Teleparallel gravity and dimensional reductions of noncommutative gauge theory

Edwin Langmann*
Department of Theoretical Physics, Royal Institute of Technology, S-100 44 Stockholm, Sweden

Richard J. Szabo†
Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, Scotland

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We study dimensional reductions of noncommutative electrodynamics on flat space, which lead to gauge theories of gravitation. For a general class of such reductions, we show that the noncommutative gauge fields naturally yield a Weitzenböck geometry on spacetime and that the induced diffeomorphism invariant field theory can be made equivalent to a teleparallel formulation of gravity which macroscopically describes general relativity. The Planck length is determined in this setting by the Yang-Mills coupling constant and the noncommutativity scale. The effective field theory can also contain higher curvature and non-local terms which are characteristic of string theory. Some applications to D-brane dynamics and generalizations to include the coupling of ordinary Yang-Mills theory to gravity are also described.

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I. INTRODUCTION

Yang-Mills theory on a flat noncommutative space arises as special decoupling limits of string theory [1,2] and M theory [2,3]. In string theory it represents the low-energy effective field theory induced on D-branes in the presence of a constant background supergravity B field. The inherent non-locality of the interactions in this field theory lead to many exotic effects that do not arise in ordinary quantum field theory, and which can be attributed to "stringy" properties of the model. It is thereby believed that, as field theories, these models can provide an effective description of many of the non-local effects in string theory, but within a much simpler setting. As string theory is a candidate for a unified quantum theory of the fundamental interactions, and in particular of gravitation, it is natural to seek ways to realize this unification in the context of noncommutative gauge theories. Gravity has been previously discussed using the framework of noncommutative geometry in [4], while the unification of Einstein gravity and Yang-Mills theory is obtained in [5] from a spectral action defined on an almost-commutative geometry. In this paper we will describe a particular way that gravitation can be seen to arise in noncommutative Yang-Mills theory on *flat* space.

There are several hints that gravitation is naturally contained in the gauge invariant dynamics of noncommutative Yang-Mills theory. In [6] the strong coupling supergravity dual of maximally supersymmetric noncommutative Yang-Mills theory in four dimensions [7] was studied and it was shown that the effective supergravity Hamiltonian has a unique zero energy bound state which can be identified with a massless scalar field in four dimensions. The ten dimensional supergravity interaction is then of the form of a four dimensional graviton exchange interaction and one may therefore identify the Newtonian gravitational potential in noncommutative gauge theory. The Planck length is determined in this setting by the scale of noncommutativity. Furthermore, noncommutative Yang-Mills theory at one-loop level gives rise to long-range forces which can be interpreted as gravitational interactions in superstring theory [8,9].

At a more fundamental level, general covariance emerges in certain ways from the extended symmetry group that noncommutative gauge theories possess. It can be seen to emerge from the low-energy limit of a closed string vertex operator algebra as a consequence of the holomorphic and anti-holomorphic mixing between closed string states [10]. The diffeomorphism group of the target space acts on the vertex operator algebra by inner automorphisms and thereby determines a gauge symmetry of the induced noncommutative gauge theory. Furthermore, noncommutative Yang-Mills theory can be nonperturbatively regularized and studied by means of twisted large N reduced models [8,11–13]. This correspondence identifies the noncommutative gauge group as a certain large N limit [14] of the unitary Lie group $U(N)$ which is equivalent to the symplectomorphism group of flat space [15]. The noncommutative gauge group can thereby be described as a certain deformation of the symplectomorphism group [or equivalently $U(\infty)$] [16] and the noncommutative gauge theory can be regarded as ordinary Yang-Mills theory with this extended, infinite dimensional gauge symmetry group [17]. Other attempts at interpreting diffeomorphism invariance in noncommutative Yang-Mills theory using reduced models of gauge theory can be found in [18,19].

These features are all consequences of the fact that noncommutative gauge transformations mix internal and spacetime symmetries, and are thereby very different from ordinary gauge symmetries. In the case of noncommutative Yang-Mills theory on flat infinite space, a global translation of the spacetime coordinates can be realized as a local gauge transformation [13,20,21], up to a global symmetry of the field theory. The main consequence of this property is that all gauge invariant operators are non-local in the sense that their translational invariance requires them to be averaged over

*Email address: langmann@theophys.kth.se
†Email address: R.J.Szabo@ma.hw.ac.uk

spacetime. Such constructions are reminescent of general relativity. In addition, the noncommutative gauge symmetry allows for extended gauge invariant operators [21] which are constructed from open Wilson line observables [12,13,22]. These observables exhibit many of the "stringy" features of noncommutative gauge theory [21,23]. They can also be used to construct the appropriate gauge invariant operators that couple noncommutative gauge fields on a D-brane to massless closed string modes in flat space [24], and thereby yield explicit expressions for gauge theory operators dual to bulk supergravity fields in this case.

The fact that the group of global translations is contained in the group of noncommutative gauge transformations is thereby naturally linked with the suggestion that noncommutative Yang-Mills theory may contain gravity. The idea of representing general relativity as a gauge theory of some kind is of course an old idea [25–28]. (See [29] for more recent reviews of the gauge theory approaches to gravitation.) Such models are based on constructing gauge theories with structure groups given by spacetime symmetry groups, such as the Poincaré group, in such a way that the mixing of gauge and spacetime symmetries enables the unambiguous identification of gauge transformations as general coordinate transformations. If noncommutative gauge theory is to contain gravitation in a gauge invariant dynamical way, then its gauge group should admit a local translational symmetry corresponding to general coordinate transformations in flat space. While there are general arguments which imply that this is not the case [16,19], it could be that a particular reduction of noncommutative gauge theory captures the qualitative manner in which noncommutative gauge transformations realize general covariance. In this paper we shall discuss one such possibility. We will show how noncommutative U(1) Yang-Mills theory on *flat* space $\mathbb{R}^n\times\mathbb{R}^n$ can generate a theory of gravitation on $\mathbb{R}^n$. The basic observation underlying the construction is that the algebra of functions on $\mathbb{R}^{2n}$, with a Lie bracket defined in terms of the deformed product of the noncommutative theory, contains the Lie algebra vect($\mathbb{R}^n$) of vector fields on $\mathbb{R}^n$.[1] We show that it is possible to consistently restrict the noncommutative Yang-Mills fields so as to obtain a local field theory whose symmetry group contains diffeomorphism invariance. This construction shows how noncommutative gauge symmetries give very natural and explicit realizations of the mixing of spacetime and internal symmetries required in the old gauge models of gravity. It is also reminescent of a noncommutative version of brane world constructions which localize ten dimensional supergravity down to four dimensions along some noncommutative directions [30].

Gauge theories whose structure group is the group of translations of spacetime lead to teleparallel theories of gravity [31]. These models are built via an explicit realization of Einstein's principle of absolute parallelism. They are defined by a non-trivial vierbein field which can be used to construct a linear connection that carries non-vanishing torsion, but is flat. Such a connection defines what is known as a Weitzenböck geometry on spacetime. The vanishing of the curvature of the connection implies that parallel transport in such a geometry is path independent, and so the geometry yields an absolute parallelism. Teleparallelism thereby attributes gravitation to torsion, rather than to curvature as in general relativity. This class of gravitational theories is thereby a very natural candidate for the effective noncommutative field theory of gravitation, which is induced on flat spacetime. We will see in the following how the gauge fields of the dimensionally reduced noncommutative Yang-Mills theory naturally map onto a Weitzenböck structure of spacetime. A teleparallel theory of gravity can also be viewed as the zero curvature reduction of a Poincaré gauge theory [25,27,28] which induces an Einstein-Cartan spacetime characterized by connections with both non-vanishing torsion and curvature. The zero torsion limit of an Einstein-Cartan structure is of course a Riemannian structure and is associated with ordinary Einstein general relativity. The Weitzenböck geometry is in this sense complementary to the usual Riemannian geometry. More general gauge theories of gravitation can be found in [29,32]. From the present point of view then, noncommutative Yang-Mills theory naturally induces gravitation through a torsioned spacetime, and its full unreduced dynamics may induce gravity on the entire spacetime through the gauging of some more complicated spacetime group, as in [32].

A teleparallel gauge theory of gravity describes the dynamical content of spacetime via a Lagrangian which is quadratic in the torsion tensor $T_{\mu\nu\lambda}$ of a Cartan connection. The most general such Lagrangian is given by[2]

$$\mathcal{L}_T=\frac{1}{16\pi G_{\rm N}}(\tau_1 T_{\mu\nu\lambda}T^{\mu\nu\lambda}+\tau_2 T_{\mu\nu\lambda}T^{\mu\lambda\nu}+\tau_3 T_{\mu\nu}{}^{\nu}T_{\lambda}{}^{\mu\lambda}), \tag{1.1}$$

where $G_{\rm N}$ is the Newtonian gravitational constant and τ_i, $i=1,2,3$, are arbitrary parameters. For generic values of τ_i the field theory defined by Eq. (1.1) is diffeomorphism invariant, but it is not equivalent to Einstein gravity [26]. However, one can demand that the theory (1.1) yields the same results as general relativity in the linearized weak field approximation. It may be shown that there is a one-parameter family of Lagrangians of the form (1.1), defined by the parametric equation

$$2\tau_1+\tau_2=1,\quad \tau_3=-1, \tag{1.2}$$

which defines a consistent theory that agrees with all known gravitational experiments [33]. For such parameter values the Lagrangian (1.1) represents a physically viable gravitational theory which is empirically indistinguishable from general relativity. For the particular solution $\tau_1=\frac{1}{4},\ \tau_2=\frac{1}{2}$ of

[1]In the following, by diffeomorphism invariance we will mean invariance under the connected diffeomorphism group, i.e. under the Lie algebra vect($\mathbb{R}^n$) of infinitesimal diffeomorphisms. In this paper we will not consider any global aspects of the gauge symmetries.

[2]In this paper an implicit summation over repeated upper and lower indices is always understood, except when noted otherwise.

Eq. (1.2), the Lagrangian (1.1) coincides, modulo a total divergence, with the Einstein-Hilbert Lagrangian [28,29,34], and the resulting gauge theory is completely equivalent to Einstein gravity at least for macroscopic, spinless matter. In addition, the Weitzenböck geometry possesses many salient features which makes it particularly well-suited for certain analyses. For instance, it enables a pure tensorial proof of the positivity of the energy in general relativity [35], it yields a natural introduction of Ashtekar variables [36], and it is the natural setting in which to study torsion [37], notably at the quantum level, in systems where torsion is naturally induced, such as in the gravitational coupling to spinor fields.

In this paper we will show that the dimensional reduction of noncommutative gauge theory contains the Lagrangian (1.1) for the particular family of teleparallel theories of gravity defined by $\tau_2=\tau_3$. Thus for the solution $\tau_1=-\tau_2=1$ of Eq. (1.2), it contains a macroscopic description of general relativity. Whether or not these latter constants really arise is a dynamical issue that must be treated by regarding the dimensionally reduced-noncommutative field theory as an effective theory, for example, induced from string theory. We shall not address this problem in the present analysis, except to present a group theoretical argument for the naturality of this choice of parameters. From this result we will determine the gravitational constant in terms of the gauge coupling constant and the noncommutativity scale. We will also find a host of other possible terms in the total Lagrangian which we will attribute to higher curvature and non-local couplings that are characteristic of string theory. Indeed, the particular theory induced by the standard, flat space noncommutative Yang-Mills theory on a D-brane is a special case of the more general construction. In that case we will find quite naturally that the gravitational theory can be invariant only under the volume-preserving coordinate transformations of spacetime, a fact anticipated from string theoretical considerations. We will also describe how the present construction can be generalized to include the coupling of gravity to ordinary gauge fields. These results all show that, at the level of the full unreduced Yang-Mills theory, noncommutative gauge symmetry naturally contains gravitation and also all other possible commutative gauge theories, at least at the somewhat simplified level of dimensional reduction and the principle of absolute parallelism. The constructions shed some light on how the full gauge invariant dynamics of noncommutative Yang-Mills theory incorporates gravitation. At a more pragmatic level, noncommutative Yang-Mills theories give a very natural and systematic way of inducing gauge models of gravity in which the mixing between spacetime and internal degrees of freedom is contained in the gauge invariant dynamics from the onset.

The structure of the remainder of this paper is as follows. In Sec. II we describe the general model of noncommutative U(1) Yang-Mills theory and its gauge invariant dimensional reductions. In Sec. III we describe a particular family of dimensional reductions and compute the induced actions. In Sec. IV we construct a Weitzenböck structure on spacetime from the dimensionally reduced gauge fields, and in Sec. V we relate the leading low-energy dynamics of the induced Lagrangian to a teleparallel theory of gravity. In Sec. VI we specialize the construction to "naive" dimensional reductions and describe the natural relationships to D-branes and volume-preserving diffeomorphisms. In Sec. VII we describe the dynamics induced by certain auxiliary fields which are required to complete the space of noncommutative gauge fields under the reductions. We show that they effectively lead to non-local effects, which are thereby attributed to stringy properties of the induced gravitational model. In Sec. VIII we describe how to generalize the construction to incorporate ordinary gauge fields coupled to the induced gravity theory, and in Sec. IX we conclude with some possible extensions and further analyses of the model presented in this paper. Appendix A at the end of the paper contains various identities which are used to derive quantities in the main text.

II. GENERALIZED NONCOMMUTATIVE ELECTRODYNAMICS

Consider noncommutative U(1) Yang-Mills theory on flat Euclidean space $\mathbb{R}^{2n}$, whose local coordinates are denoted $\xi=(\xi^A)_{A=1}^{2n}$. The star-product on the algebra $C^\infty(\mathbb{R}^{2n})$ of smooth functions f,g: $\mathbb{R}^{2n}\to\mathbb{C}$ is defined by[3]

$$f\star g(\xi)=f(\xi)(\exp\Delta)g(\xi), \tag{2.1}$$

where $\triangle$ is the skew-adjoint bi-differential operator

$$\triangle=\frac{1}{2}\sum_{A<B}\Theta^{AB}(\overleftarrow{\partial}_A\overrightarrow{\partial}_B-\overleftarrow{\partial}_B\overrightarrow{\partial}_A), \tag{2.2}$$

$\partial_A=\partial/\partial\xi^A$, and $\Theta^{AB}=-\Theta^{BA}$ are real-valued deformation parameters of mass dimension -2. The star-product is defined such that Eq. (2.1) is real valued if the functions f and g are. It is associative, noncommutative, and it satisfies the usual Leibniz rule with respect to ordinary differentiation. This implies, in the usual way, that the star commutator

$$[f,g]_\star=f\star g-g\star f \tag{2.3}$$

defines a Lie algebra structure on $C^\infty(\mathbb{R}^{2n})$. In particular, it satisfies the Leibniz rule and the Jacobi identity. The star product of a function with itself can be represented as

$$f\star f(\xi)=f(\xi)(\cosh\Delta)f(\xi), \tag{2.4}$$

while the star commutator of two functions is given by

$$[f,g]_\star(\xi)=f(\xi)(2\sinh\Delta)g(\xi). \tag{2.5}$$

Let us consider the space $\mathcal{YM}$ of U(1) gauge fields on $\mathbb{R}^{2n}$,

$$\mathcal{A}=\mathcal{A}_A d\xi^A, \tag{2.6}$$

[3] Later, when we come to the construction of action functionals, we will need to restrict the space $C^\infty(\mathbb{R}^{2n})$ to its subalgebra consisting of functions which decay sufficiently fast at infinity. Such restrictions can be imposed straightforwardly and so we will not always spell this out explicitly.

where $\mathcal{A}_A \in C^\infty(\mathbb{R}^{2n})$ is a real-valued function. Let $\mathbf{g} \subset C^\infty(\mathbb{R}^{2n})$ be a linear subalgebra of functions, closed under star commutation, which parametrize the infinitesimal, local star-gauge transformations defined by

$$\delta_\alpha \mathcal{A}_A = \partial_A \alpha + e\,\mathrm{ad}_\alpha(\mathcal{A}_A), \quad \alpha \in \mathbf{g}, \tag{2.7}$$

where

$$\mathrm{ad}_\alpha(f) = [\alpha, f]_\star, \quad f \in C^\infty(\mathbb{R}^{2n}), \tag{2.8}$$

denotes the adjoint action of the Lie algebra $\mathbf{g}$ on $C^\infty(\mathbb{R}^{2n})$. We will require that the space $\mathcal{YM}$ is invariant under these transformations. The gauge coupling constant e in Eq. (2.7) will be related later on to the gravitational coupling constant. The gauge transformation (2.7) is defined such that the skew-adjoint covariant derivative

$$\mathcal{D}_A = \partial_A + e\,\mathrm{ad}_{\mathcal{A}_A} \tag{2.9}$$

has the simple transformation property $\delta_\alpha \mathcal{D}_A = e\mathcal{D}_A \alpha$. This and the properties of the star product imply that the linear map $\alpha \mapsto \delta_\alpha$ is a representation of the Lie algebra $\mathbf{g}$,

$$[\delta_\alpha, \delta_\beta] = \delta_{[\alpha,\beta]_\star} \quad \forall \alpha, \beta \in \mathbf{g}. \tag{2.10}$$

It also implies, as usual, that the noncommutative field strength tensor, defined by

$$\mathcal{F}_{AB} = \frac{1}{e}(\mathcal{D} \wedge \mathcal{D})_{AB} = \partial_A \mathcal{A}_B - \partial_B \mathcal{A}_A + e[\mathcal{A}_A, \mathcal{A}_B]_\star, \tag{2.11}$$

transforms homogeneously under star-gauge transformations,

$$\delta_\alpha \mathcal{F}_{AB} = \mathrm{ad}_\alpha(\mathcal{F}_{AB}). \tag{2.12}$$

Let G_{AB} be a flat metric on $\mathbb{R}^{2n}$. Then, since the star commutator (2.5) is a total derivative, the standard action for noncommutative Yang-Mills theory, defined by

$$I_{\mathrm{NCYM}} = \frac{1}{2}\int_{\mathbb{R}^{2n}} d^{2n}\xi \sqrt{\det G}\, G^{AA'} G^{BB'} \mathcal{F}_{AB} \star \mathcal{F}_{A'B'}(\xi), \tag{2.13}$$

is trivially gauge invariant. In the following we will consider reductions of noncommutative gauge theory on $\mathbb{R}^{2n}$ by imposing certain constraints on the spaces $\mathcal{YM}$ and $\mathbf{g}$ and using a generalized action of the form

$$I^W_{\mathrm{NCYM}} = \frac{1}{2}\int_{\mathbb{R}^{2n}} d^{2n}\xi\, W^{AA'BB'}(\xi) \mathcal{F}_{AB} \star \mathcal{F}_{A'B'}(\xi). \tag{2.14}$$

Here $W^{AA'BB'}(\xi)$ are tensor weight functions of rank four with the symmetries

$$W^{AA'BB'} = W^{A'ABB'} = W^{AA'B'B} = W^{BB'AA'}. \tag{2.15}$$

We will require that they transform under star gauge transformations (2.7) such that the action (2.14) is gauge invariant. A sufficient condition for this is

$$\int_{\mathbb{R}^{2n}} d^{2n}\xi [\delta_\alpha W^{AA'BB'}(\xi) f(\xi) + W^{AA'BB'}(\xi)\mathrm{ad}_\alpha(f)(\xi)] = 0 \tag{2.16}$$

for all functions $\alpha \in \mathbf{g}$, $f \in C^\infty(\mathbb{R}^{2n})$ and for each set of indices A, A', B, B'. The functions $W^{AA'BB'}$ generically break the global Lorentz symmetry which is possessed by the conventional action (2.13). They are introduced in order to properly maintain gauge invariance and Lorentz invariance in the ensuing dimensional reductions. The basic point is that the field strength tensor $\mathcal{F}_{AB}$ which appears in Eq. (2.13) corresponds to an irreducible representation, i.e. the rank two antisymmetric representation $\Lambda^2(2n)$, of the Lorentz group of $\mathbb{R}^{2n}$. This will not be so after dimensional reduction, and the tensor densities will essentially enforce the decomposition of the reduced field strengths into irreducible representations of the reduced Lorentz group which may then be combined into the required singlets. Note that the key to this is that, because of Eqs. (2.12) and (2.16), each term in Eq. (2.14) is individually gauge invariant. The condition (2.16) will be used later on to determine an explicit form for the tensor density $W^{AA'BB'}$ in terms of the noncommutative gauge fields.

We will also consider the minimal coupling of the noncommutative gauge theory (2.14) to scalar matter fields. The standard method can be generalized in a straightforward manner. We assume that the scalar bosons are described by real-valued functions $\Phi \in C^\infty(\mathbb{R}^{2n})$ which transform under the infinitesimal adjoint action of the star gauge symmetry group,

$$\delta_\alpha \Phi = \mathrm{ad}_\alpha(\Phi), \quad \alpha \in \mathbf{g}. \tag{2.17}$$

Then, by the usual arguments, the action

$$I_{\mathrm{B}} = \frac{1}{2}\int_{\mathbb{R}^{2n}} d^{2n}\xi\, W(\xi)[G^{AB}\mathcal{D}_A \Phi \star \mathcal{D}_B \Phi(\xi) + m^2 \Phi \star \Phi(\xi)] \tag{2.18}$$

is gauge invariant if the scalar density $W(\xi)$ has the star-gauge transformation property

$$\int_{\mathbb{R}^{2n}} d^{2n}\xi [\delta_\alpha W(\xi) f(\xi) + W(\xi)\mathrm{ad}_\alpha(f)(\xi)] = 0 \qquad \forall \alpha \in \mathbf{g}, \quad f \in C^\infty(\mathbb{R}^{2n}). \tag{2.19}$$

Only a single function W is required for the matter part of the action because its Lorentz invariance properties will not become an issue in the reduction. Note that the scalar fields decouple from the Yang-Mills fields in the commutative limit where all Θ^{AB} vanish.

III. DIMENSIONAL REDUCTION

We will now describe a particular reduction of the generic noncommutative Yang-Mills theory of the previous section. We will denote the local coordinates of $\mathbb{R}^{2n}$ by $\xi = (x^\mu, y^a)$, where $\mu, a = 1, \ldots, n$, and we break the Lorentz symmetry of $\mathbb{R}^{2n}$ to the direct product of the Lorentz groups of $\mathbb{R}^n_x$ and

$\mathbb{R}^n_y$. We will take the noncommutativity parameters to be of the block form

$$\Theta^{AB}=\begin{pmatrix}\theta^{\mu\nu} & \theta^{\mu b}\\ \theta^{a\nu} & \theta^{ab}\end{pmatrix} \quad \text{with} \quad \theta^{\mu\nu}=\theta^{ab}=0, \tag{3.1}$$

and assume that $(\theta^{\mu b})$ is an invertible $n\times n$ matrix. The flat metric of $\mathbb{R}^{2n}$ is taken to be

$$G^{AB}=\begin{pmatrix}\eta^{\mu\nu} & \eta^{\mu b}\\ \eta^{a\nu} & \eta^{ab}\end{pmatrix} \quad \text{with} \quad \eta^{\mu b}=\eta^{a\nu}=0, \tag{3.2}$$

where $(\eta^{\mu\nu})=(\eta^{ab})=\mathrm{diag}(1,-1,\ldots,-1)$. The vanishing of the diagonal blocks $\theta^{\mu\nu}$ will be tantamount to the construction of a quantum field theory on a *commutative* space $\mathbb{R}^n_x$. The y^a can be interpreted as local coordinates on the cotangent bundle of $\mathbb{R}^n_x$ (see the next section), so that the condition $\theta^{ab}=0$ is tantamount to the commutativity of the corresponding "momentum" space. The noncommutativity $\theta^{a\nu},\theta^{\mu b}$ between the coordinate and "momentum" variables will enable the construction of diffeomorphism generators via star commutators below. Having non-vanishing $\theta^{\mu\nu}$ would lead to some noncommutative field theory, but we shall not consider this possibility here. In fact, in that case the noncommutative model only makes sense in string theory [38], so that keeping $\theta^{\mu\nu}=0$ allows us to define a quantum field theory in Minkowski signature without having to worry about the problems of non-unitarity and non-covariance that plague noncommutative field theories on non-Euclidean spacetimes. The bi-differential operator (2.2) which defines the star product is then given by

$$\Delta=\frac{1}{2}\,\theta^{\mu a}(\overleftarrow{\partial}_\mu\overrightarrow{\partial}_a-\overleftarrow{\partial}_a\overrightarrow{\partial}_\mu). \tag{3.3}$$

Let us consider now the linear subspace $\mathbf{g}$ of smooth functions α on $\mathbb{R}^{2n}$ which are linear in the coordinates y,

$$\alpha(\xi)=\alpha_a(x)y^a. \tag{3.4}$$

Using Eq. (A4) we then find that the star commutator of any two elements $\alpha,\beta\in\mathbf{g}$ is given by

$$[\alpha,\beta]_\star(\xi)=([\alpha,\beta]_\star)_a(x)y^a,$$
$$([\alpha,\beta]_\star)_a(x)=\theta^{\mu b}[\beta_b(x)\partial_\mu\alpha_a(x)-\alpha_b(x)\partial_\mu\beta_a(x)]. \tag{3.5}$$

Thus $\mathbf{g}$ is a Lie algebra with respect to the star commutator. If we now define the invertible map

$$\mathbf{g}\to\mathrm{vect}(\mathbb{R}^n_x)$$
$$\alpha\mapsto X_\alpha=-\theta^{\mu a}\alpha_a\,\frac{\partial}{\partial x^\mu} \tag{3.6}$$

onto the linear space of vector fields on $\mathbb{R}^n_x$, then Eq. (3.5) implies that it defines a representation of the Lie algebra $\mathbf{g}$,

$$[X_\alpha,X_\beta]=X_{[\alpha,\beta]_\star} \quad \forall\alpha,\beta\in\mathbf{g}. \tag{3.7}$$

This shows that, via the linear isomorphism (3.6), $\mathbf{g}$ can be identified with the Lie algebra of connected diffeomorphisms of $\mathbb{R}^n_x$.

We now define a corresponding truncation of the space $\mathcal{YM}$ of Yang-Mills fields on $\mathbb{R}^{2n}$ by

$$\mathcal{A}=\omega_{\mu a}(x)y^a dx^\mu+C_a(x)dy^a. \tag{3.8}$$

The reduction (3.8) is the minimal consistent reduction, which is closed under the action of the reduced star-gauge group. It is straightforward to compute the star-gauge transformations (2.7) of the component fields in Eq. (3.8) using the identities (A3) and (A4). One thereby checks that the ansatz (3.8) is consistent, i.e. that the gauge transforms δ_α with gauge functions (3.4) preserve the subspace of $\mathcal{YM}$ of Yang-Mills fields of the form (3.8), and that the "components" of the gauge fields transform as

$$\delta_\alpha\omega_{\mu a}=\partial_\mu\alpha_a+e\,\theta^{\nu b}(\alpha_b\partial_\nu\omega_{\mu a}-\omega_{\mu b}\partial_\nu\alpha_a),$$
$$\delta_\alpha C_a=\alpha_a-e\,\theta^{\mu b}\alpha_b\partial_\mu C_a, \quad \alpha\in\mathbf{g}. \tag{3.9}$$

The curvature components (2.11) of the gauge field (3.8) are likewise easily computed with the result

$$\mathcal{F}_{\mu\nu}(\xi)=\Omega_{\mu\nu a}(x)y^a,$$
$$\Omega_{\mu\nu a}=\partial_\mu\omega_{\nu a}-\partial_\nu\omega_{\mu a}+e\,\theta^{\lambda b}(\omega_{\nu b}\partial_\lambda\omega_{\mu a}-\omega_{\mu b}\partial_\lambda\omega_{\nu a}),$$
$$\mathcal{F}_{\mu a}=\partial_\mu C_a-\omega_{\mu a}-e\,\theta^{\nu b}\omega_{\mu b}\partial_\nu C_a,$$
$$\mathcal{F}_{ab}=0. \tag{3.10}$$

For the scalar fields, the consistent minimal truncation is to functions which are independent of the y coordinates,

$$\Phi(\xi)=\phi(x). \tag{3.11}$$

Using Eq. (A3) the gauge transformation rule (2.17) then implies

$$\delta_\alpha\phi=-\theta^{\mu a}\alpha_a\partial_\mu\phi. \tag{3.12}$$

Under the isomorphism $\mathbf{g}\cong\mathrm{vect}(\mathbb{R}^n_x)$ generated by Eq. (3.6), we see that the gauge transform (3.12) coincides with the standard transformation of a scalar field under infinitesimal diffeomorphisms of $\mathbb{R}^n_x$, i.e. with the natural adjoint action $\delta_\alpha\phi=X_\alpha(\phi)$ of $\mathrm{vect}(\mathbb{R}^n_x)$ on $C^\infty(\mathbb{R}^n_x)$. The gauge covariant derivatives of the truncated fields (3.11) are similarly easily computed to be

$$\mathcal{D}_\mu\Phi=\partial_\mu\phi-e\,\theta^{\nu a}\omega_{\mu a}\partial_\nu\phi,$$
$$\mathcal{D}_a\Phi=0. \tag{3.13}$$

It remains to compute the possible action functionals (2.14) and (2.18) corresponding to the above truncation. To arrive at a gauge invariant action on $\mathbb{R}^n_x$, we make the ansatz

$$
\begin{aligned}
W^{\mu\mu'\nu\nu'}(\xi) &= \eta^{\mu\mu'}\{\eta^{\nu\nu'}[w_\omega(x)+\theta^{\lambda a}w_\lambda^\omega(x)\partial_a \\
&\quad +\theta^{\lambda a}\theta^{\lambda' b}w_{\lambda\lambda'}(x)\partial_a\partial_b] \\
&\quad +\theta^{\lambda a}\theta^{\lambda' b}w_{\lambda\lambda'}^{\nu\nu'}(x)\partial_a\partial_b\}\delta^{(n)}(y), \\
W^{\mu\nu ab}(\xi) &= \eta^{\mu\nu}\eta^{ab}w_C(x)\delta^{(n)}(y), \\
W^{\mu\nu\nu' a}(\xi) &= \eta^{\mu\nu}\theta^{\nu' a}\theta^{\lambda b}w_\lambda^M(x)\partial_b\delta^{(n)}(y), \\
W(\xi) &= w_\phi(x)\delta^{(n)}(y)
\end{aligned}
\tag{3.14}
$$

for the weight functions. The functions w in Eq. (3.14) are smooth functions in $C^\infty(\mathbb{R}_x^n)$, and the ansatz (3.14) yields well-defined action functionals over $\mathbb{R}_x^n$ provided that all component fields live in an appropriate Schwartz subspace of $C^\infty(\mathbb{R}_x^n)$. The choice (3.14) of tensor densities represents a ''minimal'' dimensional reduction which is consistent with the reductions of the fields above and naturally contains Einstein gravity in a particular limit. There are of course many other choices for the functions $W^{AA'BB'}(\xi)$ which are possible, and these will lead to different types of diffeomorphism invariant field theories. It is essentially here that there is the most freedom involved. We have made the choice which will facilitate comparison to previously known results in general relativity and in string theory. Due to the structures of the spacetime metric (3.2), of the field strengths (3.10), and the symmetries (2.15), the remaining components of the tensorial weight functions in Eq. (3.14) need not be specified.

The derivative terms $\theta^{\lambda a}\partial_a$ in Eq. (3.14) will have the overall effect of transforming an a index of $\mathbb{R}_y^n$ into a λ index of $\mathbb{R}_x^n$. The two choices of second order y derivatives in the first line of Eq. (3.14) then correspond to the irreducible decomposition of the reduced field strength tensors $\theta^{\lambda a}\Omega_{\mu\nu a}$ under the action of the Lorentz group SO(1,$n-1$) of $\mathbb{R}_x^n$. These terms come from the rank two tensor $\mathcal{F}_{AB}$ of the original noncommutative Yang-Mills theory which corresponds to the irreducible antisymmetric representation $\Lambda^2(2n)$ of the Lorentz group SO(1,2$n-1$). After dimensional reduction, it induces the rank (1,2) tensor $\theta^{\lambda a}\Omega_{\mu\nu a}$ which corresponds to the decomposable representation

$$
\Lambda^2(n)\otimes\mathbf{n}=\overline{\mathbf{n}}\oplus\overline{\mathbf{n}}\oplus\Lambda_0^{1,2}(n), \tag{3.15}
$$

with $\mathbf{n}$ the defining and $\Lambda_0^{1,2}(n)$ the traceless, antisymmetric (1,2) representation of the reduced Lorentz group SO(1,$n-1$). In other words, the restriction of the antisymmetric rank two representation of the group SO(1,2$n-1$) to its SO(1,$n-1$) subgroup is reducible and decomposes into irreducible representations according to Eq. (3.15). The reduced Yang-Mills Lagrangian should be constructed from Lorentz singlets built out of irreducible representations of SO(1,$n-1$). This requires the incorporation of the three SO(1,$n-1$) singlets corresponding to the Clebsch-Gordan decomposition (3.15). It is achieved by summing over the cyclic permutations of the three indices of the reduced field strength tensor [34], and will be enforced by the given choice (3.14).

The gauge transformation rules for the fields in Eq. (3.14) can be determined from the conditions (2.16) and (2.19). Using these constraints it is straightforward to see that, for the types of terms appearing in Eq. (3.14), the index contractions specified there are essentially unique, in that other choices are either forbidden by star-gauge invariance or else they will produce the same local Lagrangian terms in the end. In this sense, the ''minimal'' choice (3.14) is unique and star-gauge invariance forces very rigid constraints on the allowed tensor weight functions. The restrictions (2.16) and (2.19) are satisfied if the fields w in Eq. (3.14) transform as

$$
\int_{\mathbb{R}^n}d^nx[\delta_\alpha w_\Xi(x)f(x,0)+w_\Xi(x)\mathrm{ad}_\alpha(f)(x,0)]=0, \tag{3.16}
$$

$$
\int_{\mathbb{R}^n}d^nx\ \theta^{\mu a}[\delta_\alpha w_\mu^\Xi(x)\partial_a f(x,0)+w_\mu^\Xi(x)\partial_a\mathrm{ad}_\alpha(f)(x,0)]=0, \tag{3.17}
$$

$$
\int_{\mathbb{R}^n}d^nx\ \theta^{\mu a}\theta^{\nu b}[\delta_\alpha w_{\mu\nu}(x)\partial_a\partial_b f(x,0)+w_{\mu\nu}(x)\partial_a\partial_b\mathrm{ad}_\alpha(f)(x,0)]=0, \tag{3.18}
$$

$$
\int_{\mathbb{R}^n}d^nx\ \theta^{\mu a}\theta^{\nu b}[\delta_\alpha w_{\mu\nu}^{\lambda\lambda'}(x)\partial_a\partial_b f(x,0)+w_{\mu\nu}^{\lambda\lambda'}(x)\partial_a\partial_b\mathrm{ad}_\alpha(f)(x,0)]=0, \tag{3.19}
$$

for all smooth functions $f(x,y)$ which are compactly supported on $\mathbb{R}_x^n$ and quadratic in the y^a's. The index Ξ in Eq. (3.16) denotes the labels $\Xi=\omega,C,\phi$ while $\Xi=\omega,M$ in Eq. (3.17).

We will solve Eqs. (3.16)–(3.19) for the gauge transformations of the functions w appearing in Eq. (3.14) by demanding that these equations lead to local transforms of the fields w. While the non-local integral transforms are required for the distribution-valued densities W on $\mathbb{R}^{2n}$, we will seek a dimensionally reduced field theory in the following which possesses a local gauge symmetry. For instance, setting $f(\xi)=f(x)$ independent of y in Eq. (3.16), using Eq. (A3), and integrating by parts over $\mathbb{R}^n_x$ yields the local transforms

$$\delta_\alpha w_\Xi = -\partial_\mu(w_\Xi \theta^{\mu a}\alpha_a), \quad \Xi=\omega,C,\phi. \tag{3.20}$$

Setting $f(\xi)=f_a(x)y^a$ linear in y in Eq. (3.17), using Eq. (A4), and integrating by parts over $\mathbb{R}^n_x$ yields

$$\delta_\alpha w^\Xi_\mu = -\partial_\nu(w^\Xi_\mu \theta^{\nu a}\alpha_a) - \theta^{\nu a} w^\Xi_\nu \partial_\mu \alpha_a, \quad \Xi=\omega,M. \tag{3.21}$$

Finally, setting $f(\xi)=f_{ab}(x)y^a y^b$ quadratic in y in Eqs. (3.18) and (3.19), using Eqs. (A6) and (3.20), and integrating by parts over $\mathbb{R}^n_x$ gives

$$\delta_\alpha w_{\mu\nu} = -\partial_\lambda(w_{\mu\nu}\theta^{\lambda a}\alpha_a) - \theta^{\lambda a} w_{\lambda\nu}\partial_\mu\alpha_a - \theta^{\lambda a} w_{\mu\lambda}\partial_\nu\alpha_a, \tag{3.22}$$

$$\delta_\alpha w^{\lambda\lambda'}_{\mu\nu} = -\partial_\rho(w^{\lambda\lambda'}_{\mu\nu}\theta^{\rho a}\alpha_a) - \theta^{\rho a} w^{\lambda\lambda'}_{\rho\nu}\partial_\mu\alpha_a - \theta^{\rho a} w^{\lambda\lambda'}_{\mu\rho}\partial_\nu\alpha_a. \tag{3.23}$$

It should be stressed that the transforms (3.20)–(3.23) represent but a single solution of the non-local constraint equations (2.16) and (2.19). We have taken the solutions which will directly relate local star-gauge invariance to general covariance in the dimensional reduction.

Using Eqs. (A2), (A4), (3.10), and (3.14), the noncommutative Yang-Mills action (2.14) can now be expressed in terms of a local Lagrangian over the space $\mathbb{R}^n_x$ as

$$I^W_{\rm NCYM} = \int_{\mathbb{R}^n} d^n x (L_\omega + L_C + L_M), \tag{3.24}$$

where

$$\begin{aligned} L_\omega = \frac{1}{2}\eta^{\mu\mu'}\theta^{\lambda a}\theta^{\lambda' b}\Big[& 2\eta^{\nu\nu'} w_{\lambda\lambda'}\Omega_{\mu\nu a}\Omega_{\mu'\nu' b} \\ & + w^{\nu\nu'}_{\lambda\lambda'}(\Omega_{\mu\nu a}\Omega_{\mu'\nu' b} + \Omega_{\mu\nu b}\Omega_{\mu'\nu' a}) \\ & - \frac{1}{2}\eta^{\nu\nu''} w^\omega_\lambda(\Omega_{\mu\nu a}\partial_{\lambda'}\Omega_{\mu'\nu' b} - \Omega_{\mu'\nu' b}\partial_\lambda\Omega_{\mu\nu a}) \\ & - \frac{1}{4}\eta^{\nu\nu'} w_\omega(\partial_{\lambda'}\Omega_{\mu\nu a})(\partial_\lambda\Omega_{\mu'\nu' b})\Big], \end{aligned} \tag{3.25}$$

$$L_C = \frac{1}{2} w_C \eta^{\mu\nu}\eta^{ab}\mathcal{F}_{\mu a}\mathcal{F}_{\nu b}, \tag{3.26}$$

$$L_M = \theta^{\nu' a}\theta^{\lambda b}\eta^{\mu\nu} w^M_\lambda \mathcal{F}_{\mu a}\Omega_{\nu\nu' b}. \tag{3.27}$$

In a similar fashion the reduced scalar field action (2.18) can be written as

$$I_{\rm B} = \int_{\mathbb{R}^n} d^n x L_\phi, \tag{3.28}$$

where

$$L_\phi = \frac{1}{2} w_\phi [\eta^{\mu'\nu'} h^\mu_{\mu'} h^\nu_{\nu'} (\partial_\mu\phi)(\partial_\nu\phi) + m^2\phi^2] \tag{3.29}$$

and

$$h^\nu_\mu = \delta^\nu_\mu - e\,\theta^{\nu a}\omega_{\mu a}. \tag{3.30}$$

In the following sections we will give geometrical interpretations of the field theory (3.24)–(3.30) and describe its relations to gravitation.

IV. INDUCED SPACETIME GEOMETRY OF NONCOMMUTATIVE GAUGE FIELDS

The remarkable property of the field theory of the previous section is that it is diffeomorphism invariant. This follows from its construction and the isomorphism (3.6), and is solely a consequence of the star-gauge invariance of the original noncommutative Yang-Mills theory on $\mathbb{R}^{2n}$. Precisely, it comes about from the representation (3.7) of the Lie algebra (2.10) of star-gauge transformations in terms of vector fields on flat infinite spacetime $\mathbb{R}^n_x$. This means that the various fields induced in the previous section should be related in some natural way to the geometry of spacetime. In this section we will show how this relationship arises. We have already seen a hint of this diffeomorphism invariance in the transformation law (3.12) for the scalar fields, which we have mainly introduced in the present context as source fields that probe the induced spacetime geometry. The scalar field action (3.29) is in fact the easiest place to start making these geometrical associations. This analysis will clarify the way that the star-gauge symmetry of Yang-Mills theory on noncommutative spacetime is related to the presence of gravitation.

The coordinates y^a generate the algebra $C^\infty(\mathbb{R}^n_y)$ and obey the star-commutation relations

$$[y^a, y^b]_\star = 0. \tag{4.1}$$

Under a global coordinate translation $x^\mu \mapsto x^\mu + \epsilon^\mu$, the scalar fields transform infinitesimally as $\phi(x)\mapsto\phi(x)+\epsilon^\mu\partial_\mu\phi(x)$. Since

$$\partial_\mu\phi(x) = -(\theta^{-1})_{a\mu}[y^a,\phi]_\star(x), \tag{4.2}$$

the derivative operator ∂_μ is an inner derivation of the algebra $C^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_y)$ and we may identify y^a with the holonomic derivative generators $-\theta^{\mu a}\partial_\mu$ of the n-dimensional translation group T_n of $\mathbb{R}^n_x$. The standard, flat space scalar field action $\int d^n x \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is invariant under these global translations. Let us now promote the global T_n symmetry to a local gauge symmetry. This replaces global translations with local translations $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$ of the fiber coordinates of the tangent bundle. It requires, in the usual way, the replacement of the derivatives ∂_μ with the covariant derivatives

$$\boldsymbol{\nabla}_\mu = \partial_\mu + e\,\omega_{\mu a} y^a, \tag{4.3}$$

where $\omega_{\mu a}$ are gauge fields corresponding to the gauging of the translation group, i.e. to the replacement of $\mathbb{R}^n$ by the Lie algebra $\mathbf{g}$ of local gauge transformations with gauge functions of the type (3.4). Using the identification (4.2) it then follows that the kinetic terms in the scalar field action will be constructed from

$$\nabla_\mu\phi = h_\mu^\nu\partial_\nu\phi, \tag{4.4}$$

with h_μ^ν given by Eq. (3.30). The covariance requirement

$$\delta_\alpha(\nabla_\mu\phi) = X_\alpha^\nu\partial_\nu(\nabla_\mu\phi) \tag{4.5}$$

is equivalent to the gauge transformation law for the gauge fields $\omega_{\mu a}$ in Eq. (3.9).

The quantities (3.30) can thereby be identified with vierbein fields on spacetime, and we see that the noncommutative gauge theory has the effect of perturbing the trivial holonomic tetrad fields δ_μ^ν of flat space. The noncommutative gauge fields become the non-trivial parts of the vierbein fields and create curvature of spacetime. Note that in the present formalism there is no real distinction between local spacetime and frame bundle indices, because these are intertwined into the structure of the star-gauge group of the underlying noncommutative gauge theory through the mixing of internal and spacetime symmetries. In other words, the matrix $(\theta^{\mu a})$ determines a linear isomorphism between the frame and tangent bundles of $\mathbb{R}_x^n$. It is precisely this isomorphism that enables the present construction to go through. We note also how naturally the identification (4.2) of the spacetime translational symmetry as an internal gauge symmetry arises from the point of view of the original noncommutative Yang-Mills theory on $\mathbb{R}^{2n}$. Using Eqs. (3.6) and (3.9) we see that the identification of Eq. (3.30) as a vierbein field is consistent with its gauge transform

$$\delta_\alpha h_\mu^\nu = X_\alpha^\lambda\partial_\lambda h_\mu^\nu - h_\mu^\lambda\partial_\lambda X_\alpha^\nu \tag{4.6}$$

which coincides with the anticipated behavior under infinitesimal diffeomorphisms of $\mathbb{R}_x^n$. The condition (4.6) is identical to the transformation law that one obtains from Eq. (3.12) and the homogeneous transformation law (4.5) for the covariant derivatives (4.4). Note that h_μ^ν behaves as a vector under general coordinate transformations with respect to its upper index. As we will discuss in the next section, it is a vector under local Lorentz transformations with respect to its lower index.

We can now recognize the gauge transformation (3.20) as the infinitesimal diffeomorphism of a scalar density.[4] Using Eq. (4.6) this condition can thereby be used to uniquely fix, up to a constant, the functions w_Ξ in terms of the noncommutative gauge fields, and we have[5]

$$w_\Xi = \rho_\Xi \det(h_\mu^\nu), \quad \Xi = \omega, C, \phi, \tag{4.7}$$

where ρ_Ξ are arbitrary constants. Similarly, the condition (3.21) specifies that the functions w_μ^Ξ are vector densities with respect to the connected diffeomorphism group of $\mathbb{R}_x^n$, and from (4.6) we may write

$$w_\mu^\Xi = \zeta_\Xi H_\mu^\nu \nabla_\nu \det(h_\lambda^{\lambda'}) = \zeta_\Xi \det(h_\lambda^{\lambda'}) H_{\nu'}^\nu \partial_\mu h_\nu^{\nu'}, \quad \Xi = \omega, M, \tag{4.8}$$

where ζ_Ξ are arbitrary constants. Here H_ν^μ are the inverse vierbein fields which are defined by the conditions

$$H_\mu^\lambda h_\lambda^\nu = h_\mu^\lambda H_\lambda^\nu = \delta_\mu^\nu. \tag{4.9}$$

They are thereby determined explicitly in terms of the noncommutative gauge fields as the perturbation series

$$H_\nu^\mu = \delta_\nu^\mu + e\,\theta^{\mu a}\omega_{\nu a} + \sum_{k=2}^{\infty} e^k \theta^{\mu a_1}\theta^{\mu_1 a_2}\cdots\theta^{\mu_{k-1}a_k}\omega_{\mu_1 a_1}\cdots\omega_{\mu_{k-1}a_{k-1}}\omega_{\nu a_k}, \tag{4.10}$$

and they possess the infinitesimal gauge transformation property

$$\delta_\alpha H_\nu^\mu = -X_\alpha^\lambda\partial_\lambda H_\nu^\mu - H_\lambda^\mu\partial_\nu X_\alpha^\lambda. \tag{4.11}$$

Finally, we come to the rank two tensor densities. From Eq. (3.22) we may identify

$$w_{\mu\nu} = \chi_0 \det(h_\lambda^{\lambda'})\,\eta_{\mu'\nu'} H_\mu^{\mu'} H_\nu^{\nu'}, \tag{4.12}$$

while from Eq. (3.23) we have

$$w_{\mu\nu}^{\lambda\lambda'} = \chi_{\mathbf{n}} \det(h_{\mu'}^{\nu'}) H_\mu^\lambda H_\nu^{\lambda'}, \tag{4.13}$$

with χ_0 and $\chi_{\mathbf{n}}$ arbitrary constants. As we shall see shortly, the tensor density (4.12) is associated with the antisymmetric part of the Clebsch-Gordan decomposition (3.15) while Eq. (4.13) is associated with the conjugate vector parts.

We see therefore that all fields of the previous section can be fixed in terms of gauge fields of the dimensionally reduced noncommutative Yang-Mills theory. All of the natural geometrical objects of spacetime are encoded into the noncommutative gauge fields. Let us now consider the structure of the reduced field strength tensor. From the form of the Lagrangian (3.25), and of the weight functions (4.7), (4.8), (4.12) and (4.13), it follows that the natural objects to consider are the contractions

[4]For the function w_ϕ the condition (3.20) may also be naturally deduced from Eq. (3.12) and by demanding that the mass term of the Lagrangian (3.29) be invariant up to a total derivative under infinitesimal diffeomorphisms.

[5]Note that $\det(h_\mu^\nu) = \sqrt{|\det(g_{\mu\nu})|}$ is the Jacobian of the frame bundle transformation $\partial_\mu \mapsto \nabla_\mu$, where $g_{\mu\nu} = \eta_{\mu'\nu'} h_\mu^{\mu'} h_\nu^{\nu'}$ is the Riemannian metric induced by the vierbein fields.

$$T^{\lambda}_{\mu\nu} = -e\,\theta^{\lambda'a} H^{\lambda}_{\lambda'} \Omega_{\mu\nu a} = H^{\lambda}_{\lambda'} (\nabla_\mu h^{\lambda'}_{\nu} - \nabla_\nu h^{\lambda'}_{\mu}), \tag{4.14}$$

with $\nabla_\mu = h^{\mu'}_{\mu} \partial_{\mu'}$. From Eqs. (4.6) and (4.11) it follows that the curvatures (4.14) obey the homogeneous gauge transformation laws

$$\delta_\alpha T_{\mu\nu}{}^{\lambda} = X^{\lambda'}_{\alpha} \partial_{\lambda'} T_{\mu\nu}{}^{\lambda}. \tag{4.15}$$

From Eq. (4.15) one can check directly that each term in the Lagrangian (3.25) is invariant up to a total derivative under star-gauge transformations, as they should be by construction. From Eqs. (3.10) and (3.30) it follows that the curvature (4.14) naturally arises as the commutation coefficients in the closure of the commutator of covariant derivatives to a Lie algebra with respect to the given orthonormal basis of the frame bundle,

$$[\nabla_\mu, \nabla_\nu] = T^{\lambda}_{\mu\nu} \nabla_\lambda. \tag{4.16}$$

The operators ∇_μ thereby define a non-holonomic basis of the tangent bundle with non-holonomicity given by the noncommutative field strength tensor. The change of basis $\nabla_\mu = h^{\nu}_{\mu}\partial_\nu$ between the coordinate and non-coordinate frames is defined by the noncommutative gauge field.

The commutation relation (4.16) identifies $T_{\mu\nu}{}^{\lambda}$, or equivalently the noncommutative gauge field strengths $\Omega_{\mu\nu a}$, as the torsion tensor fields of vacuum spacetime induced by the presence of a gravitational field. The non-trivial tetrad field (3.30) induces a teleparallel structure on spacetime through the Weitzenböck connection

$$\Sigma^{\lambda}_{\mu\nu} = H^{\lambda}_{\lambda'} \nabla_\mu h^{\lambda'}_{\nu}. \tag{4.17}$$

The connection (4.17) satisfies the absolute parallelism condition

$$\mathsf{D}_\mu(\Sigma) h^{\lambda}_{\nu} = \nabla_\mu h^{\lambda}_{\nu} - \Sigma^{\lambda'}_{\mu\nu} h^{\lambda}_{\lambda'} = 0, \tag{4.18}$$

where $\mathsf{D}_\mu(\Sigma)$ is the Weitzenböck covariant derivative. This means that the vierbein fields define a mutually parallel system of local vector fields in the tangent spaces of $\mathbb{R}^n_x$ with respect to the tangent bundle geometry induced by $\Sigma^{\lambda}_{\mu\nu}$. The Weitzenböck connection has non-trivial torsion given by Eq. (4.14),

$$T_{\mu\nu}{}^{\lambda} = \Sigma^{\lambda}_{\mu\nu} - \Sigma^{\lambda}_{\nu\mu}, \tag{4.19}$$

but vanishing curvature,

$$R^{\mu'}{}_{\nu'\mu\nu}(\Sigma) = \nabla_\mu \Sigma^{\mu'}_{\nu'\nu} - \nabla_\nu \Sigma^{\mu'}_{\nu'\mu} + \Sigma^{\mu'}_{\lambda\mu}\Sigma^{\lambda}_{\nu'\nu} - \Sigma^{\mu'}_{\lambda\nu}\Sigma^{\lambda}_{\nu'\mu} = 0. \tag{4.20}$$

The teleparallel structure is related to a Riemannian structure on spacetime through the identity

$$\Sigma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + K_{\mu\nu}{}^{\lambda}, \tag{4.21}$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \eta_{\mu'\nu'} \eta^{\sigma\sigma'} H^{\lambda}_{\sigma} H^{\lambda'}_{\sigma'} (h^{\nu'}_{\nu} \partial_\mu h^{\mu'}_{\lambda'} + h^{\mu'}_{\mu} \partial_\nu h^{\mu'}_{\lambda'} - h^{\mu'}_{\mu} \partial_{\lambda'} h^{\nu'}_{\nu} - h^{\nu'}_{\nu} \partial_{\lambda'} h^{\mu'}_{\mu}) + H^{\lambda}_{\lambda'} \partial_\mu h^{\lambda'}_{\nu} + H^{\lambda}_{\lambda'} \partial_\nu h^{\lambda'}_{\mu} \tag{4.22}$$

is the torsion-free Levi-Civita connection of the tangent bundle, and

$$K_{\mu\nu}{}^{\lambda} = \tfrac{1}{2}\big(\eta_{\mu\mu'} \eta^{\sigma\sigma'} H^{\lambda}_{\sigma} H^{\lambda'}_{\sigma'} T_{\lambda'\nu}{}^{\mu'} + \eta_{\nu\nu'} \eta^{\sigma\sigma'} H^{\lambda}_{\sigma} H^{\lambda'}_{\sigma'} T_{\lambda'\mu}{}^{\nu'} - T_{\mu\nu}{}^{\lambda}\big) \tag{4.23}$$

is the contorsion tensor. The torsion $T_{\mu\nu}{}^{\lambda}$ measures the noncommutativity of displacements of points in the flat spacetime $\mathbb{R}^n_x$. It is dual to the Riemann curvature tensor which measures the noncommutativity of vector displacements in a curved spacetime. This follows from the identities (4.20) and (4.21) which yield the relationship

$$\begin{aligned} R^{\mu'}{}_{\nu'\mu\nu}(\Gamma) &= \partial_\mu \Gamma^{\mu'}_{\nu'\nu} - \partial_\nu \Gamma^{\mu'}_{\nu'\mu} + \Gamma^{\mu'}_{\lambda\mu}\Gamma^{\lambda}_{\nu'\nu} - \Gamma^{\mu'}_{\lambda\nu}\Gamma^{\lambda}_{\nu'\mu} \\ &= \mathsf{D}_\nu(\Gamma) K_{\nu'\mu}{}^{\mu'} - \mathsf{D}_\mu(\Gamma) K_{\nu\nu'}{}^{\mu'} \\ &\quad + K_{\nu'\mu}{}^{\lambda} K_{\nu\lambda}{}^{\mu'} - K_{\nu'\nu}{}^{\lambda} K_{\mu\lambda}{}^{\mu'} \end{aligned} \tag{4.24}$$

between the usual Riemann curvature tensor $R^{\mu'}{}_{\nu'\mu\nu}(\Gamma)$ and the torsion tensor. Here $\mathsf{D}_\nu(\Gamma)$ is the Riemannian covariant derivative constructed from the Levi-Civita connection (4.22), whose action on the contorsion tensor is given by

$$\mathsf{D}_\nu(\Gamma) K_{\nu'\mu}{}^{\mu'} = \partial_\nu K_{\nu'\mu}{}^{\mu'} + \Gamma^{\mu'}_{\lambda\nu} K_{\nu'\mu}{}^{\lambda} + \Gamma^{\lambda}_{\nu'\mu} K_{\lambda\nu}{}^{\mu'}. \tag{4.25}$$

We see therefore that the dimensionally reduced noncommutative gauge theory of the previous section gives a very natural model of a flat spacetime with a given class of metrics carrying torsion, and with gauge field strengths corresponding to the generic anholonomity of a given local orthonormal frame of the tangent bundle of $\mathbb{R}^n_x$. It is precisely in this way that the noncommutative gauge theory on flat spacetime can induce a model of curved spacetime with torsion-free metric; i.e., it induces a teleparallel Weitzenböck geometry on $\mathbb{R}^n_x$ which is characterized by a metric-compatible connection possessing vanishing curvature but non-vanishing torsion and which serves as a measure of the intensity of the gravitational field. The teleparallel structure naturally induces a Riemannian geometry on spacetime, with curvature determined by the noncommutative field strength tensor. As we have mentioned before, it is very natural that in a noncommutative gauge theory, wherein global translations can be represented by inner automorphisms of the algebra of functions on spacetime, the translation group T_n be represented as an internal gauge symmetry group. In the ensuing dimensional reduction it thereby becomes a genuine, local spacetime symmetry of the field theory. The identification of the gauge field strengths with torsion tensors is then also very natural, given the noncommutativity of the spacetime coordinates and the fact that in noncommutative geometry the star-product only yields a projective representation of the translation group T_n with cocycle determined by the noncommutativity parameters Θ^{AB} [16].

V. GRAVITATION IN NONCOMMUTATIVE YANG-MILLS THEORY

We will now describe precisely how the diffeomorphism-invariant gauge field theory (3.24)–(3.27) is related to gravity in n dimensions. For this, we use the arbitrariness in the component weight functions to set the higher derivative terms, i.e. the second and third lines of the Lagrangian (3.25) to zero, $\rho_\omega=\zeta_\omega=0$. These terms represent higher energy contributions to the field theory, which admit a rather natural string theoretical interpretation that we will describe in the next section. Furthermore, we will see in Sec. VII that the Lagrangian (3.26), (3.27) for the auxiliary gauge fields $C_a(x)$ induces non-local interaction terms for the gravitational gauge fields $\omega_{\mu a}$, and so also do not contribute to the low-energy dynamics of the field theory (3.24). We will therefore also set $\rho_C=\zeta_M=0$.

The low-energy dynamics of the dimensionally reduced noncommutative gauge theory is thereby described by the Lagrangian

$$L_0=\frac{1}{2e^2}\det(h_\sigma^{\sigma'})\,\eta^{\mu\mu'}[2\chi_0\eta^{\nu\nu'}\eta_{\lambda\lambda'}T_{\mu\nu}{}^{\lambda}T_{\mu'\nu'}{}^{\lambda'} + \chi_{\mathbf{n}}(T_{\mu\nu}{}^{\nu}T_{\mu''\nu'}{}^{\nu'}+T_{\mu\nu}{}^{\nu'}T_{\mu'\nu'}{}^{\nu})]. \tag{5.1}$$

The constant χ_0 multiplies the torsion terms that arise from the irreducible representation $\Lambda^{1,2}(n)$ in Eq. (3.15), while the terms involving $\chi_{\mathbf{n}}$ come from the conjugate vector summands $\overline{\mathbf{n}}$. The Lagrangian (5.1) belongs to the one-parameter family of teleparallel Lagrangians (1.1), (1.2) which describe physically viable gravitational models, provided that the weight couplings obey

$$\chi_{\mathbf{n}}=-2\chi_0. \tag{5.2}$$

The choice of constants Eq. (5.2) as they appear in Eq. (5.1) is quite natural from the point of view of the symmetries of the Clebsch-Gordan decomposition (3.15). In this case, the Lagrangian (5.1) represents a gravitational theory for macroscopic matter which is observationally indistinguishable from ordinary general relativity.

This identification can be used to determine the Planck scale of the induced gravitational model (5.1). For this, we note that with the choice of weight functions (3.14) the fields $\omega_{\mu a}$ have mass dimension $n/2$ and the Yang-Mills coupling constant e has mass dimension $2-n/2$. In the gauge whereby the geometry is expanded around flat spacetime, as in Eq. (3.30), the non-trivial parts of the vierbein fields should assume the form κB_μ^ν, where κ is the Planck scale and the translational gauge fields B_μ^ν have mass dimension $n/2-1$ [26,27]. To compare this with the perturbation $e\theta^{\nu a}\omega_{\mu a}$ of the trivial tetrad field in Eq. (3.30), we introduce the dimensionless noncommutativity parameters $\hat\theta^{\mu a}=\theta^{\mu a}/|\det(\theta^{\mu'a'})|^{1/n}$, which as discussed in the previous section should be thought of, within the noncommutative geometry, as a tensor mapping translation group valued quantities to quantities in the fiber spaces of the frame bundle. By comparing mass dimensions we see that we should then properly identify

$$B_\mu^\nu=|\det(\theta^{\mu'a'})|^{1/2n}\hat\theta^{\nu a}\omega_{\mu a}. \tag{5.3}$$

Note that the Yang-Mills coupling constant itself cannot be used to compensate dimensions, for instance in $n=4$ dimensions e is dimensionless. Using Eq. (3.1), the Planck scale of n-dimensional spacetime is therefore given in terms of e and the noncommutativity scale as

$$\kappa=\sqrt{16\pi G_{\rm N}}=e|\mathrm{Pfaff}(\Theta^{AB})|^{1/2n}. \tag{5.4}$$

Comparing Eqs. (5.1), (5.2) and (1.1) then fixes the mass dimension 2 weight constant χ_0 to be

$$\chi_0=|\mathrm{Pfaff}(\Theta^{AB})|^{-1/n}. \tag{5.5}$$

The induced gravitational constant (5.4) vanishes in the commutative limit and agrees with that found in [6] using the supergravity dual of noncommutative Yang-Mills theory in four dimensions.

Let us now compare the low-energy field theory that we have obtained to standard general relativity. By using the relation (4.21), the Lagrangian

$$L_{\rm GR}=\frac{\chi_0}{e^2}\det(h_\sigma^{\sigma'})\,\eta^{\mu\mu'}\left[\frac14\eta^{\nu\nu'}\eta_{\lambda\lambda'}T_{\mu\nu}{}^{\lambda}T_{\mu'\nu'}{}^{\lambda'} - T_{\mu\nu}{}^{\nu}T_{\mu'\nu'}{}^{\nu'}+\frac12 T_{\mu\nu}{}^{\nu'}T_{\mu'\nu'}{}^{\nu}\right] \tag{5.6}$$

can be expressed in terms of the Levi-Civita connection $\Gamma_{\mu\nu}^\lambda$ alone. By using Eqs. (5.4),(5.5), along with Eqs. (4.23) and (4.24) to deduce the geometrical identity

$$\begin{aligned}R(\Gamma)&=\eta^{\lambda\lambda'}H_\lambda^\nu H_{\lambda'}^{\nu'}R^\mu{}_{\nu'\mu\nu}(\Gamma)\\ &=\eta^{\mu\mu'}(T_{\mu\nu}{}^{\nu}T_{\mu'\nu'}{}^{\nu'}-\tfrac12 T_{\mu\nu}{}^{\nu'}T_{\mu'\nu'}{}^{\nu}\\ &\quad-\tfrac14\eta^{\nu\nu'}\eta_{\lambda\lambda'}T_{\mu\nu}{}^{\lambda}T_{\mu'\nu'}{}^{\lambda'}+\partial_\nu K_{\mu\mu'}{}^{\nu}\\ &\quad+K_{\mu\mu'}{}^{\nu}H_\sigma^{\sigma'}\partial_\nu h_{\sigma'}^{\sigma}),\end{aligned} \tag{5.7}$$

the Lagrangian (5.6) can be rewritten, up to a total divergence, as the standard Einstein-Hilbert Lagrangian

$$L_{\rm E}=-\frac{1}{16\pi G_{\rm N}}\det(h_\lambda^{\lambda'})R(\Gamma) \tag{5.8}$$

in the first-order Palatini formalism. The Lagrangian (5.6) defines the teleparallel formulation of general relativity, and it is completely equivalent to Einstein gravity in the absence of spinning matter fields.

The main invariance property of the particular combination of torsion tensor fields in Eq. (5.6) is its behavior under a *local* change of frame $\boldsymbol{\nabla}_\mu$. This can be represented as a local Lorentz tranformation of the vierbein fields

$$\delta_\Lambda^{(\mathrm{L})} h^{\nu}_{\mu}(x) = \Lambda_{\mu}^{\mu'}(x) h^{\nu}_{\mu'}(x),$$

$$\delta_\Lambda^{(\mathrm{L})} H^{\mu}_{\nu}(x) = -\Lambda^{\mu}_{\mu'}(x) H^{\mu'}_{\nu}(x), \tag{5.9}$$

where $\Lambda^{\mu}_{\mu'}(x)$ are locally infinitesimal elements of SO$(1,n-1)$. Under Eq. (5.9) the torsion tensor (4.14) transforms as

$$\delta_\Lambda^{(\mathrm{L})} T_{\mu\nu}{}^{\lambda} = \partial_\mu \Lambda^{\lambda}_{\nu} - \partial_\nu \Lambda^{\lambda}_{\mu}, \tag{5.10}$$

from which it can be shown that the Lagrangian (5.6) changes by a total derivative under a local change of frame [26]. The gauge field theory defined by Eq. (5.6) is thereby independent of the choice of basis of the tangent bundle used, and in particular of the decomposition (3.30) which selects a gauge choice for the vierbein fields corresponding to a background perturbation of flat spacetime. The field equations derived from Eq. (5.6) will then uniquely determine the spacetime geometry and hence the orthonormal teleparallel frame up to a global Lorentz transformation. In fact, the Einstein-Hilbert Lagrangian defines the unique teleparallel gravitational theory which possesses this local Lorentz invariance [26].

The Lagrangian (5.1), on the other hand, is only invariant under *global* Lorentz transformations. This relates to the fact that the original noncommutative gauge theory on $\mathbb{R}^{2n}$ is only invariant under a flat space Lorentz group, and in the dimensional reduction only the translational subgroup of the full Poincaré group is gauged to a local symmetry. The reduced gauge theory is thereby a dynamical theory of the prefered orthonormal teleparallel frames in which the connection coefficients vanish and the torsion tensor has the simple form (4.14). The frame $\boldsymbol{\nabla}_\mu$ is then specified only modulo some local Lorentz transformation and the parallelism is not uniquely determined. The gravitational model is therefore ambiguous because there is a whole gauge equivalence class of geometries representing the same physics. Nevertheless, with the choice of parameters (5.2), (5.5), the Lagrangian (5.1) lies in the one-parameter family of teleparallel theories (1.1), (1.2) which pass all observational and theoretical tests of Einstein gravity. We use this criterion to fix the arbitrary constants of the gravitational model (5.1), whose presence effectively encodes the long distance effects of the internal space $\mathbb{R}^n_y$.

VI. D-BRANES AND VOLUME PRESERVING DIFFEOMORPHISMS

To understand what the higher derivative terms in the Lagrangian (3.25) represent, we return to the standard action (2.13) for noncommutative Yang-Mills theory on $\mathbb{R}^{2n}$. This is the action that is induced on a flat D$(2n-1)$-brane in flat space and in the presence of a constant background B field. We can now examine the "naive" dimensional reduction of this action to an n-dimensional submanifold $\mathbb{R}^n_x \subset \mathbb{R}^{2n}$. Such a submanifold could correspond, for example, to the embedding of a flat D$(n-1)$-brane inside the D$(2n-1)$-brane with a transverse B field, which realizes the D$(n-1)$-brane as a noncommutative soliton in the worldvolume of the D$(2n-1)$-brane [39]. The reduced action is then given by

$$I^{\mathrm{red}}_{\mathrm{NCYM}} = \frac{\mathrm{vol}_y}{2}\int_{\mathbb{R}^n} d^n x\, G^{AA'} G^{BB'} \mathcal{F}_{AB} \star \mathcal{F}_{A'B'}(x,0), \tag{6.1}$$

where vol_y is the volume of the transverse space. Note that in this reduction the mass dimension of the Yang-Mills coupling constant e is $2-n$.

Within the general formalism of Sec. III, it follows that we should choose all $w_\Xi = \mathrm{vol}_y$, while $w^{\Xi}_{\mu} = w_{\mu\nu} = w^{\lambda\lambda'}_{\mu\nu} = 0$. The field theory (3.24) is then given by the local Lagrangian

$$L_D = -\frac{\mathrm{vol}_y}{2e^2}\eta^{\mu\mu'}[\eta^{\nu\nu'}\partial_{\lambda'}(h^{\lambda}_{\sigma} T_{\mu\nu}{}^{\sigma})\partial_{\lambda}(h^{\lambda'}_{\sigma'} T_{\mu'\nu'}{}^{\sigma'}) - e^2 \eta^{aa'} \mathcal{F}_{\mu a}\mathcal{F}_{\mu' a'}]. \tag{6.2}$$

However, in order for Eq. (6.2) to define a diffeomorphism invariant field theory, we still need to satisfy the star-gauge invariance conditions (2.16). While the transforms are trivially satisfied of course for the vanishing w's, the constraint (3.20) for constant w_Ξ imposes the restriction

$$\partial_\mu X^{\mu}_{\alpha} = 0 \tag{6.3}$$

on the types of diffeomorphisms which can be used for star-gauge transformations. This means that the map (3.6) is not surjective and its image consists of only volume preserving diffeomorphisms. This is expected from the fact that the component functions of the weight densities are constant, so that the only diffeomorphism invariance that one can obtain in this case are the coordinate transformations that leave the flat volume element of $\mathbb{R}^n_x$ invariant, i.e. those which are isometries of the flat Minkowski metric $\eta_{\mu\nu}$ and thereby infinitesimally satisfy Eq. (6.3). Thus, in the D-brane interpretation of the dimensional reduction presented in this paper, star-gauge invariance acts to partially gauge fix the diffeomorphism group of spacetime. One arrives at not a full theory of gravity, but rather one which is only invariant under the subgroup consisting of volume preserving diffeomorphisms. This subgroup arises as the residual symmetry of the field theory that remains after the gauge fixing.

Generally, volume preserving diffeomorphisms constitute the symmetry group which reflects the spacetime noncommutativity that arises in D-brane models [40]. For instance, they arise as the dynamical degree of freedom in matrix models [41] which comes from the discretization of the residual gauge symmetry of the 11-dimensional supermembrane [42]. They also appear as the residual symmetry after light-cone gauge fixing in p-brane theories [43], and they naturally constitute the Lie algebra of star-gauge transformations in noncommutative Yang-Mills theory on flat spacetime [16]. Here we have tied them in with the dynamics of D-branes through the effective, higher-derivative gravitational theories (6.2) that are induced in the dimensional reduction. Another way to see that general covariance in the usual noncommutative gauge theories is only consistent with volume preserving symmetries is by noting that the infinitesimal coordinate transformation $\delta_\alpha x^\mu = X^{\mu}_{\alpha}(x)$ implies the noncommutativity parameters $\theta^{\mu a}=[x^\mu,y^a]_\star$ must transform under gauge transformations as

$$\delta_\alpha\theta^{\mu a}=[X^\mu_\alpha,y^a]_\star=\theta^{\nu a}\partial_\nu X^\mu_\alpha. \tag{6.4}$$

Requiring that the noncommutative gauge symmetries preserve the supergravity background on the D-branes sets Eq. (6.4) to zero, which also leads to the isometry constraint above. This constraint further ensures that the tensor $\theta^{\mu a}$ defines a global isomorphism between the frame and tangent bundles, as is required in the construction of this paper.

The higher derivative terms in the action (3.25) can thereby be thought of as "stringy" corrections to the teleparallel gravity theory. It is tempting to speculate that they are related to the higher-curvature couplings that arise in effective supergravity actions. It is a curious fact that in this interpretation one does not arrive at an Einstein-like theory of gravity on the D-brane. This induced brane gravity deserves to be better understood, and most notably how the analysis of this section and the previous one relates to the large N supergravity results which demonstrate the existence of conventional gravitation in noncommutative gauge theory [6]. The dimensional reductions of Sec. V could indeed be related to the way that the Newtonian gravitational potential arises from a Randall-Sundrum type localization on anti-de Sitter space. Indeed, it would be interesting to understand whether or not the generalized class of noncommutative gauge theories (2.14) arises as an effective field theory of strings in some limit, or if the dimensional reductions follow from some sort of dynamical symmetry breaking mechanism in the noncommutative quantum field theory. This would presumably fix all free parameters of the induced gravitational theory (3.24).

VII. ROLE OF THE AUXILIARY FIELDS

The most important ingredient missing from the induced gravity model of Sec. V is local Lorentz invariance. This somewhat undesirable feature owes to the indistinguishability within the present formalism between spacetime and frame indices. It is in fact quite natural from the point of view of the original noncommutative gauge theory, whereby the star-gauge symmetry allows the gauging of the translation group but is independent of the invariance of the field theory under global SO$(1,2n-1)$ transformations. We may expect, however, that local Lorentz symmetry is restored in some complicated dynamical way in the reduced noncommutative gauge theory, such that the effective gauge theory contains general relativity. This problem is addressed in [19] in the context of reduced models. In this section we will briefly describe some potential steps in this direction.

The natural place to look for the extra terms required to make the Lagrangian (5.1) invariant under local frame rotations is in the terms involving the auxiliary "internal" gauge fields $C_a(x)$, whose role in the induced gravitational theory has thus far been ignored. They represent the components of the noncommutative gauge field in the internal directions, along which lies the coordinate basis y^a defining the generators of the translation group T_n that is used in the gauging prescription. They thereby represent natural candidates to induce the necessary terms that instate the frame basis independence of the diffeomorphism invariant field theory of Sec. V. Note that these fields cannot be set to zero because of their gauge transformation law in Eq. (3.9). They therefore constitute an intrinsic dynamical ingredient of the induced gravity model.

The variation of the Lagrangian L_C+L_M given by Eqs. (3.10), (3.26), (3.27), (4.7), and (4.8) with respect to the auxilliary gauge fields $C_a(x)$ yields the field equation

$$\eta^{ab}\Box C_b(x)=J^a(x), \tag{7.1}$$

where we have introduced the second order linear differential operator

$$\Box=\rho_C\eta^{\mu\nu}[(\partial_\lambda h^\lambda_\mu)\boldsymbol{\nabla}_\nu+(\boldsymbol{\nabla}_\mu h^\lambda_\nu)H^{\nu'}_\lambda\boldsymbol{\nabla}_{\nu'}+H^{\mu'}_\lambda(\boldsymbol{\nabla}_\mu h^\lambda_{\mu'})\boldsymbol{\nabla}_\nu + h^\lambda_\nu\boldsymbol{\nabla}_\mu\partial_\lambda] \tag{7.2}$$

and the fields

$$\begin{aligned}J^a=\eta^{\mu\nu}\Bigg\{&\rho_C\eta^{ab}[H^{\mu'}_\lambda(\boldsymbol{\nabla}_\nu h^\lambda_{\mu'})\omega_{\mu b}+\omega_{\mu b}\partial_\lambda h^\lambda_\nu+\boldsymbol{\nabla}_\mu\omega_{\nu a}]\\&+\frac{\zeta_M}{e}\theta^{\nu' a}[H^{\mu'}_{\lambda'}T_{\nu\nu'}{}^\lambda(\boldsymbol{\nabla}_\lambda h^{\lambda'}_{\mu'})(\partial_{\nu''}h^{\nu''}_\mu)\\&+H^{\mu'}_{\lambda'}H^{\lambda''}_{\nu''}T_{\nu\nu'}{}^\lambda(\boldsymbol{\nabla}_\lambda h^{\lambda'}_{\mu'})(\boldsymbol{\nabla}_\mu h^{\nu''}_{\lambda''})+H^{\mu'}_{\lambda'}\boldsymbol{\nabla}_\mu(h^\lambda_{\lambda''}T_{\nu\nu'}{}^{\lambda''})\\&\times(\partial_\lambda h^{\lambda'}_{\mu'})+h^\lambda_{\lambda''}T_{\nu\nu'}{}^{\lambda''}\boldsymbol{\nabla}_\mu(H^{\mu'}_{\lambda'}\partial_\lambda h^{\lambda'}_{\mu'})]\Bigg\}.\end{aligned} \tag{7.3}$$

Substituting the solution of Eq. (7.1) for the fields $C_a(x)$ into the Lagrangian L_C+L_M thereby yields the non-local effective Lagrangian

$$\begin{aligned}L_{\rm eff}=\det(h^{\sigma'}_\sigma)\Bigg[&-\frac12\eta_{ab}J^a\frac{1}{\Box}J^b+\frac{\rho_C}{2}\eta^{\mu\nu}\eta^{ab}\omega_{\mu a}\omega_{\nu b}\\&+\frac{\zeta_M}{e}\theta^{\nu' a}\eta^{\mu\nu}H^{\lambda'}_{\mu'}(\boldsymbol{\nabla}_\lambda h^{\mu'}_{\lambda'})\omega_{\mu a}T_{\nu\nu'}{}^\lambda\Bigg].\end{aligned} \tag{7.4}$$

Note that in the case $\rho_C=0$, the auxiliary fields are Lagrange multipliers which enforce a geometric constraint given by setting the fields (7.3) identically equal to zero.

By performing a gradient expansion of the operator $\Box^{-1}$, we can now study the derivative expansion of the effective Lagrangian (7.4). Higher derivative terms can be attributed to stringy corrections, as they were in the previous section. The leading order terms may then lead to the appropriate additions of terms to the Lagrangian (5.1) which makes it invariant under local Lorentz transformations. However, generically the Lagrangian (7.4) will also contain infinitely many higher derivative terms and so a minimal, low-energy model is not strictly speaking attainable with this reasoning. In fact, one can simply set the constants $\rho_C=\zeta_M=0$ and completely ignore the non-local contributions from the auxiliary fields. Their inclusion represents the possibility of obtaining a gravitational field theory which is completely equivalent to general relativity, at least on a macroscopic scale. This possibility deserves further investigation.

VIII. COUPLING GAUGE THEORY AND GRAVITY

For gauge functions of the form (3.4), we have used a principle of ''minimal consistent reduction'' to fix the fields that arise in the induced gravitational theory. This led to the choice (3.8) involving the gauge fields $\omega_{\mu a}$ which induced the non-trivial part of the tetrad fields of the induced spacetime geometry, and the auxiliary fields C_a. It is possible to consider more general Lie algebras **g** other than the minimal one consisting of the gauge functions (3.4). For instance, it is possible to define $\alpha(\xi)$ with a piece $\alpha^{(0)}(x)$ which is independent of the y coordinates,

$$\alpha(\xi)=\alpha^{(0)}(x)+\alpha_a(x)y^a. \tag{8.1}$$

Again **g** is a Lie algebra with respect to the star commutator, with ''component'' functions (3.5) and

$$\begin{aligned}([\alpha,\beta]_\star)^{(0)}(x)&=\theta^{\mu a}[\beta_a(x)\partial_\mu\alpha^{(0)}(x)-\alpha_a(x)\partial_\mu\beta^{(0)}(x)]\\&=X^\mu_\alpha(x)\partial_\mu\beta^{(0)}(x)-X^\mu_\beta(x)\partial_\mu\alpha^{(0)}(x)\end{aligned} \tag{8.2}$$

for $\alpha,\beta\in\mathbf{g}$. The smallest truncation of the space $\mathcal{YM}$ is now defined by Yang-Mills fields of the form

$$\mathcal{A}=[A_\mu(x)+\omega_{\mu a}(x)y^a]dx^\mu+C_a(x)dy^a, \tag{8.3}$$

and the star-gauge transformation rules (3.9) are supplemented with

$$\begin{aligned}\delta_\alpha A_\mu&=\partial_\mu\alpha^{(0)}+e\,\theta^{\lambda a}(\alpha_a\partial_\lambda A_\mu-\omega_{\mu a}\partial_\lambda\alpha^{(0)})\\&=\boldsymbol{\nabla}_\mu\alpha^{(0)}-eX^\nu_\alpha\partial_\nu A_\mu.\end{aligned} \tag{8.4}$$

The noncommutative field strength tensor is then modified as

$$\begin{aligned}\mathcal{F}_{\mu\nu}(\xi)&=F_{\mu\nu}(x)+e\,\theta^{\lambda a}[\omega_{\nu a}(x)\partial_\lambda A_\mu(x)-\omega_{\mu a}(x)\partial_\lambda A_\nu(x)]\\&\quad+\Omega_{\mu\nu a}(x)y^a\\&=\boldsymbol{\nabla}_\mu A_\nu(x)-\boldsymbol{\nabla}_\nu A_\mu(x)+\Omega_{\mu\nu a}(x)y^a,\end{aligned} \tag{8.5}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{8.6}$$

and the remaining components of $\mathcal{F}_{AB}$ are as in Eq. (3.10).

It follows that the choice (8.1) of gauge functions induces a model of ordinary Maxwell electrodynamics for the photon field $A_\mu(x)$ on $\mathbb{R}^n_x$ coupled to gravity. Note that the star-gauge invariance of the original Yang-Mills theory mixes up the U(1) internal symmetries with the spacetime symmetries, as is evident in the expressions (8.2), (8.4), and (8.5). In particular, from Eq. (8.4) we see that the photon field A_μ transforms covariantly under general coordinate transformations, while it is a vector under the local Lorentz group

$$\delta^{(\mathrm{L})}_\Lambda A_\mu(x)=\Lambda_\mu^{\ \nu}(x)A_\nu(x). \tag{8.7}$$

In this way one obtains a unified gauge theory which couples the gravitational theory that was studied in earlier sections to electrodynamics. Notice also that the photon field $A_\mu(x)$ does not couple to the scalar field $\phi(x)$, consistent with the fact that the scalar bosons are taken to be neutral under the extra Abelian gauge symmetry. This generalization evidently also goes through if one starts from a noncommutative Yang-Mills theory with some non-Abelian U(N) gauge group. Then one obtains a sort of non-Abelian model of gravity coupled to ordinary Yang-Mills theory. However, the star-gauge group of the simpler noncommutative electrodynamics contains all possible non-Abelian unitary gauge groups in a very precise way [16,44]. It would be interesting to extract the gravity-coupled Yang-Mills theory directly from a dimensionally reduced gauge theory of the form (2.14). There is therefore a wealth of gravitational theories that can be induced from noncommutative gauge theory, which in itself also seems to serve as the basis for a unified field theory of the fundamental forces. In all instances the type of theory that one obtains is dictated by the choice of reduced star-gauge group, i.e. the Lie algebra **g**, as well as the choice of weight functions W for the dimensional reduction. This illustrates the richness of the constraints of star-gauge invariance in noncommutative Yang-Mills theory.

IX. CONCLUSIONS

In this paper we have described a particular class of dimensional reductions of noncommutative electrodynamics which induce dynamical models of spacetime geometry involving six free parameters. Two of these parameters can be fixed by requiring that the leading, low-energy dynamics of the model be empirically equivalent to general relativity. The higher-order derivative corrections can be attributed to stringy corrections and non-local effects due to noncommutativity. The low-energy dynamics can be consistently decoupled from the high-energy modes by an appropriate choice of parameters. These results show that a certain class of teleparallel gravity theories have a very natural origin in a noncommutative gauge theory whereby diffeomorphism invariance is solely a consequence of the star-gauge invariance of the Yang-Mills theory, in the same spirit as the usual gauge theories based on the translation group of flat space. Alternatively, the present construction sheds light on the manner in which noncommutative gauge theories on flat spacetime contain gravitation. We have also described how Yang-Mills theory on a noncommutative space naturally contains a gravitational coupling of ordinary gauge theories to the geometrical model studied in most of this paper. A real advantage of this point of view of inducing gravity from noncommutative gauge theory is that in the latter theory it is straightforward to construct gauge-invariant observables. These are constructed in terms of the open and closed Wilson line operators, which are non-local in character. It would be interesting to understand these observables from the point of view of the induced gravitational theory.

It should be stressed that we have only presented a very simple model of dimensional reduction. More general reductions are possible and will induce different geometrical models. The present technique can be regarded as a systematic way to induce theories of gravitation starting only from the single, elementary principle of star-gauge invariance of noncommutative Yang-Mills theory. One extension would be to include a gauging of the full Poincaré group of spacetime. This would cure the problem of local Lorentz invariance and potentially yield a theory of gravitation which is completely equivalent to general relativity. It should be possible to find such an extended noncommutative gauge theory whose dimensional reduction yields the appropriate model with manifest local Lorentz symmetry. After an appropriate gauge fixing, this model should then reduce to the theory analyzed in this paper.

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APPENDIX: REDUCED STAR PRODUCT IDENTITIES

In this appendix we collect, for convenience, a few formulas which were used to derive the equations given in the text. Let Θ^{AB} be as in Eq. (3.1). We denote by $f^{(k)}\in C^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_y)$, $k\geqslant 0$, a function which is of degree k in the y coordinates,

$$f^{(k)}(\xi)=f_{a_1\cdots a_k}(x)y^{a_1}\cdots y^{a_k}, \tag{A1}$$

where $f_{a_1\cdots a_k}(x)$ is completely symmetric in its indices, and $f^{(0)}(\xi)=f(x)$ is independent of y.

We then have the following reduced star-product identities:

$$f^{(0)}\star g^{(0)}(\xi)=f(x)g(x); \tag{A2}$$

$$f^{(1)}\star g^{(0)}(\xi)=g(x)f_a(x)y^a-\tfrac{1}{2}\theta^{\mu a}f_a(x)\partial_\mu g(x),$$

$$g^{(0)}\star f^{(1)}(\xi)=g(x)f_a(x)y^a+\tfrac{1}{2}\theta^{\mu a}f_a(x)\partial_\mu g(x); \tag{A3}$$

$$f^{(1)}\star g^{(1)}(\xi)=f_a(x)g_b(x)y^ay^b+\tfrac{1}{2}\theta^{\mu a}\{[\partial_\mu f_b(x)]g_a(x)-f_a(x)[\partial_\mu g_b(x)]\}y^b-\tfrac{1}{4}\theta^{\mu a}\theta^{\nu b}[\partial_\nu f_a(x)][\partial_\mu g_b(x)]; \tag{A4}$$

$$\begin{aligned}[g^{(1)},f^{(1)}\star f^{(1)}]_\star(\xi)={}&2\theta^{\mu a}[\partial_\mu g_b(x)]f_a(x)f_c(x)y^by^c+[\theta^{\lambda a}\partial_\lambda g_a(x)]\Bigg\{f_b(x)f_c(x)y^by^c+\frac{\theta^{\mu b}\theta^{\nu c}}{4}\{\partial_\mu\partial_\nu[f_b(x)f_c(x)]\\&+[\partial_\nu f_b(x)][\partial_\mu f_c(x)]\}\Bigg\}+\frac{\theta^{\lambda a}}{4}\partial_\lambda\{2g_a(x)f_b(x)f_c(x)y^by^c-\theta^{\mu b}\theta^{\nu c}g_a(x)[\partial_\nu f_b(x)][\partial_\mu f_c(x)]\\&-\theta^{\mu b}\theta^{\nu c}[\partial_\nu g_b(x)]\partial_\mu[f_a(x)f_c(x)]+\theta^{\mu b}\theta^{\nu c}g_b(x)\partial_\mu\partial_\nu[f_a(x)f_c(x)]-\theta^{\mu b}\theta^{\nu c}g_a(x)\partial_\mu\partial_\nu[f_b(x)f_c(x)]\};\end{aligned} \tag{A5}$$

$$[g^{(1)},f^{(2)}]_\star(\xi)=\theta^{\mu a}[2f_{ac}(x)\partial_\mu g_b(x)-g_a(x)\partial_\mu f_{bc}(x)]y^by^c-\theta^{\mu a}\theta^{\nu b}\theta^{\lambda c}[\partial_\mu\partial_\nu g_c(x)][\partial_\lambda f_{ab}(x)]. \tag{A6}$$

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