undergoing circular motion.

IV. EQUATORIAL ORBITS

In this section I study special configurations of the binary in which the black holes are in a circular orbit and have their spins aligned or anti-aligned with the orbital angular momentum $\mathbf{L}_N$. In these scenarios there is no precession of the angular momenta: the spins remain aligned or anti-aligned with $\mathbf{L}_N$. As a result, the companion to each of the holes orbits in the hole's equatorial plane; more precisely, the external tidal field seen by each of the holes rigidly rotates about an axis parallel or antiparallel to the hole's spin axis.

In Boyer-Lindquist coordinates (t,r,θ,ϕ) centered on BH1, with $\mathbf{S}_1$ along $\theta=0$, the t- and ϕ-dependence of the companion's tidal field enter in the combination $\phi - \Omega t$. The rotation rate Ω of the tidal field as seen in LARF1 is to leading order $\Omega = (\hat{\mathbf{L}}_N \cdot \hat{\mathbf{S}}_1)\Omega_N$, where $\hat{\mathbf{L}}_N \cdot \hat{\mathbf{S}}_1 = +1(-1)$ for a prograde (retrograde) orbit. The first correction to this expression for Ω is $O(v^2)$ higher (see Eq. (3.12) in [4]), and will be ignored in this paper.

A. Instantaneous rates

In the rigid ϕ-rotation case, simple formulas given in Eqs. (7.21) of [13] (and reproduced below) specify the rates of change of black hole quantities in terms of a horizon integral I that depends on the particular perturbing gravitational fields present:

$$\frac{dS_1}{dt} = (\Omega - \Omega_{H1})I, \quad \frac{dM_1}{dt} = \Omega\frac{dS_1}{dt} = \Omega(\Omega - \Omega_{H1})I,$$

$$\frac{\kappa_1}{8\pi}\frac{dA_1}{dt} = (\Omega - \Omega_{H1})\frac{dS_1}{dt} = (\Omega - \Omega_{H1})^2 I. \qquad (9)$$

In terms of ingoing Kerr coordinates $(V,r,\theta,\tilde{\phi})$ (see e.g. [22] for a definition), I is an integral of a function of θ and $(\tilde{\phi} - \Omega V)$ over a constant-V slice of the horizon. Since $\tilde{\phi}$-rotations are isometries of the horizon metric, I is independent of V.

Consider an expansion of I in powers of $M_1\Omega$, which is $O(v^3)$ and hence much smaller than 1. The zeroth-order part $I_0 = I|_{\Omega=0}$ is independent of Ω and, in our situation of binary

black holes, can be easily obtained from the results for a stationary companion. From Eqs. (9), we have $\dot{S}_1|_{\Omega=0} = -\Omega_{H1}I_0$, where an overdot indicates a time derivative. But $\Omega=0$ corresponds to a stationary companion, and in this case we have an explicit expression for $\dot{S}_1$ in Eqs. (8). Equating $\dot{S}_1$ in Eqs. (8) to $-\Omega_{H1}I_0$ yields

$$I_0(\theta_0) = \frac{16r_{H1}}{5b^6}M_1^5M_2^2\sin^2\theta_0\left(1 - \frac{3}{4}\chi_1^2 + \frac{15}{4}\chi_1^2\sin^2\theta_0\right), \tag{10}$$

where $\theta_0 = \pi/2$ for the equatorial orbits considered in this section. The general expression (10) with a wider range of values for θ_0 will be used for non-equatorial orbits in the next section. Since the first correction to I_0 in the expansion of I in powers of $M_1\Omega$ is $O(M_1\Omega) = O(v^3)$, I will approximate I by I_0 throughout this paper.

Note that Eqs. (9) are, strictly speaking, valid only for constant rotation rates Ω. In our situation, radiation reaction drives the binary together and so Ω changes during inspiral. However, the time scale for these changes is the inspiral time scale $\tau_{\text{ins}} \sim bv^{-6}$, where "$\sim$" means "is of the order of"; this is to be compared to the time scale κ_1^{-1} on which the divergence and shear of the null generators of the horizon probe the future [14,17,13]. By assumption, χ_1 is less than or equal to 0.998; this implies $\kappa_1^{-1} < 34M_1$, so κ_1^{-1} is much smaller than τ_{ins}. Therefore Eqs. (9) are valid in our binary system to a very good approximation. The various time scales of interest to us will be discussed in more detail below.

Note also that Eqs. (9) [and, in addition, Eqs. (11), (21), and (22) below] are valid only when integrated over time intervals much longer than κ_1^{-1} (see the discussion in Sec. VI C 11 of [13]). In this paper, I am interested in integrating these equations over the entire inspiral—that is, over time intervals of order τ_{ins}—so this condition is certainly satisfied.

After putting $I_0(\pi/2)$ and $\Omega = (\hat{\mathbf{L}}_N \cdot \hat{\mathbf{S}}_1)\Omega_N$ into Eqs. (9), we obtain the following rates of change of BH1 quantities for a circular orbit with spins aligned or anti-aligned with $\mathbf{L}_N$:

$$\frac{dS_1}{dt} = (\Omega - \Omega_{H1})I_0(\pi/2)$$

$$= \left(\frac{dJ}{dt}\right)_N\frac{v^5}{4}\left(\frac{M_1}{M}\right)^3(1+3\chi_1^2)\left\{-\chi_1 + 2(\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1)\right.$$

$$\left.\times[1+(1-\chi_1^2)^{1/2}]\frac{M_1}{M}v^3\right\},$$

$$\frac{dM_1}{dt} = \Omega(\Omega - \Omega_{H1})I_0(\pi/2)$$

$$= \left(\frac{dE}{dt}\right)_N\frac{v^5}{4}\left(\frac{M_1}{M}\right)^3(1+3\chi_1^2)\left\{-(\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1)\chi_1\right.$$

$$\left.+2[1+(1-\chi_1^2)^{1/2}]\frac{M_1}{M}v^3\right\}, \tag{11}$$

$$\frac{dA_1}{dt} = 8\pi\kappa_1^{-1}(\Omega - \Omega_{H1})^2I_0(\pi/2)$$

$$= \frac{64\pi M_1^5M_2^2(1+3\chi_1^2)}{5b^6(1-\chi_1^2)^{1/2}}$$

$$\times\left\{\chi_1 - 2(\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1)[1+(1-\chi_1^2)^{1/2}]\frac{M_1}{M}v^3\right\}^2.$$

In these formulas, the Newtonian quadrupole expressions for energy and angular momentum flow to infinity are [23,24]

$$\left(\frac{dE}{dt}\right)_N = \frac{32}{5}\eta^2v^{10}, \quad \left(\frac{dJ}{dt}\right)_N = \frac{32}{5}\eta^2Mv^7, \tag{12}$$

where $v = (M/b)^{1/2}$ and $\eta = M_1M_2/M^2$. Note that energy and angular momentum absorption/emission by a rotating (non-rotating) black hole is 2.5 (4) post-Newtonian orders below the quadrupole emission (12) to infinity, as first derived in the extreme-mass-ratio limit by Poisson and Sasaki [25] and Tagoshi, Mano, and Takasugi [2]. The rates of change for BH2 are obtained by exchanging the subscripts $1\leftrightarrow2$ in the formulas (11).

The energy absorption/emission rate $\dot{M}_1$ given above agrees in the limit $M_2/M\to0$ with the lowest-order expression obtained by Tagoshi, Mano, and Takasugi [2]. Those authors have calculated this rate in the extreme-mass-ratio limit, for a circular equatorial orbit, to much higher order in v than I have done here. However, their results are not applicable to comparable-mass binaries, while the formulas (11) are.

The expressions (11) are valid even if BH1's companion is not a black hole, provided the companion's mass is substituted for M_2.

B. Total changes during inspiral

In this subsection, I integrate Eqs. (11) to calculate the total changes in M_1, S_1, and A_1 during inspiral. I take into account only the leading-order Newtonian effects of radiation reaction when computing orbital decay; given this approximation, the orbital separation b evolves as [24,22]

$$b(t) = b_0(1 - t/\tau_0)^{1/4}, \tag{13}$$

where $\tau_0 = (5/256)b_0^4(\mu M^2)^{-1}$. I also ignore all post-Newtonian corrections to the orbital angular velocity Ω_N.

It is convenient to parametrize the orbit by separation b instead of time t. The total change in a parameter, say S_1, from infinite separation to separation b is denoted $\Delta S_1(b)$ and is calculated by integrating Eqs. (11). As a first approximation, the quantities M_B and S_B $(B=1,2)$ on the right-hand sides of Eqs. (11) can be considered constants during inspiral. The reason is that the time scales for evolution of M_B and S_B are much longer than the inspiral time scale $\tau_{\text{ins}} \sim bv^{-6}$. Indeed, the time scale for evolution of the masses is $\tau_M \sim M_B/\dot{M}_B \sim bv^{-13}$, and for the spins is $\tau_S \sim S_B/\dot{S}_B \sim bv^{-10}$. So $\tau_M \gg \tau_S \gg \tau_{\text{ins}}$ and we can safely treat M_B and

TABLE I. Normalized change $\Delta S_1/M_1^2$ in spin evaluated at $b/M=100$, 20, and 6 for an equal-mass binary with $\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1=1$. For rapidly rotating holes ($\chi_1=\chi_2=0.998$), this change is also evaluated at $b/M=2$.

χ_1	$b/M=100$	$b/M=20$	$b/M=6$	$b/M=2$
0	$9.\times10^{-10}$	$2.\times10^{-7}$	$2.\times10^{-5}$	
0.5	$-7.\times10^{-7}$	$-2.\times10^{-5}$	$-2.\times10^{-4}$	
0.998	$-3.\times10^{-6}$	$-8.\times10^{-5}$	$-8.\times10^{-4}$	$-6.\times10^{-3}$

TABLE III. Normalized change $\Delta A_1/A_1$ in horizon area evaluated at $b/M=100$, 20, and 6 for an equal-mass binary with $\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1=1$. For rapidly rotating holes ($\chi_1=\chi_2=0.998$), this change is also evaluated at $b/M=2$.

χ_1	$b/M=100$	$b/M=20$	$b/M=6$	$b/M=2$
0	$6.\times10^{-13}$	$2.\times10^{-9}$	$8.\times10^{-7}$	
0.5	$2.\times10^{-7}$	$5.\times10^{-6}$	$4.\times10^{-5}$	
0.998	$5.\times10^{-5}$	$1.\times10^{-3}$	$1.\times10^{-2}$	$7.\times10^{-2}$

S_B ($B=1,2$) as constants on the right-hand sides of Eqs. (11) when integrating over inspiral.

With these approximations, the normalized changes in BH1 parameters from infinite separation to separation b are

$$\frac{\Delta S_1}{M_1^2}(b)=\frac{\eta M_1}{4M}(1+3\chi_1^2)\left\{-\frac{\chi_1}{4}\left(\frac{M}{b}\right)^2+(\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1)\right.$$

$$\left.\times[1+(1-\chi_1^2)^{1/2}]\frac{2M_1}{7M}\left(\frac{M}{b}\right)^{7/2}\right\},$$

$$\frac{\Delta M_1}{M_1}(b)=\frac{\eta}{4}\left(\frac{M_1}{M}\right)^2(1+3\chi_1^2)\left\{-(\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1)\frac{\chi_1}{7}\left(\frac{M}{b}\right)^{7/2}\right.$$

$$\left.+[1+(1-\chi_1^2)^{1/2}]\frac{M_1}{5M}\left(\frac{M}{b}\right)^5\right\}, \qquad (14)$$

$$\frac{\Delta A_1}{A_1}(b)=\frac{\eta M_1^2(1+3\chi_1^2)}{8Mr_{H1}(1-\chi_1^2)^{1/2}}\left[\frac{\chi_1^2}{2}\left(\frac{M}{b}\right)^2-(\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1)\frac{8\chi_1}{7}\right.$$

$$\left.\times\frac{r_{H1}}{M}\left(\frac{M}{b}\right)^{7/2}+\frac{4}{5}\left(\frac{r_{H1}}{M}\right)^2\left(\frac{M}{b}\right)^5\right],$$

where $r_{H1}=M_1[1+(1-\chi_1^2)^{1/2}]$. To evaluate these changes, one can put into the formulas (14) the values of M_1, S_1, and A_1 at infinite separation or, for that matter, at any separation much larger than M, because the changes in these quantities during inspiral are small. Once again, the changes for BH2 are obtained by exchanging the subscripts $1\leftrightarrow2$ in the expressions (14).

The normalized parameter changes (14), evaluated at different stages during inspiral, are displayed in Tables I–III for an equal-mass binary ($M_1=M_2$) with $\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1=1$. Since a binary composed of slowly rotating black holes is expected to be undergoing a transition from inspiral to merger by the

TABLE II. Normalized change $\Delta M_1/M_1$ in mass evaluated at $b/M=100$, 20, and 6 for an equal-mass binary with $\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_1=1$. For rapidly rotating holes ($\chi_1=\chi_2=0.998$), this change is also evaluated at $b/M=2$.

χ_1	$b/M=100$	$b/M=20$	$b/M=6$	$b/M=2$
0	$3.\times10^{-13}$	$1.\times10^{-9}$	$4.\times10^{-7}$	
0.5	$-2.\times10^{-10}$	$-5.\times10^{-8}$	$-3.\times10^{-6}$	
0.998	$-9.\times10^{-10}$	$-2.\times10^{-7}$	$-2.\times10^{-5}$	$-6.\times10^{-4}$

time it reaches $b=6M$, the endpoint of integration is chosen to be $b=6M$ when $\chi_1=0$ and 0.5. For rapidly rotating holes ($\chi_1=\chi_2=0.998$), the endpoint is chosen to be $b=2M$. The assumption $M\ll b$ is not valid at and near these endpoints. The results presented here are most accurate in the early stages of inspiral, when the black holes are widely separated, and are a rough estimate of the true parameter changes in the late stages of inspiral.

C. Effect on orbital evolution

The orbital evolution of binary black holes is affected by the absorption/emission of energy and angular momentum by the holes. In particular, the number of orbits—and hence the number of gravitational-wave cycles emitted to infinity—changes when black hole absorption/emission is accounted for. To estimate this effect, let us consider a circular, nearly Newtonian binary, with spins aligned or anti-aligned with $\mathbf{L}_N$, that is losing orbital energy and angular momentum to infinity via Newtonian quadrupole radiation (12), and to the black holes via tidal interaction as specified by Eqs. (11). Since $\dot{M}_B=\Omega\dot{S}_B=(\hat{\mathbf{L}}_N\cdot\hat{\mathbf{S}}_B)\Omega_N\dot{S}_B$ ($B=1,2$), circular, nearly Newtonian orbits remain circular. Therefore the evolution of the separation $b(t)$ is determined by setting the rate of change of Newtonian orbital energy (given by $E_{\text{orb}}=-M_1M_2/2b$) to the rate of energy loss to infinity and to the holes:

$$\frac{dE_{\text{orb}}}{dt}=\frac{M_1M_2}{2b^2}\frac{db}{dt}=-\left(\frac{dE}{dt}\right)_N-\frac{dM_1}{dt}-\frac{dM_2}{dt}, \qquad (15)$$

where $(dE/dt)_N$ is given in Eqs. (12) and $\dot{M}_1$ in Eqs. (11) [with $\dot{M}_2$ obtained by exchanging the subscripts $1\leftrightarrow2$ in Eqs. (11)].

The number of gravitational-wave cycles N_1 emitted to infinity from initial time t_i to final time t_f (corresponding to separations b_i and b_f) is given by

$$N=\int_{t_i}^{t_f}dt\,\frac{\Omega_N}{\pi}=\frac{1}{\pi}\int_{b_i}^{b_f}db\,\frac{dt}{db}\left(\frac{M}{b^3}\right)^{1/2}, \qquad (16)$$

where dt/db is determined from Eq. (15). This number is to be compared with the number of cycles N_2 obtained by ignoring black hole absorption/emission of energy and angular momentum, i.e. by setting $\dot{E}_{\text{orb}}$ equal to $-(dE/dt)_N$. The difference $\Delta N=N_1-N_2$ measures the effect of black hole absorption/emission on the binary's orbital evolution.

TABLE IV. Change ΔN in the number of gravitational-wave cycles due to black hole absorption/emission, for various values of total mass M and mass ratio M_1/M_2. The initial separation is such that the wave frequency is 10 Hz and the spins satisfy $\chi_1 = \chi_2 = 0.998$ and $\hat{\mathbf{L}}_N \cdot \hat{\mathbf{S}}_1 = \hat{\mathbf{L}}_N \cdot \hat{\mathbf{S}}_2 = 1$. The numbers without parentheses are for a final separation b_f of $6M$; those with parentheses are for b_f equal to the larger of $2M$ or the separation at which the wave frequency is 1000 Hz.

$M(M_\odot)$	$M_1/M_2 = 1$	$M_1/M_2 = 2$	$M_1/M_2 = 4$
5	0.07 (0.07)	0.11 (0.11)	0.23 (0.24)
20	0.05 (0.07)	0.07 (0.10)	0.16 (0.22)
50	0.03 (0.06)	0.05 (0.08)	0.11 (0.18)

The values of ΔN obtained by setting b_i to be the separation at which the gravitational-wave frequency is 10 Hz (the low-frequency end of the Laser Interferometer Gravitational Wave Observatory (LIGO) band), χ_1 and χ_2 to be 0.998, and the spins to be aligned with $\mathbf{L}_N$ (i.e. $\hat{\mathbf{L}}_N \cdot \hat{\mathbf{S}}_1 = \hat{\mathbf{L}}_N \cdot \hat{\mathbf{S}}_2 = 1$) are displayed in Table IV for various choices of total mass M (in units of a solar mass $M_\odot$) and mass ratio M_1/M_2. In the table, the numbers without parentheses are obtained by setting $b_f = 6M$, and those with parentheses by setting b_f to be the larger of $2M$ or the separation at which the wave frequency is 1000 Hz (the high-frequency end of the LIGO band). For non-rotating black holes ($\chi_1 = \chi_2 = 0$), the corresponding values of ΔN (with $b_f = 6M$) are all less than 10^{-2}.

The values of ΔN in Table IV indicate that black hole absorption/emission of energy and angular momentum during inspiral may not be an important effect for the detection (by LIGO and Virgo) and analysis of gravitational waves from comparable-mass black holes. Indeed, post-Newtonian corrections to the equations of motion and energy loss have far greater influence on the number of wave cycles emitted by the binary [6,7]. It should be noted, however, that black hole absorption or emission could have a much larger impact on the orbital evolution of rapidly rotating holes when they are beginning to merge, as suggested by Price and Whelan [5]. They have presented models in which the tidal torque that results from black hole absorption/emission of angular momentum plays a crucial role in the late stages of binary evolution (see Fig. 1 in [5]). The perturbative methods used in this paper (based on wide separation of the binary) are not valid in the close limit analyzed in [5].

It has been pointed out by Hughes [3] that in the extreme-mass-ratio limit, black hole absorption/emission can strongly influence the binary's orbital evolution and is an important effect for the Laser Interferometer Space Antenna.

V. NON-EQUATORIAL ORBITS

In general, binary black holes are not expected to have spins aligned with the orbital angular momentum $\mathbf{L}_N$. This misalignment causes the spins and orbit to precess in a complicated way due to spin-orbit and spin-spin coupling [26,7]. Each black hole's companion is not, in general, confined to

the hole's equatorial plane, and so the formulas in the previous section are not applicable. However, for orbits suitably close to the equatorial plane (see below for details), one can imagine using an approximation scheme in which at each instant the companion's θ-velocity is ignored; that is, the companion is taken to be rigidly rotating in the ϕ-direction at each point on the orbit. The changes in black hole parameters can then be calculated by putting the instantaneous ϕ-velocity into the rigid ϕ-rotation formulas (9) at each point on the orbit. In this section, I construct such an approximation scheme.

A. Description of orbit

The evolutions of the spins and orbit are described by the equations [26,7]

$$\dot{\mathbf{S}}_B = \mathbf{\Omega}_{\text{spin}}^{(B)} \times \mathbf{S}_B, \quad \dot{\mathbf{L}}_N = \mathbf{\Omega}_{\text{orb}} \times \mathbf{L}_N - \frac{32}{5} \eta^2 M v^7 \hat{\mathbf{L}}_N,$$

(17)

for $B = 1,2$. The orders of magnitude of the precession frequencies are $\Omega_{\text{spin}} \sim v^3 b^{-1}$ and $\Omega_{\text{orb}} \sim v^4 b^{-1}$. Since the Newtonian angular velocity is $\Omega_N \sim v b^{-1}$, both Ω_{spin} and Ω_{orb} are much smaller than Ω_N. This means that, over a few orbital periods, $\mathbf{L}_N(t)$ and $\mathbf{S}_B(t)$ do not change much due to precession. Thus, the companion's orbit as seen in LARF1 is to a good approximation confined to a single plane with normal vector $\mathbf{n} = \hat{\mathbf{L}}_N(t)$ along the instantaneous direction of the orbital angular momentum, on time scales of a few orbital periods.

In this subsection, I analyze the trajectory of a particle in a planar, circular orbit of arbitrary orientation in a fictitious Euclidean 3-space in terms of spherical coordinates. This information will be used to specify the rotation rate and orientation of the companion's tidal field as seen in LARF1. Denote the particle's radial coordinate by b, its constant (non-negative) angular velocity by ω, and the normal to its orbital plane by $\mathbf{n}$. The angle of inclination of the normal with respect to the z-axis is denoted θ_n; so $\cos\theta_n = \mathbf{n} \cdot \mathbf{e}_{\hat{z}} = n_z$. Assume the orbit is centered on the origin, so the particle's position $\mathbf{X}(t)$ at time t is given by a rotation $R(\mathbf{n}, \omega t)$ about $\mathbf{n}$, by an angle ωt, of the initial position $\mathbf{X}_0$.

In Cartesian coordinates, the particle's trajectory is given by $\mathbf{X}(t) = \mathbf{X}_0 \cos\omega t + (\mathbf{n} \times \mathbf{X}_0)\sin\omega t$. In terms of the particle's angular coordinates $\theta(t)$ and $\phi(t)$, $\mathbf{X}(t)$ is equal to $b[\sin\theta(t)\cos\phi(t), \sin\theta(t)\sin\phi(t), \cos\theta(t)]$. I choose the initial position to be in the equatorial plane, i.e. $Z_0 = \mathbf{X}_0 \cdot \mathbf{e}_{\hat{z}} = 0$. This choice does not affect the orbit-averaged quantities that I calculate later in this section.

The angular functions $\theta(t)$ and $\phi(t)$ can now be expressed in terms of $\mathbf{n}$ and ω using the relations above. The quantities of interest are $\sin^2\theta(t)$ and $\dot{\phi}(t)$, which are determined to be

$$\sin^2\theta(t) = 1 - \sin^2\theta_n \sin^2\omega t, \quad \dot{\phi}(t) = \frac{\omega\cos\theta_n}{\sin^2\theta(t)}.$$

(18)

B. Approximation scheme

Return now to our black hole binary, and go to Boyer-Lindquist coordinates (t,r,θ,ϕ) in LARF1. The companion's trajectory as seen in LARF1 will be described (to lowest order in v) by angular functions $\theta(t)$ and $\phi(t)$ given by the expressions (18) with ω replaced by Ω_N and θ_n now referring to the angle of inclination of $\mathbf{L}_N(t)$ with respect to $\mathbf{S}_1(t)$, that is, $\cos\theta_n = \hat{\mathbf{L}}_N(t) \cdot \hat{\mathbf{S}}_1(t)$. After these substitutions, we have

$$\sin^2\theta(t) = 1 - (1 - [\hat{\mathbf{L}}_N(t) \cdot \hat{\mathbf{S}}_1(t)]^2)\sin^2\Omega_N t,$$

$$\dot\phi(t) = \frac{\Omega_N \hat{\mathbf{L}}_N(t) \cdot \hat{\mathbf{S}}_1(t)}{\sin^2\theta(t)}. \tag{19}$$

Since θ_n is now time dependent, these expressions are meaningful only when used to calculate orbit-averaged quantities.

Consider the regime in which $\sin^2\theta(t)$ and $\dot\phi(t)$ are slowly varying; more precisely, require them to be approximately constant on the time scale κ_1^{-1} associated with the horizon. As noted before, this is the time scale on which the null generators of the horizon probe the future [14,17,13]. The teleological behavior of the horizon is, however, exponentially limited; that is, the influence of future events on the horizon decays exponentially in time, with decay rate κ_1 (see, e.g., the discussion of teleological Green's functions in [13]). We thus require $|\dot\phi/\ddot\phi|$ and $|(1/\sin^2\theta)d(\sin^2\theta)/dt|^{-1}$ [which are the same to leading order by Eqs. (19)] to be only several times larger than κ_1^{-1}, rather than orders of magnitude larger.

By assumption, $\chi_1 \lesssim 0.998$, so κ_1^{-1} is less than $34M_1$. Our requirement can then be expressed as

$$34\alpha M_1\Omega_N \lesssim \frac{\sin^2\theta(t)}{\sin^2\theta_n|\sin 2\Omega_N t|} \tag{20}$$

for all t, where α is a number roughly in the range 2–4. A sufficient condition for this constraint to be satisfied is $\cot^2\theta_n \geq 34\alpha M_1\Omega_N$. This requires $\hat{\mathbf{L}}_N(t)$ to be near one of the polar axes $\pm\hat{\mathbf{S}}_1(t)$, which correspond to $\theta = 0,\pi$; or, equivalently, the orbital plane must be near the equatorial plane.

We are interested in separations as small as $b = 6M$, so Ω_N can be as large as $(6^{3/2}M)^{-1}$. For this reason, I impose the constraint $\cot^2\theta_n \geq 34\alpha 6^{-3/2}$ and set α to be approximately 3, obtaining the approximate constraints $0 \leq \theta_n \lesssim \pi/9$ or $8\pi/9 \lesssim \theta_n \leq \pi$. In other words, $\hat{\mathbf{L}}_N(t)$ is within 20° cones around the polar axes, or, equivalently, the inclination angle of the orbit with respect to the equatorial plane is less than or (approx.) equal to 20°.

For the approximation scheme in this section to be valid, we require further that in the horizon's reference frame, the external tidal field should rotate primarily in the ϕ-direction and not significantly in the θ-direction. More precisely, we require $|\dot\theta| \ll |\dot\phi - \Omega_{H1}|$. The rates of change presented in Eqs. (21) and (22) below are subject to this condition. For

most values of χ_1, this condition is automatically satisfied throughout inspiral (down to $b = 6M$). Even if it is not satisfied at some point during inspiral, the restriction on θ_n discussed above ensures that the effect of the θ-motion, when integrated over inspiral, is negligible compared to that of the ϕ-motion, for almost all values of χ_1.

With the above restriction on θ_n, we can at each instant take $\sin^2\theta(t)$ and $\dot\phi(t)$ to be constant relative to the horizon time scale κ_1^{-1}, and apply the rigid ϕ-rotation formulas (9) with the instantaneous values $\theta(t)$ and $\dot\phi(t)$ put in. This yields

$$\frac{dS_1}{dt} = [\dot\phi(t) - \Omega_{H1}]I_0[\theta(t)],$$

$$\frac{dM_1}{dt} = \dot\phi(t)[\dot\phi(t) - \Omega_{H1}]I_0[\theta(t)], \tag{21}$$

$$\frac{\kappa_1}{8\pi}\frac{dA_1}{dt} = [\dot\phi(t) - \Omega_{H1}]^2 I_0[\theta(t)],$$

where $\theta(t)$ and $\dot\phi(t)$ are given by Eqs. (19) and I_0 by Eq. (10).

C. Orbit-averaged quantities

Next I would like to average these rates of change over an orbit assuming the binary's masses, spins, separation, and orbital angular momentum are approximately constant over an orbital period. This assumption is justified by the following ordering of the relevant time scales: $\Omega_N^{-1} \ll \Omega_{spin}^{-1}$, $\Omega_{orb}^{-1} \ll \tau_{ins} \ll \tau_S \ll \tau_M$. We can therefore take all the quantities on the right-hand sides of Eqs. (21) except $\theta(t)$ and $\dot\phi(t)$ to be constant, to a good approximation, when averaging over an orbit. I denote orbit averages by angular brackets $\langle\ \rangle$; so, for example, $\langle\dot{S}_1\rangle = (\Omega_N/2\pi)\int_0^{2\pi/\Omega_N}\dot{S}_1 dt$. Plugging the expressions (19) into Eqs. (21) and performing the orbit averages (as defined above) yields

$$\left\langle\frac{dS_1}{dt}\right\rangle = \frac{r_{H1}}{10b^6}M_1^5 M_2^2$$

$$\times[16(1+3\chi_1^2)\{2\Omega_N\mathcal{N}_1(t) - \Omega_{H1}[\mathcal{N}_1^2(t)+1]\}$$

$$+ 15\chi_1^2[\mathcal{N}_1^2(t)-1]$$

$$\times\{4\Omega_N\mathcal{N}_1(t) - \Omega_{H1}[3\mathcal{N}_1^2(t)+1]\}],$$

$$\left\langle\frac{dM_1}{dt}\right\rangle = \frac{2r_{H1}}{5b^6}M_1^5 M_2^2\Omega_N\mathcal{N}_1(t)\{2\Omega_N(4-3\chi_1^2)\text{sign}[\mathcal{N}_1(t)]$$

$$-8\Omega_{H1}(1+3\chi_1^2) + 15\chi_1^2\{2\Omega_N\mathcal{N}_1(t)$$

$$+ \Omega_{H1}[1-\mathcal{N}_1^2(t)]\}\}, \tag{22}$$

$$\left\langle \frac{dA_1}{dt} \right\rangle = \frac{8\pi r_{H1}^2 M_1^5 M_2^2}{5b^6(1-\chi_1^2)^{1/2}}\{(16-12\chi_1^2)\{\Omega_{H1}^2[\mathcal{N}_1^2(t)+1]$$
$$-4\Omega_{H1}\Omega_N\mathcal{N}_1(t)+2\Omega_N^2|\mathcal{N}_1(t)|\}$$
$$+15\chi_1^2\{\Omega_{H1}^2[3\mathcal{N}_1^4(t)+2\mathcal{N}_1^2(t)+3]$$
$$-8\Omega_{H1}\Omega_N\mathcal{N}_1(t)[\mathcal{N}_1^2(t)+1]+8\Omega_N^2\mathcal{N}_1^2(t)\}\},$$

where $\mathcal{N}_B(t) = \hat{\mathbf{L}}_N(t)\cdot\hat{\mathbf{S}}_B(t)$ for $B=1,2$. The corresponding expressions for BH2 can be obtained by exchanging the subscripts $1\leftrightarrow2$ in Eqs. (22). Note that these equations are valid only for $\mathcal{N}_B(t)$ suitably close to ±1, as discussed above. The formulas (22) can be applied to a black hole in a binary with any companion body (e.g. a neutron star) that has mass M_2 and is well separated from the hole.

Numerical integration of Eqs. (22) using the 2.5 post-Newtonian equations of motion for spinning bodies ([7] and references therein) yields results comparable to those in Tables I–III.

VI. DISCUSSION

Having obtained the leading-order changes in a black hole's mass and spin during inspiral [see Eqs. (14)], we must check whether these changes exceed the ambiguities inherent in the definitions of mass and spin [9]. Denote by δM and δS the magnitudes of the mass and spin ambiguities. From Eqs. (1.8) in [9],

$$\delta M \sim \frac{ML^2}{\mathcal{R}^2}, \quad \delta S \sim \frac{M^3 L}{\mathcal{R}^2}, \quad (23)$$

where M and L are the mass and size of the (isolated) body in question, and $\mathcal{R}$ is the external universe's radius of curvature. For a black hole in a binary, say BH1, $L\sim M_1$ and $\mathcal{R}^2\sim b^3/M_2$. This implies

$$\frac{\delta M_1}{M_1} \sim \frac{\delta S_1}{M_1^2} \sim \eta\frac{M_1}{M}\left(\frac{M}{b}\right)^3. \quad (24)$$

From Eqs. (14), the changes ΔM_1 and ΔS_1 from infinite separation to separation b are

$$\frac{\Delta M_1}{M_1} \sim \eta\left(\frac{M_1}{M}\right)^2\left(\frac{M}{b}\right)^{7/2}, \quad \frac{\Delta S_1}{M_1^2} \sim \eta\frac{M_1}{M}\left(\frac{M}{b}\right)^2. \quad (25)$$

So, at separation b, we have

$$\frac{\Delta M_1}{\delta M_1} \sim \frac{M_1}{M}\left(\frac{M}{b}\right)^{1/2}, \quad \frac{\Delta S_1}{\delta S_1} \sim \frac{b}{M}. \quad (26)$$

We conclude that $|\Delta S_1|$ exceeds the ambiguity δS_1 in the definition of spin, but $|\Delta M_1|$ does not rise above δM_1. Note that the concept of tidal work is unambiguous [10–12].

When analyzing initial data that contain a black hole and represent the interface between inspiral and merger, one can define and calculate the hole's mass and spin in different ways, giving different answers corresponding to the ambiguities δM and δS discussed above. Since δM is larger than $|\Delta M|$, the hole's mass can be considered constant during inspiral to the same level of accuracy as used in defining mass. On the other hand, $|\Delta S|$ exceeds δS, so the hole's spin cannot be considered constant; however, as Table I indicates, the changes in spin are small during inspiral.

The results of this work—in particular, Eqs. (14) and (22)—can be used to relate the spin and horizon area of a black hole in a particular initial data set to the spin and horizon area the hole had when infinitely separated from its companion.

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