Hypermultiplets, domain walls, and supersymmetric attractors

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We establish general properties of supersymmetric flow equations and of the superpotential of five-dimensional $\mathcal{N}=2$ gauged supergravity coupled to vector multiplets and hypermultiplets. We provide necessary and sufficient conditions for BPS domain walls and find a set of algebraic attractor equations for $\mathcal{N}=2$ critical points. As an example we describe in detail the gauging of the universal hypermultiplet and a vector multiplet. We study a two-parameter family of superpotentials with supersymmetric AdS critical points and we find, in particular, an $\mathcal{N}=2$ embedding for the UV-IR solution of Freedman, Gubser, Pilch, and Warner of the $\mathcal{N}=8$ theory. We comment on the relevance of these results for brane world constructions.

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I. INTRODUCTION AND GOALS

Among the remarkable spinoffs of the AdS-conformal field theory (CFT) correspondence between strings on AdS $\times$X and boundary superconformal theories, a lot of interest is devoted to the duality between domain wall supergravity solutions and renormalization group (RG) flows of field theory couplings.

The purpose of this paper is to find out the general properties of supersymmetric flows and vacua of $D=5$, $\mathcal{N}=2$ supergravity coupled to hypermultiplets and vector multiplets with nonconstant scalars, based on the theory of [1]. This analysis aims at following the lines of the attractor flows [2] for black holes in four and five dimensions that paved the way to finding Bogomol'nyi-Prasad-Sommerfield (BPS) solutions.

The basic interest in domain wall or supersymmetric flows is due to the possible correspondence between BPS domain walls of gauged supergravity and exact supersymmetric vacua of the fundamental M or string theory. Unlike black holes, the domain wall solutions in $D=5$ interpolating between AdS vacua do not break $D=4$ Lorentz symmetry and therefore may give interesting possibilities for the realistic vacua of our 4D world. On the other hand, the intuition gained in studies of black hole attractors may be useful for understanding the issues of stabilization of moduli at supersymmetric vacua.

The gauging of supergravity in the vector multiplet sector has been studied with respect to the supersymmetric vacua of the theory. In particular, for U(1) $D=5$ gauged supergravity the supersymmetric vacua are defined by the superpotential $W=h^I(\phi)V_I$, which has a dependence on moduli analagous to the black hole central charge $Z=h^I(\phi)q_I$. Here $h^I(\phi)$ are special coordinates, V_I are constants related to the Fayet-Iliopoulos terms, and q_I are black hole electric charges. Many critical points of both these systems are known. One always finds AdS_5 vacua (and domain walls with AdS vacua) when scalars from vector multiplets reach their fixed points. These fixed points are specified by the algebraic equation $V_I=h_I(\phi_{\rm cr})W_{\rm cr}$, where $h_I(\phi)$ are the dual special coordinates. This equation is analogous to the equation $q_I=h_I(\phi_{\rm cr})Z_{\rm cr}$, which defines the fixed scalars near the horizon of the $D=5$ electrically charged black holes. Both of these equations were derived and analyzed in [3].

The solutions without hypermultiplets are also known to have specific properties, like the fact that they always approach a UV fixed point, i.e., the AdS boundary [4,5]. This feature leads to a no-go theorem for the "alternative to compactification" Randall-Sundrum (RS) smooth scenario [6].

However, the situation may change when hypermatter is added and thus it is important to elucidate the nature of the fixed scalars in the BPS domain wall configurations in this case. The first examples of domain walls were found in the coupling with the universal hypermultiplet. The one in [7] does not have AdS critical points, whereas the one in [8] displayed one UV and one IR critical point. In the latter, the authors claimed that their model could give an $\mathcal{N}=2$ realization of the domain wall solution found by Freedman, Gubser, Pilch, and Warner [9] (FGPW) as holographic dual to a RG flow from an $\mathcal{N}=4$ to an $\mathcal{N}=1$ Yang-Mills theory. However, this description relied on a nonstandard formulation of 5D supergravity which has not been proven to be consistent.

Other U(1) gaugings of the same model were recently studied in [10–12].

This paper starts with a systematic description of supersymmetric flow equations in the presence of vector and hypermultiplets. We first solve the issue of describing both the flows and the attractor points in terms of a single superpotential W for generic (non-Abelian) gaugings. This is nontrivial, since the theory is defined in terms of the SU(2) triplet of quaternionic prepotentials $P_I^r(q)$ dressed with the $h^I(\phi)$ functions of the scalars in vector multiplets.

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We will find that *the single superpotential* $W(\phi,q)$ *is related to the norm of the dressed prepotentials* $P^r(\phi,q)$ *and controls the supersymmetric flow equations if and only if their* SU(2) *phase satisfies the constraint*

$$\partial_\phi Q^r=0 \quad \text{where} \quad P^r(\phi,q)\equiv h^I(\phi)P_I^r(q)\equiv\sqrt{2/3}\,WQ^r,$$
$$(Q^r)^2=1. \tag{1.1}$$

This may restrict the class of gauged supergravities with hypermultiplets that have BPS solutions.

We then characterize the critical points by a set of attractor equations. It parallels the attractor mechanism for moduli near the black hole horizon [2] and is also supported by an enhancement of unbroken supersymmetry near the AdS vacua. The attractor equations are simple algebraic conditions which fix the values of the moduli:

$$P_I^r(q_{\rm cr})-h_I(\phi_{\rm cr})P_{\rm cr}^r(\phi_{\rm cr},q_{\rm cr})=0,$$
$$K_{\rm cr}^X\equiv h^I(\phi_{\rm cr})K_I^X(q_{\rm cr})=0. \tag{1.2}$$

The first equation is defined by the very special geometry and is analogous to the one for black holes discussed above. The other one requires a certain combination of quaternionic Killing vectors to vanish.[1] These algebraic equations are very useful in simplifying the general analysis of critical points, as they replace the differential equations for the extrema of the superpotential. They will prove to be very useful also in our simple cases.

Our general theory will be applied to the model of the universal hypermultiplet alone as well as coupled to one vector multiplet. The full moduli space is

$$\mathcal{M}=\mathrm{O}(1,1)\times\frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times\mathrm{U}(1)}. \tag{1.3}$$

For the hypermultiplet alone, we study the properties and parametrizations of the scalar manifold $\mathrm{SU}(2,1)/\mathrm{SU}(2)\times\mathrm{U}(1)$, giving all Killing vectors and prepotentials that allow us to write down the generic scalar potential. Analyzing all U(1) gaugings systematically, we find that only one critical point arises. On general grounds we give precise conditions for determining its (UV/IR) nature, which interestingly can be tuned by the choice of the direction gauged within the compact subgroup. We compare results with other parametrizations, that, due to an ill-defined metric, can give rise to spurious singular points. Specifically, in Appendix C, we show how this happens in [12], where the parametrization of [14] was used.

Then we turn to the full model, where we analyze the most general $\mathrm{U}(1)\times\mathrm{U}(1)$ gauging. The requirements for a first critical point lead to three real parameters for the embedding in $\mathrm{SU}(2)\times\mathrm{U}(1)$. A linear relation determines whether extra possibilities exist where noncompact generators of SU(2,1) contribute to one of the U(1) generators. We further study a two-parameter subclass. We then restrict ourselves to the theories that have two different AdS critical points, as they could be extrema of RG-flows as well as of Randall-Sundrum-type smooth solutions. This leaves us with only two independent real numerical parameters β and γ. The superpotential of the two-parameter model existing on a line in the quaternionic manifold parametrized by χ is

$$W=\frac{1}{4\,\rho^2}[3+\beta+(\tfrac{3}{2}+\gamma)\,\rho^6$$
$$+(1-\beta+(\tfrac{1}{2}-\gamma)\,\rho^6)\cosh(2\,\chi)], \tag{1.4}$$

where ρ is the vector modulus.

Again, the nature of the critical points depends on the relation between the parameters. Quite remarkably, for the special values $\beta=-1$ and $\gamma=3/2$ we recover precisely the superpotential and the UV-IR AdS critical points of the kink solution of Freedman, Gubser, Pilch, and Warner [9]. This means that the FGPW flow can be described all within $D=5$, $\mathcal{N}=2$ gauged supergravity [1] coupled to one vector and one hypermultiplet, and thus the corresponding sector of the $\mathcal{N}=2$ theory yields a consistent truncation of ten-dimensional type IIB supergravity [15].

As an outcome of this analysis, we find as an interesting feature that domain walls with hypermultiplets can give rise to IR directions. This removes in principle the main obstacle for realizing a smooth supersymmetric RS scenario with a single brane (RSII), which in the case of vector multiplets only, resulted in the no-go theorem of [4,5]. The next step would be to find in a specific model two IR critical points and an interpolating solution such that the warp factor obtains a maximum. The supergravity flow equations impose the condition that $A'=\pm W$ and thus the existence of such a maximum implies a vanishing superpotential.

On the other hand, the holographic c theorem [16,9,17] imposes the monotonicity of the c function $c\sim|W|^{-3}$. This should imply that it is impossible to connect smoothly two IR points and find a smooth supersymmetric RSII. However, the only condition imposed by supergravity and by the BPS flow equations is the "monotonicity theorem" $A''\leqslant 0$. This leads to a monotonicity of the first derivative of the warp factor $A'=\pm W$ and not in general of the c function. This does not exclude flows where the superpotential reaches $W=0$. These points may signal some problem with the validity of the five-dimensional supergravity approximation of the holographic correspondence [18]. However, BPS flows crossing such points are perfectly well behaved from the supergravity perspective. In our specific study we find examples with W vanishing at some points,[2] but the flows by

[1]This requirement appeared in [1] and was then noticed in [11] for BPS instantons and also in [13].

[2]In [19] it is shown that a world-volume theory for a domain wall at such a place has problems due to unbounded fermions. This has been investigated in the context of theories with only vector multiplets. It should be investigated whether similar problems persist for fermions with transformation laws like those in hypermultiplets.

these points always lead to a naked singularity. We conclude therefore that, although no example exists at the moment, a smooth supersymmetric realization of RSII does not seem to be ruled out in the presence of vector and hypermultiplets.

Conversely, it is likely that a realistic one-brane Randall-Sundrum scenario can be constructed on the basis of any of the models discussed above with at least one IR critical point by employing a method with supersymmetric singular brane sources [20].

The paper proceeds in Sec. II with a general discussion of supersymmetric flows with an arbitrary number of vector and hypermultiplets based on the most general consistent gaugings. We start by repeating the general ingredients of the very special real and the quaternionic manifolds, and the gauging of the isometries. We provide a general constraint for gravitational stability that can be expressed as the BPS condition in a domain wall background. We spell out the requirements for a RSII scenario in terms of the concepts of a renormalization group flow. The requirement for critical points of the (super)potential can be reduced to algebraic conditions, the attractor equations. We end this section with a summary of the features to be investigated in examples.

Section III starts with motivations for studying a simple model with a vector multiplet and a hypermultiplet such as the one giving rise to the $\mathcal{N}=2$ description of the FGPW flow. Then we study properties and parametrizations of the scalar manifold, giving all Killing vectors and prepotentials that allow us to write down the generic scalar potential.

Section IV provides an analysis of gauges and flows in examples. For the toy model with only a universal hypermultiplet, only one critical point arises, and, depending on the direction gauged within the $SU(2)\times U(1)$ compact subgroup, it can have different nature. Then we study the full model and examine the possibility of finding different flows between two fixed points, proving in particular that the FGPW flow can be recovered.

We finish with some concluding remarks in Sec. V.

Our conventions are generally those of [1]. In Appendix A we present a convenient table for the reader to recall the use and range of all the indices. In Appendix B we repeat some notational issues, paying attention to reality conditions. In Appendix C we comment on the toy model in a different parametrization, useful for comparison with [12].

II. SUPERSYMMETRIC FLOW EQUATIONS AND DOMAIN WALL ATTRACTORS

A. Basic aspects of the theory

The bosonic sector of 5D, $\mathcal{N}=2$ supergravity coupled to n vector multiplets and r hypermultiplets[3] has as independent fields the fünfbein e_μ^a, the $n+1$ vectors A_μ^I with field strengths $F_{\mu\nu}^I=\partial_\mu A_\nu^I-\partial_\nu A_\mu^I+gA_\mu^J A_\nu^K f_{JK}{}^I$, the n scalars ϕ^x, and the $4r$ "hyperscalars" q^X. Full results of the action and transformation laws are in [1]. We repeat here the main ingredients (for some technical issues, see Appendix B). The bosonic part of the Lagrangian is

$$\begin{aligned}e^{-1}\mathcal{L}_{\text{bosonic}}^{\mathcal{N}=2}=&-\tfrac{1}{2}R-\tfrac{1}{4}a_{IJ}F_{\mu\nu}^I F^{J\mu\nu}-\tfrac{1}{2}g_{XY}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y\\&-\tfrac{1}{2}g_{xy}\mathcal{D}_\mu\phi^x\mathcal{D}^\mu\phi^y\\&+\frac{1}{6\sqrt{6}}C_{IJK}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}F_{\mu\nu}^I F_{\rho\sigma}^J A_\tau^K-g^2\mathcal{V}(\phi,q),\end{aligned}$$

where

$$\mathcal{D}_\mu q^X=\partial_\mu q^X+gA_\mu^I K_I^X(q),\quad \mathcal{D}_\mu\phi^x=\partial_\mu\phi^x+gA_\mu^I K_I^x(\phi). \tag{2.1}$$

Here $K_I^X(q)$ are the Killing vectors of the gauged isometries on the quaternionic scalar manifold parametrized by the hyperscalars q^X, whereas $K_I^x(\phi)$ are those of the very special manifold spanned by the ϕ^x of the vector multiplets. We will come back to these below.

The scalars of the vector multiplets can be described by a hypersurface in an $(n+1)$-dimensional space [21]

$$C_{IJK}h^I(\phi)h^J(\phi)h^K(\phi)=1. \tag{2.2}$$

The real coefficients C_{IJK} determine the metrics of "very special geometry" [22]

$$\begin{aligned}&a_{IJ}\equiv-2C_{IJK}h^K+3C_{IKL}C_{JMN}h^K h^L h^M h^N=h_I h_J+h_{xI}h_J^x,\\&g_{xy}\equiv h_x^I h_y^J a_{IJ},\quad h_I=C_{IJK}h^J h^K,\quad h_x^I\equiv-\sqrt{\tfrac{3}{2}}\partial_x h^I(\phi),\end{aligned} \tag{2.3}$$

which are further used for raising and lowering indices. A non-Abelian structure in the absence of tensor multiplets should satisfy

$$C_{L(IJ}f_{K)M}^L=0,\quad K_I^x=\sqrt{\tfrac{3}{2}}h_K f_{JI}^K h^{Jx} \tag{2.4}$$

which implies

$$h_I f_{JK}^I h^K=0\rightarrow K_I^x h^I=0. \tag{2.5}$$

The quaternionic Kähler geometry is determined by $4r$-beins f_X^{iA} (as one-forms $f^{iA}=f_X^{iA}dq^X$), with the SU(2) index $i=1,2$ and the $Sp(2r)$ index $A=1,\ldots,2r$, raised and lowered by the symplectic metrics C_{AB} and ε_{ij} (see Appendix B for conventions, reality conditions, etc). The metric on the hyperscalar space is given by

$$g_{XY}\equiv f_X^{iA}f_Y^{jB}\varepsilon_{ij}C_{AB}=f_X^{iA}f_{YiA}. \tag{2.6}$$

This implies that the vielbeins satisfy also

$$f_{iA}^X f_Y^{iA}=\delta_Y^X,\quad f_{iA}^X f_X^{jB}=\delta_i{}^j\delta_A{}^B. \tag{2.7}$$

They are covariantly constant, including Levi-Cività connection $\Gamma_{XZ}{}^Y$ on the manifold, $Sp(2r)$ connection $\omega_X{}^B{}_A$, and SU(2) connection $\omega_{Xi}{}^j$, which are all functions of the hyperscalars:

[3] We omit tensor multiplets for simplicity.

$$\partial_X f_Y^{iA} - \Gamma_{XY}{}^Z f_Z^{iA} + f_Y^{iB}\omega_{XB}{}^A + \omega_{Xk}{}^i f_Y^{kA} = 0. \tag{2.8}$$

The SU(2) curvature is

$$\mathcal{R}_{XYij} = f_{XC(i} f^C_{j)Y}, \tag{2.9}$$

and there is a connection $\omega_{\mu i}{}^j = (\partial_\mu q^X)\omega_{Xi}{}^j$ such that

$$\mathcal{R}_{XYi}{}^j = 2\partial_{[X}\omega_{Y]i}{}^j - 2\omega_{[X|i|}{}^k \omega_{Y]k}{}^j = i\mathcal{R}^r_{XY}(\sigma_r)_i{}^j,$$
$$\mathcal{R}^r = d\omega^r - \varepsilon^{rst}\omega^s\omega^t \tag{2.10}$$

with $r=1,2,3$ and real $\mathcal{R}^r_{XY}$ [see Eq. (B2)]. The SU(2) curvatures have a product relation that reflects that they are proportional to the three complex structures of the quaternionic space

$$\mathcal{R}^r_{XY}\mathcal{R}^{sYZ} = -\tfrac{1}{4}\delta^{rs}\delta_X{}^Z - \tfrac{1}{2}\varepsilon^{rst}\mathcal{R}^t{}_X{}^Z. \tag{2.11}$$

The Killing vectors on the hyperscalars K_I^X can be obtained from an SU(2) triplet of real prepotentials $P_I^r(q)$ that are defined by the relation [23–25,1]

$$\mathcal{R}^r_{XY}K_I^Y = D_X P_I^r, \quad D_X P_I^r \equiv \partial_X P_I^r + 2\varepsilon^{rst}\omega_X^s P_I^t. \tag{2.12}$$

These yield [using Eq. (2.11)]

$$K_I^Z = -\tfrac{4}{3}\mathcal{R}^{r\,ZX} D_X P_I^r. \tag{2.13}$$

These prepotentials satisfy the constraint

$$\tfrac{1}{2}\mathcal{R}^r_{XY}K_I^X K_J^Y - \varepsilon^{rst}P_I^s P_J^t + \tfrac{1}{2}f_{IJ}^K P_K^r = 0. \tag{2.14}$$

In local supersymmetry, the prepotentials are defined uniquely from the Killing vectors. Indeed,[4]

$$P_I^r = \frac{1}{2r}D_X K_{IY}\mathcal{R}^{XYr} \tag{2.15}$$

satisfies Eq. (2.12), and any covariantly constant shift $P_I^{(0)r}$ is excluded as the integrability condition $\varepsilon^{rst}\mathcal{R}^s_{XY}P^{(0)t}=0$ implies that $P_I^{(0)r}=0$. As in four dimensions [25], these shifts are interpreted as the analogues of the Fayet-Iliopoulos (FI) terms for the $D=4$, $\mathcal{N}=1$ theories. However, in local supersymmetry we thus find the absence of the FI term except when there are no hypermultiplets [or in rigid supersymmetry where the SU(2) curvature vanishes].

We will also need the bosonic part of the supersymmetry transformations of the fermions, which are (with vanishing vectors)

$$\delta_\epsilon\psi_{\mu i} = D_\mu(\omega)\epsilon_i + i\frac{1}{\sqrt{6}}g\gamma_\mu P_{ij}\epsilon^j$$
$$= \partial_\mu\epsilon_i + \tfrac{1}{4}\gamma^{ab}\omega_{\mu,ab}\epsilon_i - \omega_{\mu i}{}^j\epsilon_j + i\frac{1}{\sqrt{6}}g\gamma_\mu P_{ij}\epsilon^j,$$
$$\delta_\epsilon\lambda_i^x = -i\tfrac{1}{2}(\not\partial\varphi^x)\epsilon_i + g\epsilon^j P^x_{ij},$$
$$\delta_\epsilon\zeta^A = -i\tfrac{1}{2}f^A_{iX}(\not\partial q^X)\epsilon^i + g\epsilon^i\mathcal{N}_i^A, \tag{2.16}$$

where [as for all triplets $P_{ij} = iP^r(\sigma^r)_{ij}$; see Eq. (B2)]

$$P^r \equiv h^I(\phi)P_I^r(q), \quad P_x^r \equiv -\sqrt{\tfrac{3}{2}}\partial_x P^r = h_x^I P_I^r,$$
$$\mathcal{N}_i^A \equiv \frac{\sqrt{6}}{4}f^A_{iX}K^X = \frac{2}{\sqrt{6}}f^A_{iX}\mathcal{R}^{r\,YX}D_Y P^r,$$
$$K^X \equiv h^I(\phi)K_I^X(q). \tag{2.17}$$

The scalar potential is given by

$$\mathcal{V} = -4P^rP^r + 2P_x^rP_y^r g^{xy} + 2\mathcal{N}_{iA}\mathcal{N}^{iA}. \tag{2.18}$$

This expression can be understood as in all supersymmetric theories (see [26] for a proof in four dimensions) by squaring the scalar part of the supersymmetry transformations of the fermions using their kinetic terms. The kinetic terms of the fermions are

$$e^{-1}\mathcal{L}^{\mathcal{N}=2}_{\text{ferm,kin}} = -\tfrac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - \tfrac{1}{2}\bar{\lambda}^i_x\gamma^\nu\partial_\nu\lambda_i^x - \bar{\zeta}^A\gamma^\nu\partial_\nu\zeta_A. \tag{2.19}$$

This defines the metric to be used to square the supersymmetry transformations:

$$-\tfrac{1}{2}(\delta_{\epsilon_1,sc}\bar{\psi}_\mu)\gamma^{\mu\nu\rho}(\delta_{\epsilon_2,sc}\psi_\rho) - \tfrac{1}{2}(\delta_{\epsilon_1,sc}\bar{\lambda}^i_x)\gamma^\nu(\delta_{\epsilon_2,sc}\lambda_i^x)$$
$$-(\delta_{\epsilon_1,sc}\bar{\zeta}^A)\gamma^\nu(\delta_{\epsilon_2,sc}\zeta_A) = \tfrac{1}{4}g^2\bar{\epsilon}_1\gamma^\nu\epsilon_2\,\mathcal{V}. \tag{2.20}$$

The gravitino gives the negative contribution to the potential, while the gauginos and hyperinos give the positive contributions.

We introduce the scalar "superpotential" function W, which can be read off the gravitino supersymmetry transformation, by[5]

$$W = \sqrt{\tfrac{1}{3}P_{ij}P^{ij}} = \sqrt{\tfrac{2}{3}P^rP^r}, \tag{2.21}$$

such that the potential gets, under certain conditions, the form that has been put forward for gravitational stability:

$$\mathcal{V} = -6W^2 + \frac{9}{2}g^{\Lambda\Sigma}\partial_\Lambda W\partial_\Sigma W, \tag{2.22}$$

[4]This formula can also be derived from the harmonicity property of the quaternionic prepotential $D^XD_XP_I^r = 2rP_I^r$ [13].

[5]As a convention, we pick a positive definite W.

where $g_{\Lambda\Sigma}$ is the metric of the complete scalar manifold, involving the scalars of vector multiplets as well of hypermultiplets. It is easy to see that in this case critical points of W are also critical points of $\mathcal{V}$.

For one scalar, a proof of gravitational stability was found in [27] (even without supersymmetry) in four dimensions, and extended to higher dimensions and to the multiscalar case in [28] for potentials that are a function of the "superpotential" as in Eq. (2.22). However, more general potentials are also compatible with the gravitational stability. More recently, this issue has been revived in [17,29].

The negative part of the potential (2.18) straightforwardly takes the form of the first term in Eq. (2.22). For the contribution of the hypermultiplets, the form $g^{XY}\partial_X W \partial_Y W$ follows from

$$\partial_X W = \frac{2}{3W} P^r D_X P_r = \frac{2}{3W} P^r \mathcal{R}^{\;r}_{XY} K^Y \tag{2.23}$$

and Eq. (2.11). However, for the vector multiplets the analogous expression cannot be obtained in general. Using the decomposition of the vector P^r in its norm and phases

$$P^r = \sqrt{\tfrac{3}{2}}\, W Q^r, \qquad Q^r Q^r = 1, \tag{2.24}$$

one sees that the term $2P^r_x P^{rx}$ in Eq. (2.18) gets the form of the vector multiplet contribution in Eq. (2.22) if

$$(\partial_x Q^r)(\partial^x Q^r) = 0 \Rightarrow \partial_x Q^r = 0. \tag{2.25}$$

This condition[6] is satisfied in several cases. When there are no hyperscalars and only Abelian vector multiplets, the constraints (2.12) and (2.14) imply that the Q^r are constants. Also, when there are no physical vector multiplets, this condition is obviously satisfied. We will see below that Eq. (2.25) is related to a condition of unbroken supersymmetry. In the explicit example that we will show in Sec. IV, there will be flows where Q^r is independent of the scalars in vector multiplets, such that again Eq. (2.25) is satisfied.

B. BPS equations in a domain wall background

We are looking for supersymmetric domain wall solutions that preserve half of the original supersymmetries of the $\mathcal{N}=2$ supergravity. Thus we use as a generic ansatz for the metric

$$ds^2 = a(x^5)^2 dx^{\underline{\mu}} dx^{\underline{\nu}} \eta_{\underline{\mu\nu}} + (dx^5)^2, \tag{2.26}$$

where $\underline{\mu},\underline{\nu} = 0,1,2,3$, which respects four-dimensional Poincaré invariance, and we model this solution by allowing the scalars to vary along the fifth direction x^5. These solutions are obtained when we require that the supersymmetry transformation rules on this background vanish for some Killing spinor parameter ϵ^i.

[6]This constraint is equivalent to the one found in [8].

When all vectors are vanishing, the relevant supersymmetry flow equations for the gravitinos ψ^i, the gaugini λ^x_i, and the hyperini ζ^A are

$$\begin{aligned}
\delta_\epsilon \psi_{\underline{\mu} i} &= \partial_{\underline{\mu}} \epsilon_i + \gamma_{\underline{\mu}} \left(\frac{1}{2}\frac{a'}{a}\gamma_5 \epsilon_i + \frac{i}{\sqrt{6}} g P_{ij} \epsilon^j \right),\\
\delta_\epsilon \psi_{5i} &= \epsilon_i' - q^{X\prime} \omega_{Xi}{}^j \epsilon_j + \frac{i}{\sqrt{6}} g \gamma_5 P_{ij} \epsilon^j,\\
\delta_\epsilon \lambda^x_i &= -\frac{i}{2}\gamma^5 \epsilon_i \phi^{x\prime} + g P^x_{ij} \epsilon^j,\\
\delta_\epsilon \zeta^A &= f^A_{iX}\left[-\frac{i}{2}\gamma^5 q^{X\prime} - \frac{2g}{\sqrt{6}} \mathcal{R}^{r\,XY} (D_Y P)^r \right] \epsilon^i,
\end{aligned} \tag{2.27}$$

where the prime is a derivative with respect to x^5, and we have assumed that q^X depends only on x^5.

The equation $\delta\psi^i_5 = 0$ gives just the dependence of the Killing spinor on the fifth coordinate. We assume[7] also that the Killing spinor does not depend on $x^{\underline{\mu}}$.

The first Killing equation gives

$$i\frac{a'}{a}\gamma_5 \epsilon_i = \sqrt{\frac{2}{3}} g P_{ij} \epsilon^j, \tag{2.28}$$

whose consistency as a projector equation requires

$$\left[\delta^k_i \left(\frac{a'}{a}\right)^2 - g^2 \frac{2}{3} P_{ij} P^{jk} \right] \epsilon_k = 0. \tag{2.29}$$

This can then be easily written in terms of W as

$$\left[\left(\frac{a'}{a}\right)^2 - g^2 W^2 \right] \epsilon_i = 0. \tag{2.30}$$

For any preserved supersymmetry, this gives us an equation relating the warp factor and the superpotential (with $g>0$):

$$gW = \left|\frac{a'}{a}\right| = \pm \frac{a'}{a}. \tag{2.31}$$

Using the notation (2.24), the projection (2.28) is [we further keep consistently the upper and lower signs as they appear in Eq. (2.31)]

$$\gamma_5 \epsilon_i = \pm Q^r \sigma^r_{ij} \epsilon^j. \tag{2.32}$$

The gaugino equation, after using Eq. (2.28), gives rise to the condition

[7]In some cases there may be other solutions. At the critical points, the supersymmetry is doubled, the extra Killing spinors being of the type with extra dependence on the $x^{\underline{\mu}}$. Here we restrict ourselves to solutions with Killing spinors that do not depend on $x^{\underline{\mu}}$.

$$P_{ij}\epsilon^j\phi^{x\prime}=-3\frac{a'}{a}g^{xy}\partial_y P_{ij}\epsilon^j. \tag{2.33}$$

Using the decomposition of P^r in Eq. (2.24) one finds

$$WQ^r\phi^{x\prime}=-3\frac{a'}{a}g^{xy}(Q^r\partial_y W+W\partial_y Q^r). \tag{2.34}$$

Since $Q^r\partial_x Q^r=0$, the two pieces on the right are orthogonal to each other and so we derive as independent conditions

$$\partial_y Q^r=0 \tag{2.35}$$

and [using Eq. (2.31)]

$$\phi^{x\prime}=\mp 3gg^{xy}\partial_y W. \tag{2.36}$$

The first condition is Eq. (2.25), and we thus find that the BPS condition is equivalent to requiring that the potential can be written in the stability form (2.22). Notice that the projection given in Eq. (2.32) therefore only depends on the hypermultiplets.[8]

The formula (2.36) can be generalized to the hypermultiplets. In view of this, we turn to the hyperino Killing equation, the last of Eq. (2.27). For the first term, we can already use Eq. (2.32). We multiply the transformation of the hyperinos by f^Y_{Aj}. Equations (2.6) and (2.9) lead to

$$f_{YjA}f^{iA}_X=\tfrac{1}{2}g_{YX}\delta_j{}^i+\mathcal{R}_{YXj}{}^i. \tag{2.37}$$

This gives

$$0=[g_{YX}\delta_j{}^i+2i\mathcal{R}_{YX}{}^r\sigma^r{}_j{}^i]\left[\pm\frac{1}{2}iQ^s\sigma^s{}_i{}^k q^{X\prime}+\frac{g\sqrt{6}}{4}K^X\delta_i{}^k\right]\epsilon_k, \tag{2.38}$$

This we write as a matrix equation $\mathcal{A}_{Yj}{}^k\epsilon_k=0$:

$$0=[\mathcal{A}^0_Y\delta_j{}^k+\mathcal{A}^r_Y i(\sigma^r)_j{}^k]\epsilon_k,$$

$$\mathcal{A}^0_Y\equiv\frac{g\sqrt{6}}{4}K_Y\mp\mathcal{R}^r_{YX}Q^r q^{X\prime},$$

$$\mathcal{A}^r_Y\equiv\pm\frac{1}{2}Q^r q'_Y\mp\varepsilon^{rst}\mathcal{R}^s_{YX}Q^t q^{X\prime}+\frac{g\sqrt{6}}{2}\mathcal{R}^r_{YX}K^X. \tag{2.39}$$

The reality of these quantities implies that the determinant of the matrix $\mathcal{A}_Y$ is

$$\det\ \mathcal{A}_Y=(\mathcal{A}^0_Y)^2+(\mathcal{A}^r_Y)^2. \tag{2.40}$$

If there are any preserved supersymmetries, then this determinant has to be zero. Therefore, $\mathcal{A}^0_Y=\mathcal{A}^r_Y=0$. However, it is easy to see that the condition $\mathcal{A}^0_Y=0$ implies $\mathcal{A}^r_Y=0$, and is thus the remaining necessary and sufficient condition. With Eq. (2.23), this implies

$$g\partial_X W=\sqrt{\tfrac{2}{3}}gQ^r\mathcal{R}^r_{XY}K^Y=\mp\tfrac{1}{3}g_{XY}q^{Y\prime}. \tag{2.41}$$

One can also show that this equation is sufficient for the Killing equations.

We have obtained the same condition for the scalars of hypermultiplets as for vector multiplets, and we can write collectively for all the scalar fields $\phi^\Lambda=\{\phi^x,q^X\}$

$$\phi^{\Lambda\prime}=\mp 3g\, g^{\Lambda\Sigma}\partial_\Sigma W. \tag{2.42}$$

This equation, together with the constraint (2.35) and the flow equation for the warp factor (2.31), completely describes our supersymmetric flow.[9] A solution to these equations is also a solution of the full set of equations of motion.

The fact that BPS states are described by Eqs. (2.42) and (2.31) can also be seen from the expression of the energy functional. Once the (2.35) condition is satisfied, such a functional can be written as

$$E=\int_{-\infty}^{+\infty}dx^5\, a^4\left[\frac{1}{2}(\phi^{\Lambda\prime}\mp 3g\partial^\Lambda W)^2-6\left(\frac{a'}{a}\mp gW\right)^2\right]$$
$$\mp 3g\int_{-\infty}^{+\infty}dx^5\frac{\partial}{\partial x^5}(a^4W)+4\int_{-\infty}^{+\infty}dx^5\frac{\partial}{\partial x^5}(a^3a'). \tag{2.43}$$

C. Renormalization group flow

The above formulas can be used to obtain the equations that give the dependence of the scalars on the warp factor *a*. Using the chain rule, the relevant supersymmetry flow equations for all the scalar fields reduce to

$$\beta^\Lambda\equiv a\frac{\partial}{\partial a}\phi^\Lambda=a\frac{\partial x^5}{\partial a}\frac{\partial\phi^\Lambda}{\partial x^5}$$
$$=\mp 3g\frac{a}{a'}g^{\Lambda\Sigma}\partial_\Sigma W$$
$$=-3g^{\Lambda\Sigma}\frac{\partial_\Sigma W}{W}. \tag{2.44}$$

The notation as a beta function follows from the interpretation as a conformal field theory, where the scalars play the role of coupling constants and the warp factor *a* is playing the role of an energy scale.

[8]In the presence of tensor multiplets, the gaugino supersymmetry (SUSY) rule would have been modified by an additional term $\delta'_\epsilon\lambda^x_i=gW^x\epsilon_i$. However, this would have been put to zero by the gaugino projector equation $\mathcal{A}^x{}_k{}^i\epsilon_i=(\mathcal{A}^{0x}\epsilon_k+\mathcal{A}^{rx}i(\sigma_r)_k{}^i)\epsilon_i=0$ where $\mathcal{A}^{0x}\equiv W^x$ and $\mathcal{A}^{rx}$ was implicit in Eq. (2.33).

[9]This flow equation also appears in [8]. However, there it was derived using a condition that is stronger than the one we need. Our condition is the one that also implies the stability form of the potential.

This same function can be used to determine the nature of the critical points ϕ^*. Whether ϕ^* has to be interpreted as UV or IR in the dual CFT can be inferred from the expansion of Eq. (2.44)

$$a\frac{\partial}{\partial a}\phi^{\Lambda}=(\phi^{\Sigma}-\phi^{\Sigma}*)\frac{\partial\beta^{\Lambda}}{\partial\phi^{\Sigma}}\bigg|_{\phi*}. \tag{2.45}$$

This tells us that any time the matrix

$$\mathcal{U}_{\Sigma}{}^{\Lambda}\equiv-\frac{\partial\beta^{\Lambda}}{\partial\phi^{\Sigma}}\bigg|_{\phi*}=\frac{3}{W}g^{\Lambda\Xi}\frac{\partial^2 W}{\partial\phi^{\Sigma}\partial\phi^{\Xi}}\bigg|_{\phi*} \tag{2.46}$$

has a positive eigenvalue ϕ^* is a UV critical point, whereas when it has negative eigenvalues ϕ^* is IR.

The eigenvalues of $\mathcal{U}$ are the conformal weights E_0 of the associated operators in the conformal picture. One can obtain a general formula [30] for $\mathcal{U}$:

$$\mathcal{U}=\begin{pmatrix}\frac{3}{2}\delta_X{}^Y-\frac{1}{W^2}\mathcal{J}_X{}^Z\mathcal{L}_Z{}^Y & \frac{1}{W^2}\mathcal{J}_{XZ}\partial^y K^Z\\ -\frac{1}{W^2}(\partial_x K^Z)\mathcal{J}_Z{}^Y & 2\delta_x{}^y\end{pmatrix}, \tag{2.47}$$

where the first entry corresponds to hypermultiplets and the second to vector multiplets. The quantities $\mathcal{J}$ and $\mathcal{L}$ are defined as[10]

$$D_X K_Y=\mathcal{J}_{XY}+\mathcal{L}_{XY},\quad \mathcal{J}_{XY}=2P^r\mathcal{R}^r_{XY}. \tag{2.48}$$

They commute, $\mathcal{J}^2$ is proportional to (minus) the unit matrix, and the trace of $\mathcal{JL}$ is zero:

$$\mathcal{J}_X{}^Y\mathcal{J}_Y{}^Z=-\tfrac{3}{2}W^2\delta_X{}^Z,\quad \mathcal{J}_X{}^Y\mathcal{L}_Y{}^Z=\mathcal{L}_X{}^Y\mathcal{J}_Y{}^Z,$$
$$\mathcal{J}_X{}^Y\mathcal{L}_Y{}^X=0. \tag{2.49}$$

The decomposition of DK in Eq. (2.48) is a split of the isometries in SU(2) and USp($2r$) parts.

The lower right entry of Eq. (2.47), a consequence of the basic equations of very special geometry, is the statement that for only vector multiplets there are only UV critical points, preventing the RS scenarios [4,5]. The other entries imply that the appearance of IR directions can be due to two different mechanisms [8]. One is the presence of the hypermultiplets, if the upper left entry gets negative values, whereas the other is due to the possibility of mixing between vector and hypermultiplets. To have negative eigenvalues due to the hypermultiplets only, the $\mathcal{L}$ matrix has to get large. This means that the gauging has to be "mainly" outside the SU(2) group. We will see this explicitly in the examples of Sec. IV, where the orthogonal part to SU(2) is a U(1) group.

An immediate consequence of Eq. (2.47) is that

$$\mathrm{Tr}\,\mathcal{U}=6\,r+2\,n. \tag{2.50}$$

[10]This splitting was also put in evidence in [13].

The right-hand side is thus the sum of all the eigenvalues. This implies that there are no pure IR fixed points, i.e., there are at most fixed points for which flows in particular directions are of the IR type.

These same eigenvalues are related to the scalar masses through the mass matrix [9]

$$\mathcal{M}_{\Lambda}{}^{\Sigma}=W_{\rm cr}^2\;\mathcal{U}_{\Lambda}{}^{\Delta}(\mathcal{U}_{\Delta}{}^{\Sigma}-4\delta_{\Delta}{}^{\Sigma}). \tag{2.51}$$

The scaling dimensions of the dual conformal fields are therefore the eigenvalues of $\mathcal{U}$.

Equations (2.31) and (2.42) also lead directly to the monotonicity theorem for A'. Indeed, defining

$$A=\ln a,\quad A'=\pm gW, \tag{2.52}$$

we have directly that

$$A''=\pm gW'=-3g^2(\partial_{\Lambda}W)g^{\Lambda\Sigma}(\partial_{\Sigma}W)\leqslant 0. \tag{2.53}$$

Therefore A' is a monotonically decreasing function. In the usual holographic correspondence, this is related to the monotonicity of the c function [16,9,17,31,32].

The above issues can be applied to address the question of the existence of *smooth Randall-Sundrum* scenarios. In such a scenario, the scalars should get to a constant value at $x^5=\pm\infty$, and with Eq. (2.42) this means that W should have an extremum at $x^5=\pm\infty$, i.e., with Eq. (2.44), a zero of the beta function and thus a critical point. For a RS scenario the warp factor should be small there, i.e., it should be a critical point for a small energy scale, an IR critical point. Thus, we need a solution that interpolates between *two IR critical points* for $x^5=\pm\infty$, getting to a maximum of the warp factor A at the center of the domain wall, placed, for instance, at $x^5=0$. This requires that at the same point *W should be zero.*

This situation can in principle be realized without violating the condition (2.53). Indeed, take a W that decreases to zero at $x_5=0$ from positive x_5. With a smooth flow, one might expect that W changes sign, as its derivative is nonzero at this point. Note, however, that our W is always positive due to its definition as the norm of the SU(2) vector P^r. This is necessary because in the geometry of hypermultiplets the local SU(2) is essential, and W has to be an invariant function. It thus bumps up again and increases. But at the same time the unit vector Q^r jumps to its negative. In this way, $P^r=WQ^r$ behaves smoothly, leading to a smooth flow despite the apparent jumps. Because of the sign switches in ∂W and Q, for negative x_5, one must take consistently all the different signs in Eqs. (2.31)–(2.42). Note that the two sign flips combine such that the projection of the Killing spinor in Eq. (2.32) will not change. Then W will increase again for negative x_5 and the monotonicity of the warp factor will not be violated.

Of course in the holographic interpretation of the c theorem, the central charge would blow up or the height function would become singular at the zero of the superpotential [11], and the dual field theory would be ill defined at that point. In spite of this, the supergravity monotonicity theorem can further be satisfied with increasing W, if it was decreasing at the other side of $x^5=0$.

The interesting points are thus the zeros that we just discussed, and the critical points, where $\partial_\Lambda W=0$. We now turn to discussing the properties of the latter.

D. Enhancement of unbroken supersymmetry and algebraic attractor equations

The search for critical points can be nicely formalized as an attractor mechanism, which was discovered in [2] and was studied in great detail in the absence of hypermultiplets. So far only partial investigations exist for the coupling of both vector multiplets and hypermultiplets, in the context of domain walls [8,11,33] and in the context of the BPS instantons [34]. At the fixed points of the solution the moduli are defined by the condition $\mathcal{N}_{iA}=0$ [1], which implies $K^X=0$, as can be understood from Eqs. (2.17) and (2.23). This fact was also observed in [34,8,11,13].

Here we will use the fact derived in the previous section, that the Killing spinor projector Q^r must satisfy Eq. (2.25). Only in such case does the superpotential W control the flow equations. *Using the enhancement of supersymmetry near the critical points we will derive all necessary and sufficient conditions for critical points.* Our method follows [2,35,3], where the geometric tools of special geometry in $D=4$ and very special geometry in $D=5$ were used to convert the BPS differential equations into algebraic ones and where enhancement of unbroken supersymmetry played an important role.

Consider the domain wall solutions of the previous subsection in the limit where the scalars are frozen:

$$q^{X\prime}=0, \quad \phi^{x\prime}=0, \quad \frac{a'}{a}=\pm W_{\rm cr}=\text{const}. \tag{2.54}$$

If const $\neq 0$, the flow tends to the AdS horizon in case of the IR critical point and to the boundary of the AdS space in case of the UV critical point. This becomes clear when the metric is rewritten as $ds^2=a^2(dx^{\underline{\mu}})^2+(1/W^2)(da/a)^2$. For constant *nonvanishing* W, small a define the horizon of the AdS space whereas large a correspond to its boundary.

The gravitino supersymmetry transformation at the critical point (2.54) acquires a second Killing spinor. This is the same doubling that always occurs in the AdS background near the black hole horizon. One finds that

$$\delta_\epsilon \psi_\mu=0, \quad (\epsilon_i)_{\rm attr}\neq 0, \tag{2.55}$$

without restrictions on the Killing spinors, i.e., they have eight real components.

In analyzing the equations we will have to be careful that we are inside the domain of validity of our coordinate system. In particular, this means that g_{xy}, g_{XY}, f_X^{iA}, and $\mathcal{R}^{r\,XY}$ are neither vanishing nor infinite. We will be able to invert these geometric objects using the rules of very special and quaternionic geometry. The procedure is analogous to the steps performed in the previous section to find the solutions with $\mathcal{N}=1$ unbroken supersymmetry. Now we will specify it to the case of frozen moduli and $\mathcal{N}=2$ unbroken supersymmetry.

By direct inspection of the supersymmetry transformations we observe that the first term in the gaugino and hyperino transformation vanishes and we get

$$\delta_\epsilon \lambda_i^x = g P^x_{ij}\epsilon^j=0,$$

$$\delta_\epsilon \zeta^A = f^A_{iX}\left[\frac{\sqrt{6}g}{4}K^X\right]\epsilon^i=0. \tag{2.56}$$

The first of these equations for $\epsilon_i\neq 0$ can be satisfied if and only if

$$(\partial_y P^r)_{\rm attr}=0, \tag{2.57}$$

which, for AdS vacua, can also be written as

$$(Q^r\partial_y W+W\partial_y Q^r)=0 \Rightarrow (\partial_y W)_{\rm attr}=0 \quad \text{and} \quad (\partial_y Q^r)_{\rm attr}=0. \tag{2.58}$$

The implication follows from the same argument as for Eq. (2.35).

Finally, we have to derive the necessary and sufficient conditions to satisfy the hyperino equation also. Evaluating Eq. (2.39) at the attractor point $q'^X=0$ for $\epsilon_i\neq 0$ requires

$$(\mathcal{A}^0_Y)_{\rm attr}\equiv \frac{g\sqrt{6}}{4}K_Y=0, \quad (\mathcal{A}^r_Y)_{\rm attr}\equiv \frac{g\sqrt{6}}{2}\mathcal{R}^r{}_{YX}K^X=0. \tag{2.59}$$

The solution of this equation is

$$(K^X)_{\rm attr}\equiv h^I(\phi)K^X_I(q)=0. \tag{2.60}$$

As previously noticed, this is an algebraic equation that defines the fixed values of the scalars at the critical point and solves $\delta_\epsilon \zeta^A=0$.

The algebraic rather than differential nature of this condition stimulates us to look for an algebraic equation also in the vector multiplet sector of the theory. Indeed, such an algebraic attractor equation was known to be valid for AdS critical points in theories without hypermultiplets [3] and we now try to generalize it. We start with Eq. (2.57) and multiply this equation by $g^{xy}\partial_y h_I$. Using the fact that (h_{Ix},h_I) forms an $(n+1)\times(n+1)$ invertible matrix in very special geometry (2.3), this equation becomes

$$P^r_I=C_{IJK}h^Jh^KP^r=h_IP^r. \tag{2.61}$$

So far the result is valid for any critical point. If we now restrict ourselves to the AdS ones, we can multiply Eq. (2.61) by Q^r and get

$$h_IW=C_{IJK}\tilde{h}^I\tilde{h}^K=P_I, \quad P_I(q)\equiv\sqrt{\tfrac{2}{3}}P^r_IQ^r,$$

$$\tilde{h}^I(\phi,q)=h^I\sqrt{W}. \tag{2.62}$$

Note that P_I depends only on quaternions. This type of algebraic attractor equation with constant values of P_I was used in an efficient way in various situations before. In particular, it was used in calculations of the entropy of Calabi-Yau black holes and the warp factor of Calabi-Yau domain walls near the critical points. We have shown here that in the presence of the hypermultiplets the analogous algebraic equations with quaternion-dependent P_I are valid at the critical points. Thus the algebraic equation $h_I P^r = P_I^r$ is equivalent to the differential equation $\partial_x W = 0$, since for supersymmetric flows the $\partial_x Q^r = 0$ condition is satisfied. If we multiply this by h^I we will get an identity $P^r = P^r$; however, Eq. (2.61) is not satisfied in general but only at the fixed points where all scalars are constant.

Thus we have the system of algebraic equations, defining the critical points:

$$h_I(\phi_{\rm cr}) P^r(\phi_{\rm cr}, q_{\rm cr}) = P_I^r(q_{\rm cr}), \quad K^X(\phi_{\rm cr}, q_{\rm cr}) = 0. \tag{2.63}$$

They are equivalent to the system of differential equations minimizing the superpotential. These equations, together with $\partial_x Q^r = 0$, are equivalent to minimizing the triplet of the prepotentials

$$(\partial_x P^r)_{\rm cr} = 0, \quad (D_X P^r)_{\rm cr} = 0. \tag{2.64}$$

The advantage of having algebraic rather than differential equations defining the critical points is already obvious in the simple examples that we consider in the models below, but it will be even more essential in cases with arbitrarily many moduli.

As a conclusion of this section, we can summarize the relevant equations to be examined in the specific examples. Given a scalar manifold, we have to look for the following special points.

(1) *Fixed points*. These are points where $\partial_\Lambda W = 0$. They are determined by algebraic equations:

$$\text{Fixed points:} \quad K^X \equiv h^I(\phi) K_I^X(q) = 0,$$
$$P_I^r(q) = h_I(\phi) P^r(\phi, q). \tag{2.65}$$

In particular, for the AdS case, the eigenvalues of the matrix (2.46) determine whether they are UV (eigenvalues positive), or whether some eigenvalues are negative. In the latter case, they can be used as IR fixed points, and represent the values of the scalars at $x^5 = \pm\infty$ in the RS scenario.

(2) *Zeros.* These determine the values of the scalars on the place of the domain wall, i.e., where the warp factor reaches an extremum:

$$\text{Zeros:} \quad P^r = h^I(\phi) P_I^r(q) = 0. \tag{2.66}$$

Note that the presence of zeros implies that the β function diverges. This indicates that the AdS/CFT correspondence breaks down at this point. These zeros are necessary for Randall-Sundrum domain walls but are thus pathological for applications as renormalization group flows.

III. A MODEL WITH A VECTOR MULTIPLET AND A HYPERMULTIPLET

In this and the next section we want to specify the results obtained so far to two detailed examples. The simplest model that can show all the main features of this kind of analysis is given by supergravity coupled to one vector and one hypermultiplet. Thus, as a first step, we describe in full detail the toy model based on the universal hypermultiplet alone. Then we analyze the complete model, whose moduli space is given by the scalar manifold

$$\mathcal{M} = \mathrm{O}(1,1) \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times \mathrm{U}(1)}. \tag{3.1}$$

Actually, in this example we focus on one important supersymmetric domain wall solution that was previously discussed as the dual to the renormalization group flow describing the deformation from an $\mathcal{N}=4$ to an $\mathcal{N}=1$ super Yang-Mills theory with SU(2) flavor group [9].

This solution (at least numerically) was originally obtained inside the $\mathcal{N}=8$ gauged supergravity theory, but we will show that it can also have a consistent description in the standard $\mathcal{N}=2$ one. Notice that this same flow was claimed to be present also in a truncated $\mathcal{N}=4$ gauged supergravity coupled to two tensor multiplets, but the relevant model has only recently been constructed in [36]. The older Ref. [37] dealt only with the coupling to vector multiplets and gauging of the $\mathrm{SU}(2)_R$ group, whereas [38,39] discussed the $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ gauging without any matter coupling.

More precisely, we will show that *the FGPW flow can be consistently retrieved in the* $\mathcal{N}=2$ *supergravity with one massless graviton multiplet, a massless vector multiplet, and one hypermultiplet with the gauging of a* $\mathrm{U}(1)\times\mathrm{U}(1)$ *symmetry of the scalar manifold* (3.1).

In the decomposition [9] of the $\mathcal{N}=8$ graviton multiplet into $\mathcal{N}=2$ multiplets (which is completely valid only at the infrared fixed point), the supergravity fields are arranged into representations of the $\mathrm{SU}(2,2|1)\times\mathrm{SU}(2)_I$ residual superalgebra. Retaining only $\mathrm{SU}(2)_I$ singlets leaves us with one graviton multiplet, one hypermultiplet, one massive vector multiplet, and one massive gravitino multiplet. However [9], since the only scalars that change along the flow are the two belonging to the massive vector multiplet, it is expected that the theory can be further consistently truncated to one containing the graviton and massive vector multiplets only, but how can we describe such couplings in the standard framework [1]?

The representations of the $\mathrm{SU}(2,2|1)$ supergroup not only include a massless short graviton multiplet and an arbitrary number of (massless) short vector and (massive) tensor and hypermultiplets, but also present a wide spectrum of long and semilong supermultiplets. While in the general theory [1] the couplings and interactions of short multiplets are explicitly described, those of massive vector multiplets can arise as the result of a Higgs mechanism where a massless vector eats a scalar coming from a hypermultiplet.

Since the UV fixed point should correspond to the $\mathcal{N}=8$ supersymmetric theory, both the graviphoton and gauge vectors must be massless there, as they are both gauge vectors of $\mathrm{U}(1)\times\mathrm{U}(1)\subset\mathrm{SU}(4)$. Then, along the flow, only one of them (or at most a combination of the two) will remain massless, while the other will gain a mass, breaking the residual invariance to the $\mathrm{U}(1)_R$ subgroup and giving rise to the massive vector multiplet described above. This means that we can further decompose the long vector multiplet into a massless one plus a hypermultiplet, which is exactly the content of the model we are going to analyze now.

To complete the characterization of the flow we only need some information to sort out which $U(1)\times U(1)$ subgroup of the isometry group of the manifold $\mathcal{M}$ has to be gauged. This can be understood by examining the mechanism that gives the mass to one of the two vectors.

The vector mass terms come from the kinetic terms of the hypermultiplet scalars. Indeed, due to the gauged covariant derivatives (2.1), the kinetic term for such scalars is

$$-\tfrac{1}{2}(\partial_\mu q^X+gA_\mu^I K_I^X)^2, \tag{3.2}$$

and therefore for a U(1) gauging one has a term like $g^2 A_\mu A^\mu K^2$ in the action, where A_μ is a linear combination of gauge vectors and K^X is the corresponding Killing vector of the gauged isometry. This, of course, will act as a mass term for the A_μ vector any time the Killing vector has a nonzero norm.

It is therefore quite easy now to identify the isometries to be gauged in order to obtain the FGPW flow. They are those associated with Killing vectors K_I^X that have vanishing norm at the UV fixed point, and such that along the flow the norm of a combination of them still remains zero.

A. The scalar manifold

We now turn to the description of the parametrization and of the isometries of the scalar manifold (3.1). The O(1,1) factor is relative to the vector multiplet scalar ρ, and is given by a very special manifold characterized by $n_V+1=2$ functions $h^I(\rho)$ constrained to the surface (2.2). Its essential geometric quantities, the C constants that determine the embedding of this manifold in the ambient space and also fix the Chern-Simons coupling, can be chosen to be all but C_{011} equal to zero. Then we take[11] [40]

$$C_{011}=\frac{\sqrt{3}}{2},\quad h^0=\frac{1}{\sqrt{3}}\rho^4,\quad h^1=\sqrt{\frac{2}{3}}\frac{1}{\rho^2},$$

$$g_{\rho\rho}=\frac{12}{\rho^2},\quad a_{00}=\frac{1}{\rho^8},\quad a_{11}=\rho^4,\quad a_{01}=0. \tag{3.3}$$

The metric $g_{\rho\rho}$ is well behaved for $\rho\neq 0$ and, for definiteness, we choose the branch $\rho>0$.

Much more can be said about the second factor of $\mathcal{M}$, and due to its fundamental role we would like to describe it in some more detail. It is known that the quaternionic Kähler space $SU(2,1)/[SU(2)\times U(1)]$, classically parametrized by the universal hypermultiplet, is also a Kähler manifold [41]. This means that it can be derived from a Kähler potential, which is usually taken to be

$$\mathcal{K}=-\tfrac{1}{2}\log(S+\bar{S}-2C\bar{C}). \tag{3.4}$$

In addition to the many parametrizations existing in the literature, the one that will prove convenient to us follows very closely the notations of [7], with some further redefinitions to match the conventions of [1]. We thus call the four hyperscalars $q^X=\{V,\sigma,\theta,\tau\}$, which are related to the previous variables by

$$S=V+(\theta^2+\tau^2)+i\sigma,\quad C=\theta-i\tau. \tag{3.5}$$

The domain of the manifold is covered by $V>0$. Note that this parametrization of the universal hypermultiplet is the one that comes out naturally from the Calabi-Yau compactifications of M theory [42,7] and thus one can hope to explicitly see how gauging of isometries can be obtained from such a higher-dimensional description.

Let us define the following one-forms:

$$u=\frac{d\theta+id\tau}{\sqrt{V}},\quad v=\frac{1}{2V}[dV+i(d\sigma-2\tau d\theta+2\theta d\tau)], \tag{3.6}$$

which will be very useful in the whole construction. The quaternionic vielbeins $f^{iA}=f_X^{iA}dq^X$ are then chosen to be $[\epsilon_{12}=C_{12}=+1$ are the conventions for the $SU(2)\times USp(2)$ metrics]

$$f^{iA}=\begin{pmatrix} u & -v\\ \bar{v} & \bar{u}\end{pmatrix},\quad f_{iA}=\begin{pmatrix} \bar{u} & -\bar{v}\\ v & u\end{pmatrix}. \tag{3.7}$$

The metric is then given by $g=f^{iA}\otimes f_{iA}=2u\otimes\bar{u}+2v\otimes\bar{v}$ and reads

$$ds^2=\frac{dV^2}{2V^2}+\frac{1}{2V^2}(d\sigma+2\theta\,d\tau-2\tau\,d\theta)^2+\frac{2}{V}(d\tau^2+d\theta^2). \tag{3.8}$$

The determinant for such a metric is $1/V^6$ and therefore the metric is positive definite and well behaved for any value of the coordinates except $V=0$. Since in the Calabi-Yau derivation V acquires the meaning of the volume of the Calabi-Yau manifold, we restrict it to the positive branch $V>0$.

From the vielbeins we can derive the SU(2) curvature:

$$\mathcal{R}_i{}^j=-\frac{1}{2}f_{iA}\wedge f^{jA}=-\frac{1}{2}\begin{pmatrix} -(v\bar{v}+u\bar{u}) & 2\overline{uv}\\ 2vu & (v\bar{v}+u\bar{u})\end{pmatrix}. \tag{3.9}$$

Using the triplet of curvatures as in Eq. (2.10),

$$\mathcal{R}^1=-\frac{1}{2V^{3/2}}[(d\sigma+2\theta\,d\tau)d\,\theta-d\tau\,dV],$$

$$\mathcal{R}^2=-\frac{1}{2V^{3/2}}[(d\sigma-2\tau d\theta)d\tau+d\theta dV],$$

[11]The numerical factors are partly chosen for convenience and partly to satisfy the request that there exists a point of the manifold where the metric a_{IJ} can be put in the form of a delta δ_{IJ}.

$$\mathcal{R}^3 = -\frac{1}{V} d\theta d\tau + \frac{1}{4V^2}[(d\sigma - 2\tau d\theta + 2\theta d\tau)dV]. \tag{3.10}$$

These can be derived from the following SU(2) connections:

$$\omega^1 = -\frac{d\tau}{\sqrt{V}}, \quad \omega^2 = \frac{d\theta}{\sqrt{V}}, \quad \omega^3 = -\frac{1}{4V}(d\sigma - 2\tau d\theta + 2\theta d\tau). \tag{3.11}$$

B. The isometries

The metric (3.8) has an SU(2,1) isometry group generated by the following eight Killing vectors k_α^X:

$$\vec{k}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{k}_2 = \begin{pmatrix} 0 \\ 2\theta \\ 0 \\ 1 \end{pmatrix}, \quad \vec{k}_3 = \begin{pmatrix} 0 \\ -2\tau \\ 1 \\ 0 \end{pmatrix}, \quad \vec{k}_4 = \begin{pmatrix} 0 \\ 0 \\ -\tau \\ \theta \end{pmatrix},$$

$$\vec{k}_5 = \begin{pmatrix} V \\ \sigma \\ \theta/2 \\ \tau/2 \end{pmatrix}, \quad \vec{k}_6 = \begin{pmatrix} 2V\sigma \\ \sigma^2 - (V + \theta^2 + \tau^2)^2 \\ \sigma\theta - \tau(V + \theta^2 + \tau^2) \\ \sigma\tau + \theta(V + \theta^2 + \tau^2) \end{pmatrix},$$

$$\vec{k}_7 = \begin{pmatrix} -2V\theta \\ -\sigma\theta + V\tau + \tau(\theta^2 + \tau^2) \\ \frac{1}{2}V - \theta^2/2 + 3\tau^2/2 \\ -2\theta\tau - \sigma/2 \end{pmatrix},$$

$$\vec{k}_8 = \begin{pmatrix} -2V\tau \\ -\sigma\tau - V\theta - \theta(\theta^2 + \tau^2) \\ -2\theta\tau + \sigma/2 \\ \frac{1}{2}V + 3\theta^2/2 - \tau^2/2 \end{pmatrix}. \tag{3.12}$$

The first three correspond to some constant shift of the coordinates; in particular, the first one was analyzed in [7], where $\sigma \to \sigma + c$, whereas the second and third correspond to the shifts $\theta \to \theta + a$, $\tau \to \tau + b$, and $\sigma \to \sigma - 2a\tau + 2b\theta$. The fourth Killing vector is the generator of the rotation symmetry between the θ and τ coordinates: $\theta \to \cos\phi\,\theta - \sin\phi\,\tau$, $\tau \to \sin\phi\,\tau + \cos\phi\,\theta$, which is the one considered in [8]. Finally, the fifth Killing vector is the generator of dilatations while the remaining three are other complicated isometries of the metric.

The commutators of these vectors confirm that they really close the SU(2,1) algebra. To this purpose, it is easier to recast them in the following combinations:

$$SU(2)\begin{cases} T_1 = \frac{1}{4}(k_2 - 2k_8), \\ T_2 = \frac{1}{4}(k_3 - 2k_7), \\ T_3 = \frac{1}{4}(k_1 + k_6 - 3k_4), \end{cases}$$

$$U(1)\left\{ T_8 = \frac{\sqrt{3}}{4}(k_4 + k_1 + k_6), \right.$$

$$\frac{SU(2,1)}{U(2)}\begin{cases} T_4 = ik_5, \\ T_5 = -i\frac{1}{2}(k_1 - k_6), \\ T_6 = -i\frac{1}{4}(k_3 + 2k_7), \\ T_7 = -i\frac{1}{4}(k_2 + 2k_8). \end{cases} \tag{3.13}$$

These generators satisfy the SU(3) commutation relations

$$T_\alpha^Y \partial_Y T_\beta^X - T_\beta^Y \partial_Y T_\alpha^X = -f_{\alpha\beta}{}^\gamma T_\gamma^X. \tag{3.14}$$

The factors of i in Eq. (3.13) allow us to have completely antisymmetric structure constants $f_{\alpha\beta\gamma} = f_{\alpha\beta}{}^\gamma$ with $f_{123} = 1$, $f_{147} = 1/2$, $f_{156} = -1/2$, $f_{246} = 1/2$, $f_{257} = 1/2$, $f_{345} = 1/2$, $f_{367} = -1/2$, $f_{458} = \sqrt{3}/2$, $f_{678} = \sqrt{3}/2$. The generators T_4, T_5, T_6, and T_7 are imaginary, such that the real algebra is SU(2,1).

The relation (2.15) leads directly to the prepotentials

$$\vec{P} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{4V} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{\sqrt{V}} \\ 0 \\ -\frac{\theta}{V} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \frac{1}{\sqrt{V}} \\ \frac{\tau}{V} \end{pmatrix}, \quad \begin{pmatrix} -\frac{\theta}{\sqrt{V}} \\ -\frac{\tau}{\sqrt{V}} \\ \frac{1}{2} - \frac{\theta^2 + \tau^2}{2V} \end{pmatrix},$$

$$\begin{pmatrix} -\frac{\tau}{2\sqrt{V}} \\ \frac{\theta}{2\sqrt{V}} \\ -\frac{\sigma}{4V} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{\sqrt{V}}[\sigma\tau + \theta(-V + \theta^2 + \tau^2)] \\ \frac{1}{\sqrt{V}}[\sigma\theta - \tau(-V + \theta^2 + \tau^2)] \\ -\frac{V}{4} - \frac{1}{4V}[\sigma^2 + (\theta^2 + \tau^2)^2] + \frac{3}{2}(\theta^2 + \tau^2) \end{pmatrix},$$

$$\begin{pmatrix} \dfrac{4\,\theta\tau+\sigma}{2\sqrt{V}} \\ \dfrac{3\,\tau^2-\theta^2}{2\sqrt{V}}-\dfrac{\sqrt{V}}{2} \\ -\dfrac{3}{2}\tau+\dfrac{1}{2V}[\sigma\theta+\tau(\theta^2+\tau^2)] \end{pmatrix}, \quad \begin{pmatrix} -\dfrac{3\,\theta^2-\tau^2}{2\sqrt{V}}+\dfrac{\sqrt{V}}{2} \\ \dfrac{\sigma-4\,\theta\tau}{2\sqrt{V}} \\ \dfrac{3}{2}\theta+\dfrac{1}{2V}[\sigma\tau-\theta(\theta^2+\tau^2)] \end{pmatrix}. \tag{3.15}$$

As we have seen in the previous sections, these prepotentials are indeed necessary in order to write an explicit expression for the potential (2.18) and superpotential (2.21) of the gauged theory and thus to solve the supersymmetry flow equations (2.42).

IV. GAUGING AND THE FLOWS

In this section, we will analyze the flows that can be obtained in this model. This requires, as pointed out in Sec. II, that we perform a specific gauging and search for critical points and for zeros of the corresponding potential.

A. Toy model with only a hypermultiplet

We now turn to analyzing the supersymmetric flows that can be obtained in our model with one vector multiplet and one hypermultiplet. To this end, it is interesting to start with a preliminary study on the vacua obtained by considering only a U(1) gauging of the universal hypermultiplet manifold, when no other vectors but the graviphoton are present. This will give us some important hints.

As for this case $h^0=1$ is the only component of h^I, the conditions for critical points (2.65) just reduce to the vanishing of a certain linear combination K of the Killing vectors that give rise to the gauged isometry. In other words, a critical point should be left invariant under the U(1) generated by K. Therefore, such U(1) must be part of the isotropy group of the manifold,[12] i.e.,

$$\mathrm{U}(1)_{\mathrm{gauge}}\subset \mathrm{SU}(2)\times \mathrm{U}(1). \tag{4.1}$$

In a symmetric space any point is equivalent, and for convenience we have chosen the basis (3.13) such that the generators T_1, T_2, T_3, and T_8 are the isotropy group of the point

$$(V,\sigma,\theta,\tau)=(1,0,0,0). \tag{4.2}$$

The Killing vector that we consider is given by

$$K=\sum_{r=1}^{3}\alpha_r T_r+\beta\frac{1}{\sqrt{3}}T_8, \tag{4.3}$$

[12]This important feature was not realized in any of the revisions of [12], where isometries were gauged outside the compact subgroup.

and the same constant parameters $\{\alpha_r,\beta\}$ are used to define the gauged prepotential P^r. Correspondingly, we find

$$W^2|_{(1,0,0,0)}=\tfrac{1}{6}\alpha_r\alpha_r. \tag{4.4}$$

The value of W is indeed determined only by the gauging of the SU(2) part of the isotropy group of the corresponding point. Therefore, any gauging in the direction of T_8 alone can give rise to a supersymmetric vacuum of the theory corresponding to a Minkowski space. This same feature can also be observed for the gauging of an SU(2)×U(1) group in the $\mathcal{N}=4$ theory, namely, the gauging of the U(1) isometry alone gives rise only to Minkowski vacua [38,36]. This implies that, if a dual field theory could be built, it would be in a confined phase.

The matrix $\mathcal{J}$ in Eq. (2.48) is at the base point (4.2) proportional to α_r, while $\mathcal{L}$ is proportional to β, these being the gaugings in the SU(2) and orthogonal directions, respectively. The eigenvalues of $\mathcal{J}\mathcal{L}$ are twice $\pm\alpha\beta/4$ (where α is the length of the vector α_r) such that the matrix of second derivatives (2.46) according to Eq. (2.47) satisfies

$$\text{eigenvalues } \mathcal{U}_X{}^Y|_{(1,0,0,0)}=\frac{3}{2}\left\{1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha},1-\frac{\beta}{\alpha},1-\frac{\beta}{\alpha}\right\}, \tag{4.5}$$

and this tells us the nature of the critical point. As an example, consider the gauging in the direction of $\alpha=\alpha_3$. Then the first two eigenvalues correspond to the (V,σ) directions, and the latter two are of the (θ,τ) directions.

In general, the obtained eigenvalues reflect the supersymmetry structure of the universal hypermultiplet. We know that the above eigenvalues are related to the masses of the fields and therefore also to their conformal dimension E_0 (actually, given the eigenvalues δ_k, the relation is $E_0=|\delta_k|$ or $E_0=|4-\delta_k|$). For a hypermultiplet two of the scalars must have the same E_0 and two must have E_0+1. In this case this is realized by the fact that two have $E_0=\frac{3}{2}\pm 3\beta/2\alpha$ and two have $E_0=\frac{5}{2}\pm 3\beta/2\alpha$.

The critical point (4.2) appears as isolated whenever $|\beta|\neq\alpha$, where $\alpha>0$ is the length of the vector α_r. If $|\beta|=\alpha$, then there is a two-dimensional plane where $K=0$. More precisely, we have to distinguish three regions of the parameter space $\{\alpha_r,\beta\}$.

(1) If $|\beta|<\alpha$, then the base point is an UV critical point.

(2) If $|\beta|=\alpha$ then there is a plane of critical points. This plane is parametrized by θ and τ with

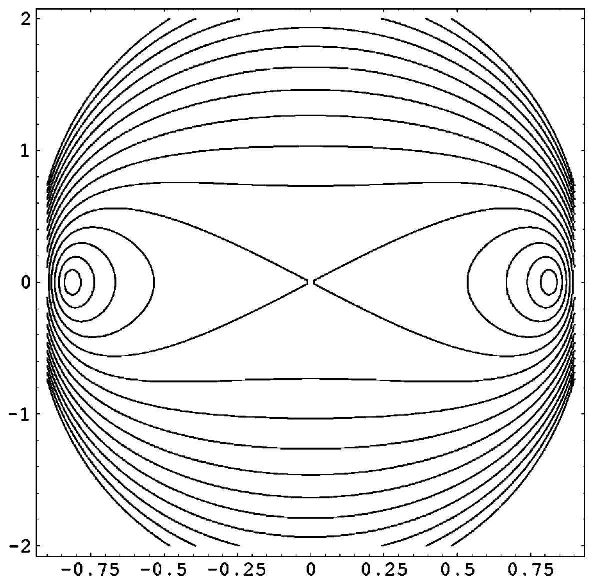

FIG. 1. Contours of constant W in the plane (τ,σ) with $V=1-\tau^2$, $\theta=0$, for $\alpha_1=\alpha_2=0$, $\alpha_3=\sqrt{6}$, and $\beta=2\sqrt{6}$.

$$V=1-\theta^2-\tau^2+\frac{2(\alpha_1\theta-\alpha_2\tau)}{\alpha_3+\beta},$$

$$\sigma=-\frac{2(\alpha_2\theta+\alpha_1\tau)}{\alpha_3+\beta}. \tag{4.6}$$

In the case $\beta=-\alpha_3$ the plane is $\theta=\tau=0$. In the orthogonal directions, W is increasing, i.e., this plane is of the UV type.

(3) If $|\beta|>\alpha$, then W decreases in two directions. The critical point is an IR critical point for flow in these directions. In this case, W also has a line of zeros. For example, for gauging in the direction $\alpha=\alpha_3$ we find the zeros for

$$V=\frac{\beta-\alpha}{\alpha+\beta},\quad \sigma=0,\quad \theta^2+\tau^2=\frac{2\alpha}{\beta+\alpha}\quad \text{for } \beta>\alpha>0. \tag{4.7}$$

The last case shows the aspects of the toy model of the universal hypermultiplet that were not known for vector multiplets only. Let us now choose $\alpha_1=\alpha_2=0$, which we can safely do due to the SU(2) invariance. The superpotential can be written as

$$W=\frac{1}{8\sqrt{6}V}(\{(\alpha+\beta)[1+\sigma^2+(V+\xi^2)^2]+2(3\alpha-\beta)(V-\xi^2)\}^2+128\alpha(\alpha-\beta)\xi^2V)^{1/2}, \tag{4.8}$$

where

$$\xi^2\equiv\theta^2+\tau^2. \tag{4.9}$$

We plot in Fig. 1 contours of constant W, where the base point is in the middle. One sees that W increases in the vertical direction from this point. The central line in the horizontal direction represents the locus

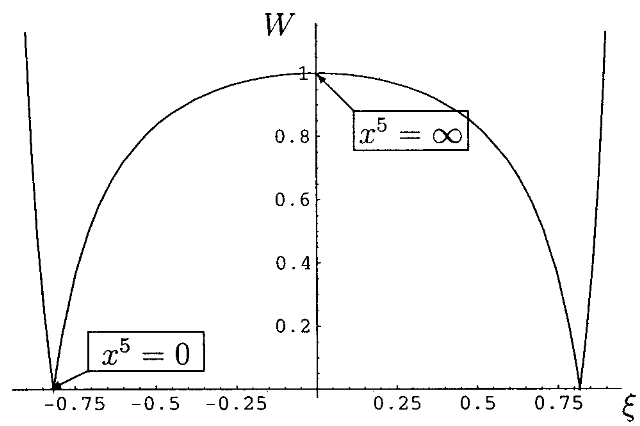

FIG. 2. W as a function of ξ along the line (4.10), for $\alpha_1=\alpha_2=0$, $\alpha_3=\sqrt{6}$, and $\beta=2\sqrt{6}$. Note that the apparent singular point at $\xi=\pm\sqrt{2/3}=0.82$ is in fact just a regular point, as W is the norm of a vector. We indicate the corresponding values of x^5 for the flow.

$$\sigma=0,\quad V+\xi^2=1\quad \text{with}\quad 0<V\leqslant 1\quad \text{or}\quad -1<\xi<1,$$

$$\text{or}\quad V=1-\tanh^2\chi,\quad \xi=\tanh\chi\quad \text{with}\quad -\infty<\chi<\infty. \tag{4.10}$$

In view of Eq. (4.9), this "line" is actually a plane in the full quaternionic manifold, but as we often use the parametrization (4.10) we will call it a line. The parametrization by χ, involving a hyperbolic function, was introduced in [9], where this variable was called φ_1. On this line, the first two components of the Killing vector are zero, and the beta functions of the two field combinations vanish:

$$\beta^{\sigma}=\beta^{V+\xi^2}=0. \tag{4.11}$$

Here, the Killing vector has as the only nonzero components $K^{\theta}=(\alpha-\beta)\tau/2$ and $K^{\tau}=-(\alpha-\beta)\theta/2$. These are indeed never vanishing for $\beta\neq\alpha$ except at the base point (4.2). On this line, the superpotential reduces to

$$W_{\rm line}=\frac{|\alpha(2-\xi^2)-\beta\xi^2|}{2\sqrt{6}(1-\xi^2)}. \tag{4.12}$$

This means that there is a zero for

$$\xi^2=\frac{2\alpha}{\alpha+\beta}, \tag{4.13}$$

which is in the domain of definition if $\beta>\alpha>0$. We plot the superpotential on this line for such a case in Fig. 2, exhibiting the zeros, which are in a circle in the (θ,τ) plane.

We will now consider a flow on that line. The critical point at $\xi=0$ is used for the asymptotic values of the zeros at $x=+\infty$ and the zero of the superpotential is placed at $x^5=0$. The BPS equation (2.31) leads on both sides of the zero to (we put here $g=1$)

$$A'=\frac{\alpha(2-\xi^2)-\beta\xi^2}{2\sqrt{6}(1-\xi^2)}. \tag{4.14}$$

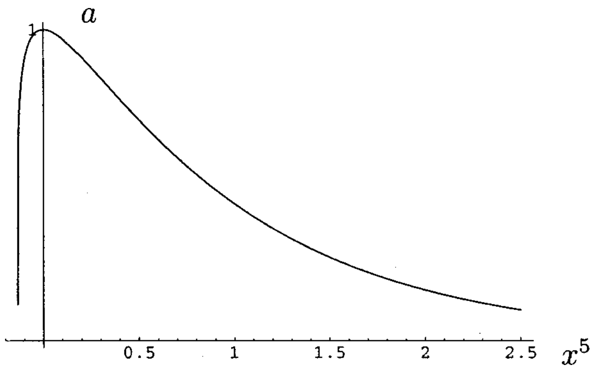

FIG. 3. Graph of the warp factor $a(x^5)=e^A$ for the values $\alpha=\sqrt{6}$ and $\beta=2\sqrt{6}$.

Note that the sign flip has disappeared. For the other BPS equation (2.42), we use the inverse metric to obtain[13]

$$V'=3\partial^V W=\sqrt{\tfrac{3}{2}}(\beta-\alpha)\xi^2=\sqrt{\tfrac{3}{2}}(\beta-\alpha)(1-V), \tag{4.15}$$

where again the sign flipping has disappeared.

Choosing the integration constant such that the zero is $x^5=0$ and that the warp factor reaches the value $a(0)=1$, the solution is given by

$$\xi^2(x^5)=\frac{2\alpha}{\alpha+\beta}\ e^{-\sqrt{3/2}\,(\beta-\alpha)\,x^5},$$

$$A(x^5)=-\frac{\alpha x^5}{\sqrt{6}}+\frac{1}{6}\log\frac{\alpha+\beta-2\alpha e^{-\sqrt{3/2}\,(\beta-\alpha)\,x^5}}{\beta-\alpha}. \tag{4.16}$$

This flow has a singular point at $x^5=-\sqrt{\tfrac{2}{3}}[1/(\beta-\alpha)]\log[(\alpha+\beta)/2\alpha]$, where the border of the quaternionic manifold is indeed reached as $\xi^2=1$. Here $A\to-\infty$. At the other end, one reaches the fixed AdS critical point for $x^5\to+\infty$, where the behavior of the warp factor is $A\to-(\alpha/\sqrt{6})x^5=-W_{\rm cr}x^5$, which is the asymptotic of an IR AdS fixed point.

We display in Fig. 3 the exponential of the warp factor for such a solution, showing that it is perfectly well behaved and continuous at the point $x^5=0$ where it reaches its maximum equal to 1.

As W has no other critical points, one can have zeros only in that case. In [12], where the parametrization of [14] was used, some misleading results were obtained. We discuss this in Appendix C.

These features regarding the case with one hypermultiplet and no vector multiplets can be generalized to arbitrary quaternionic Kähler homogeneous spaces G/[SU(2)×K]. There can only be one connected region with critical points, and the eigenvalues of the Hessian are always spread evenly from 3/2 as in Eq. (4.5). To get IR directions in the critical point the gauging has to be ''mainly'' in the direction of K, while some gauging in the direction SU(2) is necessary for a nonzero W at the critical point. All this will be shown in [30].

[13]Alternatively, one can use that the metric reduced to the line is $ds^2=(2/V^2)(d\tau^2+d\theta^2)$, to obtain, e.g., $\tau'=-(1/2)\sqrt{3/2}(\beta-\alpha)\tau$, also leading to Eq. (4.15).

B. The full model and the FGPW flow

We now consider the full model, with a vector multiplet and a hypermultiplet. Taking into account the graviphoton, we have two vector fields, and therefore we can gauge two Abelian isometries U(1)×U(1). As explained before, we should find the FGPW flow between an UV and an IR fixed point by a specific U(1)×U(1) gauging.

We will start by considering the requirements imposed by the presence of *a first critical point*, for which we will choose the base point

$$\text{c.p. 1:}\quad q=\{1,0,0,0\},\quad \rho=1. \tag{4.17}$$

We must first solve the condition $K_{\rm c.p.1}=(h^0K_0+h^1K_1)_{\rm c.p.1}=0$. As in the toy model and as a general result, this implies that $K_{\rm c.p.1}$ generates a U(1) inside the isotropy group of the scalar manifold. In our specific case, this results in the absence of contributions from the noncompact generators in the combination $K_0+\sqrt{2}K_1$. Furthermore, we can use the SU(2) invariance to choose just one direction in SU(2). We again take the generator T_3.

Since we now also have a vector multiplet scalar, the critical points of the superpotential are defined by both attractor equations (2.65), and one must also satisfy the requirement for the prepotentials $h_IP^r=P^r_I$. Due to the self-consistency of this equation (multiplying it by h^I), this gives only one triplet of requirements:

$$h_0P^r_1=h_1P^r_0\quad \text{at the critical point.} \tag{4.18}$$

Only the generators of the SU(2) part of the isotropy group contribute to the prepotentials, and the three generators have three independent prepotentials. Therefore, this condition does not lead to any constraint on the noncompact generators. However, it does give conditions on the generators in the SU(2) part of the isotropy group that imply the absence of T_1 and T_2 from K_0 as well as K_1. Furthermore, it fixes the relative weight of T_3 in both generators. As a result of the attractor equations at c.p. 1, we can parametrize the generators as

$$K_0=\sqrt{2}\alpha\left(\frac{1}{2}T_3+\frac{1}{\sqrt{3}}\gamma T_8+T_{\rm nc}\right),$$

$$K_1=\alpha\left(T_3+\frac{1}{\sqrt{3}}\beta T_8-T_{\rm nc}\right), \tag{4.19}$$

where $\alpha>0$ and $T_{\rm nc}$ is a linear combination of the noncompact Killing vectors.

In order to obtain two independent U(1)'s we still must impose the requirement that the two generators commute. It can be easily seen that this is equivalent to requiring a vanishing commutator of $T_{\rm nc}$ with the combination $\frac{3}{2}T_3+[(\beta+\gamma)/\sqrt{3}]T_8$. Thus for a general quaternionic manifold the generators whose roots lie in the direction defined by this generator are the ones that can survive, if there are any. Therefore noncompact generators are possible only if $\beta+\gamma=\pm\frac{3}{2}$. We formulated the analysis such that it is suitable for an easy generalization to situations with one vector multiplet and an arbitrary homogeneous quaternionic manifold.

As the prepotentials depend only on the SU(2) part, we find that the value of W at the critical point is proportional to α:

$$W_{\rm c.p.1}=\tfrac{1}{2}\alpha. \tag{4.20}$$

We now turn to analyze the (IR/UV) properties of the critical point, computing the eigenvalues of the matrix (2.47). In the matrices in Eq. (2.48), $\mathcal{J}$ depends only on the gauging in the SU(2) direction, i.e., it is proportional to α, while $\mathcal{L}$ is a function of the U(1) gauging, i.e., it is proportional to $\beta+\gamma$. This implies that in the generic case we find

$$\mathcal{U}_{\rm c.p.1}=\text{diagonal}\{\tfrac{3}{2}+\beta+\gamma,\tfrac{3}{2}+\beta+\gamma,\tfrac{3}{2}-\beta-\gamma,\ \tfrac{3}{2}-\beta-\gamma,\,2\}. \tag{4.21}$$

Thus, noncompact generators can enter when this matrix has zero eigenvalues. From the form of Eq. (2.47), one can see that the noncompact symmetries can contribute only to the off-diagonal elements that are in the direction of the zero modes of the pure hypermultiplet part of $\mathcal{U}$. They therefore modify the part (0,0,2) of Eq. (4.21). The result is that the eigenvalues are then

$$\text{eigenvalues}\ \ \mathcal{U}_{\rm c.p.1}=\{3,3,0,1+\sqrt{1+6a^2},1-\sqrt{1+6a^2}\}, \tag{4.22}$$

where a is the weight with which the noncompact generators appear (e.g., $T_{\rm nc}=aT_4$).

We should remark that the critical point c.p. 1 does not in general preserve the full $U(1)\times U(1)$ gauged symmetry. Indeed, the gauge invariance could be spoiled by the presence of a mass term for the gauge vector coming from the kinetic part of the hyperscalars (3.2). Since the mass is related to the norm of the Killing vectors of the gauged isometries, we see that the invariance is broken anytime $K_I^X K_{IX}\neq 0$. This happens whenever we turn on the noncompact generators. In our case, $T_{\rm nc}\neq 0$ implies that at the c.p. 1 the $U(1)\times U(1)$ isometry is broken to the U(1) generated by $K_0+\sqrt{2}K_1$.

As our present interest is specifically aimed at reproducing the FGPW flow, from now on we restrict consideration to $T_{\rm nc}=0$. This allows the existence of a single point where the full $U(1)\times U(1)$ gauge symmetry is preserved.

We now go back to analyzing the *BPS flows* that can originate from c.p. 1. Remember that with mixed vector and hypermultiplets they have to satisfy the requirement

$$\partial_\rho Q^r=0. \tag{4.23}$$

This is not satisfied for a generic point in the manifold. However, it is satisfied on the line (4.10) and thus we further consider a flow along it. In order for the flow to be consistent with the restriction, the flow equation (2.42) should not drive the fields off the line and this translates into requiring that $\beta^\sigma=\beta^{V+\xi^2}=0$. These conditions are indeed satisfied. Along the line, the quantities K, W, and Q simplify to[14]

$$K^X|_{\rm line}=\frac{\alpha}{\sqrt{6}\rho^2}\left[1-\beta+\rho^6\left(\frac{1}{2}-\gamma\right)\right](0,0,\tau,-\theta),$$

$$\begin{aligned}W|_{\rm line}=&\frac{\alpha}{6\rho^2(1-\xi^2)}\big|(2+\rho^6)(1-\tfrac{1}{2}\xi^2)-(\beta\\&+\gamma\rho^6)\xi^2\big|,\end{aligned} \tag{4.24}$$

$$Q^r|_{\rm line}=\pm(2\theta\sqrt{1-\xi^2},2\tau\sqrt{1-\xi^2},-1+2\xi^2).$$

We now fix the end point of the flow at *a second critical point* and we have again two requirements: the vanishing of the Killing vector, and the requirement (4.18). These now fix the values of ρ and ξ^2 for the critical point.

The value of ρ for the second critical point follows directly from the first of Eqs. (4.24), while we use Eq. (4.18) to fix also ξ^2. Along the line, the prepotentials are given by

$$\begin{pmatrix}P_0^r\\P_1^r\end{pmatrix}\Bigg|_{\rm line}=\frac{\alpha Q^r}{4(1-\xi^2)}\begin{pmatrix}(1/\sqrt{2})(2-\xi^2-2\gamma\xi^2)\\2-\xi^2-\beta\xi^2\end{pmatrix}. \tag{4.25}$$

Note that they now involve a ξ-dependent mixing of the parameters that appear in Eq. (4.19), but are all proportional to the same Q^r. This comes about because, as we mentioned before, the relevant part is the SU(2) part of the isotropy group. This isotropy group rotates within the full SU(2,1) while we move over the line. With only T_3 and T_8 used in our gauging, the effective SU(2) still has its first two components zero. That is why all entries in Eq. (4.25) are proportional to the same matrix Q^r. The third component of the SU(2) part of the isotropy group is a linear combination of T_3 and T_8, leading to the mixture in Eq. (4.25). Using $h_1/h_0=\sqrt{2}\rho^6$, we find for the second critical point[15]

$$\text{c.p. 2:}\ \ q=\{V_{\rm cr},0,\theta_{\rm cr},\tau_{\rm cr}\},\quad \rho=\rho_{\rm cr},\quad \rho_{\rm cr}^6=\frac{2(\beta-1)}{1-2\gamma}, \tag{4.26}$$

$$\xi_{\rm cr}^2=\theta_{\rm cr}^2+\tau_{\rm cr}^2=\frac{2(1-\rho_{\rm cr}^6)}{3\beta-1-2\rho_{\rm cr}^6}=\frac{6-4(\beta+\gamma)}{3-\beta+2\gamma-6\beta\gamma},$$

$$V_{\rm cr}=\frac{3(\beta-1)}{3\beta-1-2\rho_{\rm cr}^6}.$$

[14]The $\pm$ in the expression for Q^r is dependent on whether the expression in W of which we have to take the modulus is positive or negative.

[15]As mentioned above, we always have a full circle of critical points for these values of $\theta_{\rm cr}$ and $\tau_{\rm cr}$.

We find for the value of W at this second critical point

$$W_{\text{c.p.2}}=\frac{\alpha(\beta-2\gamma)}{3(1-2\gamma)\rho_{\text{cr}}^{2}}=\frac{\alpha(2+\rho_{\text{cr}}^{6})}{6\rho_{\text{cr}}^{2}}, \tag{4.27}$$

such that

$$\frac{W_{\text{c.p.1}}^{3}}{W_{\text{c.p.2}}^{3}}=\frac{27\rho_{\text{cr}}^{6}}{(2+\rho_{\text{cr}}^{6})^{3}}=\frac{27(1-\beta)(1-2\gamma)^{2}}{4(2\gamma-\beta)^{3}}. \tag{4.28}$$

The condition that the critical point is in the domain can be written as

$$(\rho_{\text{cr}}-1)(1-\beta)>0. \tag{4.29}$$

Notice that this is exactly the condition that the third and fourth entries of Eq. (4.21) (θ and τ directions) are positive. These are the ones that are relevant when one computes the eigenvalues of the matrix (2.47) restricted to the line, and *excludes the possibility that the first critical point has IR directions along the flow*.

To summarize, we have presented above a two-parameter model, where β and γ are free parameters and α defines an overall normalization related to the AdS radius.

We will now give more details of the models for specific values of the parameters, leading also to the identification of the FGPW potential as a part of our two-parameter model. We will further restrict consideration to the branch $\rho>1$ and thus $\beta<1$. For definiteness we now choose the value of ρ at the second critical point as in [9]:

$$\rho_{\text{cr}}^{6}=2\Rightarrow\gamma=1-\frac{\beta}{2}. \tag{4.30}$$

With this choice, the second critical point is thus at

$$\text{c.p. 2:}\quad q=\{V_{\text{cr}},0,\theta_{\text{cr}},\tau_{\text{cr}}\},\qquad \rho_{\text{cr}}=2^{1/6},$$
$$V_{\text{cr}}=\frac{3(1-\beta)}{5-3\beta},\qquad \xi_{\text{cr}}^{2}=\theta_{\text{cr}}^{2}+\tau_{\text{cr}}^{2}=\frac{2}{5-3\beta}. \tag{4.31}$$

We are therefore left with a one-parameter family of models with two critical points where β fixes the ratios of the gaugings. A second critical point appears only if $\beta<1$.

We can also compute the value of the cosmological constant W^2 at the critical points:

$$W_{\text{c.p.1}}^{2}=\frac{\alpha^{2}}{4},\quad W_{\text{c.p.2}}^{2}=\frac{2^{4/3}}{9}\alpha^{2},\quad \frac{W_{\text{c.p.1}}^{3}}{W_{\text{c.p.2}}^{3}}=\frac{27}{32}, \tag{4.32}$$

the relation that is found also in [9], and which we generalize here for arbitrary β.

Now that we have two critical points we still have to discuss their nature. As we have already excluded the possibility that the first critical point has IR directions along the flow, we are interested in whether these models have interesting applications for renormalization group flows. We are indeed interested in having an UV and an IR critical point such that we can reproduce the FGPW flow.

To understand the nature of such points we have to look again at the Hessian matrices of the superpotential and at their eigenvalues. At the first critical point (4.17), the $\mathcal{U}$ matrix (4.21) reduces to

$$\mathcal{U}_{\text{c.p.1}}=\text{diagonal}\left(\tfrac{5}{2}+\tfrac{1}{2}\beta,\tfrac{5}{2}+\tfrac{1}{2}\beta,\tfrac{1}{2}-\tfrac{1}{2}\beta,\tfrac{1}{2}-\tfrac{1}{2}\beta,2\right). \tag{4.33}$$

The value of the vector scalar sector is still the characteristic one of the very special vector scalar manifold [4,5]. The values in the hypermultiplet sector follow instead the pattern outlined by the formula (2.47). As mentioned already, the third and fourth entries are positive if we demand the existence of the second critical point ($\beta<1$). If, in addition, one satisfies the more stringent constraint $\beta<-5$, then there are two IR directions, which, however, are not along the flow.

At the other critical point the eigenvalues are given by

$$\text{eigenvalues }\ \mathcal{U}_{\text{c.p.2}}=\{0,3,3,1+\tfrac{1}{2}\sqrt{19-9\beta},1-\tfrac{1}{2}\sqrt{19-9\beta}\}, \tag{4.34}$$

no matter which point is chosen on the critical circle. With the limit $\beta<1$ that we already obtained, this implies that there is always one IR direction. If the first critical point has an IR direction $\beta<-5$, then both critical points have an IR direction. This is thus the first example of a model with two IR fixed points. However, the flow along the line that we consider does not connect in these directions. It is difficult to indicate a flow along another line that would connect the two, or to exclude this possibility.

Note that Eq. (4.34) is of the form (4.22), with $a^2=(15-9\beta)/24$. This can be understood as follows. As we stressed before, any point in the manifold is equivalent. Thus what we find at the second critical point should fit in the pattern that we discussed for the first critical point. However, the generators T_3 and T_8 that appear in Eq. (4.19) are not in the isotropy group of any other point of the manifold. Thus, in this second critical point, the generator K_I with $T_{\text{nc}}=0$ should be interpreted as having a part T_3' [giving rise to Eq. (4.25)], a part T_8', and a part $T_{\text{nc}}'\neq 0$. Therefore this second critical point should fall in the general analysis for a critical point with gaugings outside the isotropy group, i.e., the eigenvalues should be of the form (4.22) where a measures the amount in which the generators T_3 and T_8 contribute to T_{nc}'. This principle could be used for an alternative, group-theoretical, analysis of the possibilities for the second critical point.

The contribution of $a^2=(15-9\beta)/4$ to the eigenvalues arises in $\mathcal{U}$ by the mixing of the scalars of the vector multiplet and hypermultiplets.

This means that when vector multiplets are added to hypermultiplets, the appearance of IR directions can be due to two different mechanisms. One is the presence of the hypermultiplets, which allows values as in Eq. (4.22), which would lead here to $(3,3,1,1,0)$. The other is due to the possible breaking of the gauge symmetry, which occurs if their generators are outside the isotropy group of the critical point, such that K_I^Z in the off-diagonal elements of Eq. (2.47) is nonvanishing. This shifts the (1,1) eigenvalues to the values in Eq. (4.34).

The resulting form of the superpotential is, using the parametrization in Eq. (4.10),

$$W=\frac{\alpha}{4\rho^2}\left[1+\frac{1}{3}\beta+\frac{\rho^6}{6}(5-\beta)-\frac{(1-\beta)}{6}(\rho^6-2)\cosh(2\chi)\right]. \tag{4.35}$$

It is striking that for the choice $\alpha=3$ and $\beta=-1$ one can retrieve the superpotential[16] presented in [9]

$$W=-\frac{1}{4\rho^2}[(\rho^6-2)\cosh(2\chi)-(2+3\rho^6)], \tag{4.36}$$

and therefore also the flow. For this value of β we have indeed one IR and one UV critical point:

$$\text{UV:}\quad \text{CP1:}\ \ \rho=1,\quad \xi=0,\qquad (\chi=0),$$

$$\text{IR:}\quad \text{CP2:}\ \ \rho=2^{1/6},\quad \xi=\pm\tfrac{1}{2},\quad (\chi=\pm\tfrac{1}{2}\log 3), \tag{4.37}$$

and the value of W at these critical points is $W_{\text{c.p.1}}=\frac{3}{2}$ for the UV and $W_{\text{c.p.2}}=2^{2/3}$ for the IR, both representing AdS vacua.

C. FGPW flow in $\mathcal{N}=2$ theory

Let us now give a closer inspection to the FGPW model embedded in $\mathcal{N}=2$ theory. In terms of the scalar manifold isometries, the FGPW flow can be retrieved by gauging those generated by

$$K_{(0)}\equiv\frac{3}{\sqrt{2}}(k_1+k_6)=\frac{3}{\sqrt{2}}(T_3+\sqrt{3}T_8) \tag{4.38}$$

and by

$$K_{(1)}\equiv-3\,k_4=\sqrt{3}(\sqrt{3}T_3-T_8). \tag{4.39}$$

The corresponding superpotential (in terms of all the coordinates) depends on θ and τ via the combination ξ in Eq. (4.9) and is

$$\begin{aligned}W=-\frac{1}{4\,\rho^2\,V}\{&16\,(V+\xi^2)^2+2\sigma^2\,[1+(V+\xi^2)^2]+\rho^{12}[\sigma^4\\&+V^4+4V^3\,\xi^2+(1+\xi^4)^2+2V^2\,(1+3\xi^4)]\\&+8\rho^6\,(V-\xi^2)\,[1+\sigma^2+(V+\xi^2)^2]\\&+4V[\xi^6-3\xi^2]\}^{1/2}.\end{aligned} \tag{4.40}$$

[16]In [9] W was chosen to be the definition $W=-|W|$ rather than the one given in Eq. (2.21), and thus differs by a sign from ours. To compare our flow equations with those of [9], we have to take our coupling constant to be 2/3 rather than 2, as in [9].

It is therefore more convenient to study its stationary points in the variables $\{V,\sigma,\xi,\rho\}$.

A general analysis of the Eq. (4.40) gives indeed only the two expected critical points at

$$\text{c.p. 1:}\ \ \sigma=\xi^2=0,\quad V=1,\quad \rho\equiv e^{\phi_3/\sqrt{6}}=1 \tag{4.41}$$

and at

$$\text{c.p. 2:}\ \ \sigma=0,\quad V=\tfrac{3}{4},\quad \rho=2^{1/6},\quad \xi^2=\tfrac{1}{4}. \tag{4.42}$$

The first is the expected isolated UV point. As explained above, we can identify it as an UV fixed point by considering the leading contributions to the β function of the couplings $\phi=\{V,\sigma,\theta,\tau;\rho\}$ that are encoded in the eigenvalues of the matrix (2.46),

$$\mathcal{U}_{\text{c.p.1}}=\begin{pmatrix}2&&&&\\&2&&&\\&&1&&\\&&&1&\\&&&&2\end{pmatrix}, \tag{4.43}$$

which are all positive.

The second critical point (4.42) actually represents a whole circle of saddle points. We show here the explicit form of the Hessian matrix at the point $\theta=\frac{1}{2}$ and $\tau=0$,

$$\mathcal{U}_{\text{c.p.2}}=\begin{pmatrix}\frac{9}{4}&&\frac{3}{4}&&2^{1/6}/3\\&3&&\frac{3}{4}&\\\frac{9}{4}&&\frac{3}{4}&&-2^{1/6}\\&&&0&\\9/2^{7/6}&&-9/2^{7/6}&&2\end{pmatrix}, \tag{4.44}$$

whose eigenvalues are $0,3,3,(1+\sqrt{7})$ and $(1-\sqrt{7})$, which is Eq. (4.22) with $a=1$.

As foreseen above, here the mechanism that determines the appearance of an infrared direction is different from the one shown for the hypermultiplets alone. In this case indeed the negative eigenvalue comes from the mixed partial derivatives with respect to the hyperscalars V, θ and the vector multiplet scalar ρ. The null eigenvalue is related to the massless Goldstone scalar that is eaten by the vector combination which becomes massive at this point.

We also want to point out here that the presence of a whole line of critical points should be connected to the fact that in the dual CFT one expects to have a line of exactly marginal perturbations [43] to the theory at such an IR point.

After identification of the correct UV and IR end points, we now turn to the second important guideline for the identification of the FGPW discussed in Sec. III, which was related to the mass terms for the gauged vector fields, and thus to the norm of the gauged Killing vectors. In the FGPW example, at the UV fixed point both the graviphoton and the gauge vector are massless, whereas at the IR point only the graviphoton is still massless and is the gauge vector of the residual $U(1)_R$ symmetry.

In order to translate these facts into our present language, we observe that all along the flow where $\sigma=0$ and $V+\xi^2=1$, the two Killing vectors $K_{(0)}$ and $K_{(1)}$ are proportional to one another and are equal to

$$K_{(1)}=-\sqrt{2}\,K_{(0)}=3\begin{pmatrix}0\\0\\\tau\\-\theta\end{pmatrix}, \tag{4.45}$$

and this translates the statement that along the flow the combination $V+\theta^2+\tau^2$ and the σ field should remain constant. It is straightforward to see that $\delta_{K_{(0)}}(V+\theta^2+\tau^2)=\delta_{K_{(0)}}\sigma=0$ and the same for $K_{(1)}$.

Equation (4.45) then allows us to identify the graviphoton with the gauge vector of the $U(1)_R\subset U(1)\times U(1)\subset SU(2)\times U(1)$ symmetry generated by

$$K_R\equiv\sqrt{2}\,K_{(0)}+K_{(1)}\quad(=0\ \text{ along the flow}). \tag{4.46}$$

Let us then analyze the relevant supersymmetry transformations at the IR and UV fixed points in order to identify which mixture of the vector fields $\mathcal{A}^I_\mu$ gives rise to the graviphoton and to the extra vector field.

We find that upon defining

$$\mathcal{A}_\mu\equiv\frac{2^{1/3}}{\sqrt{6}}\left(\frac{A^0_\mu}{\sqrt{2}}+2\,A^1_\mu\right), \tag{4.47}$$

$$\mathcal{B}_\mu\equiv\frac{2^{1/3}}{\sqrt{6}}(\sqrt{2}A^1_\mu-A^0_\mu), \tag{4.48}$$

the SUSY transformations at the IR point reduce to (at leading order in the Fermi fields)

$$\delta_\epsilon\,\psi_{\mu\,i}=D_\mu\epsilon_i+\frac{i}{4\sqrt{6}}(\gamma_{\mu\nu\rho}\epsilon_i-4g_{\mu\nu}\gamma_\rho\epsilon_i)\,\mathcal{F}^{\nu\rho}+\cdots, \tag{4.49}$$

$$\delta_\epsilon\mathcal{A}_\mu=i\,\frac{\sqrt{6}}{4}\overline{\psi}^i_\mu\epsilon_i\,,$$

$$\delta_\epsilon\mathcal{B}_\mu=-\frac{1}{2}\overline{\epsilon}^i\gamma_\mu\lambda_i\,, \tag{4.50}$$

where we have also defined $\mathcal{F}=d\mathcal{A}$. Therefore, by these equations one identifies the graviphoton field with the $\mathcal{A}_\mu$ combination, and the vector at the head of the massive vector multiplet with $\mathcal{B}_\mu$.

At this IR point, for the mechanism we showed, the true massive vector $\mathcal{B}_\mu$ is given by an appropriate sum of Eq. (4.48) and $D_\mu q$, where q is the right combination of the hyperscalars that acts as a Goldstone boson. Therefore the full supersymmetry transformation rule for $\mathcal{B}_\mu$ will also contain a term of the type $D_\mu(\delta_\epsilon q)$.

If we associate with the graviphoton its Killing vector proportional to K_R, we find that the $U(1)_R$ symmetry gauged by the graviphoton is generated by

$$K_{\mathcal{A}}=\frac{2^{2/3}}{\sqrt{6}}K_R\,, \tag{4.51}$$

while the generator of the (broken) $U(1)_{\mathcal{B}}$ isometry associated with the massive vector is given by

$$K_{\mathcal{B}}=\frac{2^{2/3}}{\sqrt{6}}\left(\frac{1}{\sqrt{2}}K_{(1)}-2K_{(0)}\right). \tag{4.52}$$

At the UV fixed point one can again rewrite the supersymmetry rules as in Eqs. (4.49) and (4.50), provided that now one makes the identifications

$$\mathcal{A}_\mu=\frac{1}{\sqrt{3}}(A^0_\mu+\sqrt{2}A^1_\mu), \tag{4.53}$$

$$\mathcal{B}_\mu=\frac{1}{\sqrt{3}}(A^1_\mu-\sqrt{2}A^0_\mu). \tag{4.54}$$

Again, the graviphoton field is identified with the $\mathcal{A}_\mu$ combination, and the gauge vector with $\mathcal{B}_\mu$.

If we relate to these vectors the Killing generators of the U(1) isometries that they gauge [we remark that at this point they are both massless and that both are gauge vectors of U(1) isometries], one can see that the graviphoton gauges the U(1) generated by

$$K_{\mathcal{A}}=\frac{1}{\sqrt{3}}(K_{(0)}+\sqrt{2}K_{(1)}), \tag{4.55}$$

whereas the massless vector gauges that generated by

$$K_{\mathcal{B}}=\frac{1}{\sqrt{3}}(K_{(1)}-\sqrt{2}K_{(0)}). \tag{4.56}$$

This means that along the flow the R symmetry gauged by the graviphoton has rotated.

The interpretation of this fact in terms of the dual CFT [9] is the following. When one adds a mass term to the CFT at the UV point, the R current connected to the graviphoton becomes anomalous (i.e., the original graviphoton acquires a mass along the flow) whereas the nonanomalous one is the combination of the original graviphoton and the other gauge vector that keeps massless. As stated in [8], this latter couples to the Konishi current, and therefore the one that couples to the anomaly-free $U(1)_R$ current must be a combination of the original graviphoton and this latter.

In detail, at the UV fixed point, both $K^{UV}_{\mathcal{A}}$ and $K^{UV}_{\mathcal{B}}$ are equal to 0, but, as we move away from it, θ and (or) τ will vary and then they will no longer be 0. For the sake of simplicity, choosing the $\theta=\frac{1}{2}$ and $\tau=0$ point between the IR points (4.42), we can also keep $\tau=0$ for all the flow (indeed, for such conditions $\beta^{\tau}=0$ also, no matter what the θ, V, and ρ values) and parametrize the other variables as $\theta=\xi$ and $V=1-\xi^2$. Then the Killing vector of the broken $\mathrm{U}(1)_{\mathcal{B}}$ isometry will be parametrized by

$$K_{\mathcal{B}}=-3\sqrt{3}\,2^{-1/3}\begin{pmatrix}0\\0\\0\\\xi\end{pmatrix}, \tag{4.57}$$

and this will therefore give rise to a mass term proportional to $\frac{1}{2}m_{\mathcal{B}}^2=\frac{1}{2}g_{XY}K_{\mathcal{B}}^XK_{\mathcal{B}}^Y$ for the $\mathcal{B}_\mu$ vector field (with kinetic term $-\frac{1}{4}F_{\mathcal{B}}^2$):

$$m_{\mathcal{B}}^2=27\times 2^{1/3}g^2\frac{\xi^2}{(1-\xi^2)^2}. \tag{4.58}$$

This is precisely 0 at the UV fixed point and flows to $6\times 2^{4/3}g^2$ at the IR. This means also that at this point its conformal dimension is given by

$$E_0=2+\sqrt{1+\frac{m^2}{W_{\mathrm{c.p.2}}^2}}=2+\sqrt{7}, \tag{4.59}$$

which was the expected one for this massive vector [9].

An interesting example regarding flows in the theory with two hypermultiplets spanning the $\mathrm{G}_{2,2}/[\mathrm{SU}(2)\times\mathrm{SU}(2)]$ manifold has recently been investigated in [44], where, using $\mathcal{N}=8$ supergravity, it has been shown that the effective mass term of the vector fields reduces to the second derivative of the warp factor. Although Eq. (4.58) does not comply with this request, we still have to take into account the contribution coming from the noncanonical normalization of the kinetic term. In fact, we have seen in the previous section that the fields that have the interpretation of graviphoton and (gauge) vector rotate along the flow. Thus, one can still expect the equation $m^2\sim A''$ to arise upon performing some nontrivial field redefinitions.

Presently, the vector field kinetic term has in front a function of the vector modulus ρ. This signals the mixing between the two vectors to be resolved. One hopes that a suitable rescaling of the vector fields yielding the standard normalization of the kinetic terms will also do the job of disentangling this mixing and of giving the correct relation between the effective mass function and the warp factor.

V. DISCUSSION AND OUTLOOK

Generic gauged supergravities were not considered relevant from the perspective of a string theory in a flat background until very recently. Indeed, due to the discovery of the AdS-CFT correspondence, the $\mathrm{AdS}_5\times\mathrm{X}^5$ background of string theory and the gauged supergravity in 5D came into the spotlight both in the maximally supersymmetric ($X^5=S^5$) and in the lower supersymmetric ($X^5=T^{11}$) cases [45].

The latter will be useful in trying to better identify the supergravity details that allow one to select precisely the dual conformal field theory operators. This will continue the analysis of the previous section along the lines of [44].

This paper has uncovered the properties of a general class of 5D $\mathcal{N}=2$ gauged supergravities, which have a rich structure of vacua and interpolating flows. As a first general result, we have specified the set of conditions under which supergravity models coupled to both vector and hypermultiplets, with Abelian and non-Abelian gauging, give rise to $\mathcal{N}=1$ BPS domain walls connecting different critical points. As more specific results, we performed a systematic study of U(1) gaugings of the toy model with the universal hypermultiplet as well as a thorough analysis of a simple model with one vector multiplet and one hypermultiplet. We studied a family of $\mathcal{N}=2$ supergravity potentials with nontrivial vacua that are parametrized by two real numbers. As another interesting result, this model is found to produce, for $\beta=-1$ and $\gamma=3/2$, an $\mathcal{N}=2$ description for the kink solution of [9] previously known within the $\mathcal{N}=8$ theory, and therefore offers a two-parameter generalization of this case. The dual gauge field theory side of the models with arbitrary β and γ is certainly worthy of investigation.

It will be quite natural to apply our apparatus for the search for flows and critical points in more complicated examples. The first one is a simple model with no vector but two hypermultiplets [46], which has been shown to lead to the RG flow proposed by Girardello, Petrini, Porrati, and Zaffaroni [32]. The second kind of example is given by models where non-Abelian gaugings can be explicitly performed and thus need coupling to more vector multiplets. Another line of investigation concerns the realization of models dual to flows from conformal to nonconformal field theories that would need the presence of both AdS and Minkowski vacua simultaneously in the same model.

A further class of models is the one possessing diverse IR fixed points. These are aimed at improving the understanding of a possible supersymmetric realization of the smooth Randall-Sundrum scenario. Regarding this subject, our work has come to suggest the following picture. In the presence of hypermultiplets and vector multiplets 5D $\mathcal{N}=2$ gauged supergravities may have IR AdS_5 fixed points which eliminate the first reason for the no-go theorem proved in [4,5] for vector multiplets where only UV critical points exist.

We also found that the interpolation between two IR fixed points (if such examples are found in the future) for the smooth solution must proceed through the point where the W superpotential vanishes. As emphasized in our discussion, this would not disagree with the monotonicity theorem for the warp factor $A''\leq 0$. This led us to conclude that a smooth RSII scenario can take place in the presence of vectors and hypermultiplets.

On the other hand, the conjectured holographic c theorem is violated since the c function $c\sim W^{-3}$ blows up at $W=0$. This poses some problems for the validity of the AdS-CFT correspondence at such points. However, the general understanding[17] is that at the vanishing points of W the physics may not be captured by field theory but by supergravity, and therefore the violation of the holographic c theorem signals that gravity near the wall cannot be replaced by field theory. This obviously does not happen at $|x^5|\to\infty$ where the c theorem is expected to be valid and a dual field theory is well defined.

In this paper we found the general rules for critical points and zeros of the superpotential. In more general models, possibly with the use of other scalar manifolds and different gaugings, one may try to find a smooth supersymmetric domain wall solution of the RS type.

If the search for a smooth RS scenario remains open, an alternative strategy would be to introduce some brane sources as in [20]. This procedure is expected to be quite straightforward as it will require extending the supersymmetric brane action in a theory with vector multiplets to include also the hypermultiplets. A more difficult step would be to find the natural mechanism for appearance of such brane sources in string theory with $O8$-planes and $D8$-branes, stabilized moduli, and additional fluxes along the lines suggested in [47].

One could also explore how much of our analysis of 5D can be exported to 4D and 6D, where the geometry described by hypermultiplets is still quaternionic and where a lot of work has been done for $\mathcal{N}=2$ supergravity coupled with vector multiplets only.

Another issue that would be very interesting to discuss is the 11-dimensional origin of the theory at hand. It is indeed known that five-dimensional supergravity with gauging of the U(1) isometry generated by k_1 can be obtained from M-theory compactifications on Calabi-Yau manifolds in the presence of G fluxes [7,48].

The same question could be addressed for the new gaugings proposed in this paper. At first sight their higher dimensional origin seems quite mysterious, as other isometries are involved, in addition to the shift in the σ scalar field that was discussed in [7,48].

A related issue is how much of our analysis of domain walls and supersymmetric vacua may survive in the exact M or string theory rather than in classical 5D supergravity. The experience with supersymmetric black hole attractors and quantization of charges suggests the following possibility. Our domain wall solutions interpolating between supersymmetric vacua may be hoped to be exact solutions of quantum theory at most for a restricted values of gauging parameters. At the level of 5D classical supergravity these parameters may take arbitrary continuous values. Classically, there are no restrictions on the parameters of our model. Originally, before gauging, they are just parameters of global symmetries of ungauged supergravity. These symmetries may well be broken by quantum effects like instantons. Therefore it would be inconsistent in the presence of quantum corrections to perform the gauging for continuous gauging parameters. Only for discrete values of the parameters do we expect the solutions to be valid when account is taken of quantum corrections. It is likely that the clarification of the 11D origin of the 5D models, taking into account anomalies, fluxes, and quantized charges of M-branes, will shed some light on breaking of continuous symmetries of gauged supergravities to their discrete subgroups. In such case the method developed here may provide exact supersymmetric vacua of M or string theory.

[17]This argument was suggested in discussions with both M. Porrati and L. Susskind.

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APPENDIX A: INDICES

We used in this paper the following indices to describe n vector multiplets and r hypermultiplets:

μ	$0,\ldots,3,5$	local spacetime
$\underline{\mu}$	$0,\ldots,3$	4D local spacetime
i	1,2	SU(2) doublets
r	1,2,3	SU(2) triplets
I	$0,\ldots,n$	vectors
x	$1,\ldots,n$	scalars in vector multiplets
A	$1,\ldots,2r$	symplectic index for hypermultiplets
X	$1,\ldots,4r$	scalars in hypermultiplets
Λ	$1,\ldots,n+4r$	all scalars
α	$1,\ldots,8$	SU(2,1) isometries (A1)

APPENDIX B: REALITY CONDITIONS AND SU(2) NOTATIONS

We first repeat that SU(2) doublet indices $i,j,\ldots$ are raised or lowered using the NW-SE convention by $\varepsilon_{ij}=\varepsilon^{ij}$, with $\varepsilon_{12}=1$. The same applies to the $\mathrm{Sp}(2r)$ indices $A,B,\ldots$, where a constant antisymmetric matrix C_{AB} is used, satisfying $C_{AB}C^{CB}=\delta_A{}^C$, with $C^{AB}=(C_{AB})^*$. By a redefinition, this matrix can be brought into the standard form

$\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. These matrices also enter into reality conditions. Reality can be replaced by ‘‘charge conjugation.’’ The charge conjugation of a scalar [a scalar also in spinor space, an SU(2) scalar as well as an Sp($2r$) scalar] is just its complex conjugate. Charge conjugation does not change the order of spinors. For a symplectic Majorana spinor, the charge conjugate is equal to the spinor itself. However, for a bispinor, one has to introduce a minus sign. Thus, e.g., for Majorana spinors $(\bar{\lambda}\xi)^* = (\bar{\lambda}\xi)^C = -(\bar{\lambda})^C \xi^C = -\bar{\lambda}\xi$.

Gamma matrices are ‘‘imaginary’’ under this charge conjugation: $\gamma_a^C = -\gamma_a$. For any object that has SU(2) indices or Sp($2r$) indices, the definition of charge conjugation uses the symplectic metric $(V_i)^C = \varepsilon_{ij}(V_j)^* = (V^i)^*$ and $(V^i)^C = \varepsilon^{ij}(V^j)^* = -(V_i)^*$ or, similarly, $(V_A)^C = C_{AB}(V_B)^*$. All the quantities that we introduce in the text are real with respect to this charge conjugation, e.g.,

$$f_{iA}^X = (f_{iA}^X)^C = \varepsilon_{ij} C_{AB} (f_{jB}^X)^*. \tag{B1}$$

Symmetric matrices in SU(2) space can be expanded in three components as

$$R_{(ij)} = iR^r(\sigma_r)_{ij} \quad \text{or} \quad R^r = \tfrac{1}{2} i R_{(ij)}(\sigma^r)^{ij}. \tag{B2}$$

Invariance of R_{ij} under charge conjugation translates into reality of R^r. The usual σ matrices are $(\sigma^r)_i{}^j$, and $(\sigma_r)_{ij}$ is defined from the NW-SE contraction convention: $(\sigma^r)_{ij} \equiv (\sigma^r)_i{}^k \varepsilon_{kj}$. This leads, e.g., to $R_{ij}R^{ij} = 2R^rR^r$.

APPENDIX C: TOY MODEL IN ANOTHER PARAMETRIZATION

The manifold SU(2,1)/[SU(2)×U(1)] can be viewed as an open ball in real four-dimensional space. Written in complex coordinates z_1 and z_2, the domain is $|z_1|^2 + |z_2|^2 < 1$. A useful parametrization has been introduced in [14], and used in [12] to discuss the toy model that we treated in Sec. IV A. The variables z_1 and z_2 are written as functions of variables r, θ, φ, ψ as[18]

$$z_1 = r(\cos\tfrac{1}{2}\theta)e^{i(\psi+\varphi)/2}, \quad z_2 = r(\sin\tfrac{1}{2}\theta)e^{i(\psi-\varphi)/2}. \tag{C1}$$

The manifold is covered by

$$0 \le r < 1, \quad 0 \le \theta < \pi, \quad 0 \le \varphi, \psi < 2\pi. \tag{C2}$$

The determinant of the metric is

$$\det g = \frac{r^6 \sin^2\theta}{4(1-r^2)^6}. \tag{C3}$$

Thus in this parametrization the metric is singular in $r=0$ and for $\theta=0$. These belong to the manifold, and thus need special care.

[18]The relation to the variables in Sec. III is $z_1 = (1-S)/(1+S)$ and $z_2 = 2C/(1+S)$.

In this parametrization, the SU(2) (parameters Λ^r) and U(1) (parameter Λ^4) isometries that vanish at the origin take a simple form on the z variables:

$$\delta\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} = \begin{pmatrix} -K_{r1} \\ K_{r2} \end{pmatrix}\Lambda^r = \tfrac{1}{2} i[(\sigma_r)\Lambda^r + \mathbb{1}_2\Lambda^4]\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix}. \tag{C4}$$

We gauge with $K = \alpha_r K_r + \beta K_4$. Apart from the critical point at the origin, vanishing Killing vectors occur only if there is a zero mode of the determinant of transformations, i.e., if $|\alpha| = \beta$. We find two equations:

$$\begin{aligned} &(\alpha_3+\beta)z_1 - (\alpha_1 - i\alpha_2)z_2 = 0,\\ &(\alpha_1 + i\alpha_2)z_1 + (\alpha_3 - \beta)z_2 = 0. \end{aligned} \tag{C5}$$

One of the two defines the (real) two-dimensional plane of critical points, and then the other is automatically satisfied if $|\alpha| = \beta$. In terms of the angular coordinates, the critical line is at

$$\begin{aligned} &e^{i\varphi}\cot\tfrac{1}{2}\theta = \frac{\alpha_1 + i\alpha_2}{\alpha_3 + \beta}, \quad e^{2i\varphi} = \frac{\alpha_1 - i\alpha_2}{\alpha_1 + i\alpha_2},\\ &\cot^2\tfrac{1}{2}\theta = \frac{(\alpha_1)^2 + (\alpha_2)^2}{(\alpha_3+\beta)^2}. \end{aligned} \tag{C6}$$

Although there is clearly no difference in the choice of the direction in SU(2) space, the choice of angular coordinates makes the gauging in the direction α_3 difficult. For example, the Killing vectors in the angular coordinates are

$$\begin{aligned} &K_1 = (0, -\sin\varphi, -\cos\varphi\cot\theta, \cos\varphi\sin^{-1}\theta),\\ &K_2 = (0, -\cos\varphi, \sin\varphi\cot\theta, -\sin\varphi\sin^{-1}\theta),\\ &K_3 = (0,0,-1,0),\\ &K_4 = (0,0,0,-1). \end{aligned} \tag{C7}$$

This gives the impression that $\alpha_3 K_3 + \beta K_4$ never vanishes, not even at $r=0$. However, in this case, the two combinations of z mentioned above are $z_1 = 0$ and $z_2 = 0$. The latter is the line $\theta=0$ where the parametrization degenerates. Note that these singularities are coordinate singularities. There is nothing generically different for gauging in the direction ‘‘3,’’ as this direction is equivalent to the others in the symmetric space. The different features that are mentioned in [12] are artifacts of the parametrization, which is singular at $r=0$ and at $\theta=0$. It is precisely at $\theta=0$ that these authors obtain different results from ours.

To avoid the singularities, and for showing the main features, we will further concentrate on gauging in direction ‘‘1’’ for the SU(2) and the U(1) direction; thus

$$\alpha_2 = \alpha_3 = 0, \quad \alpha_1\beta > 0. \tag{C8}$$

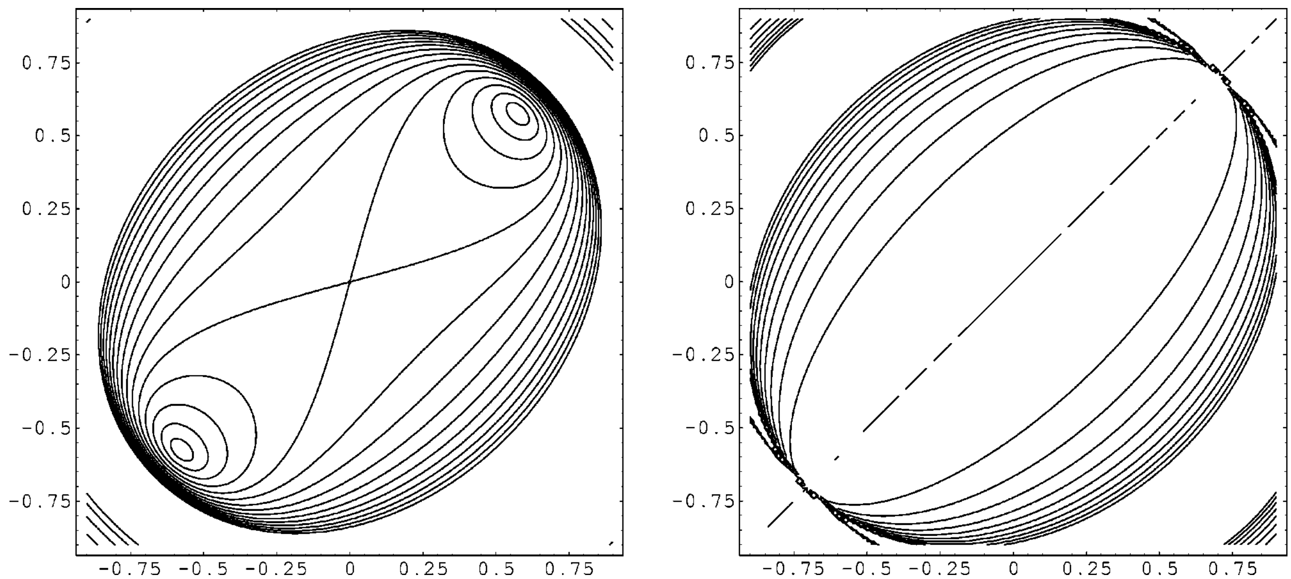

FIG. 4. Contours of constant W in the plane $(\mathrm{Re}\, z_1, \mathrm{Re}\, z_2)$ for $\alpha_1=\sqrt{3/2}$ and $\beta=2\sqrt{3/2}$ (left) and for $\alpha_1=\beta=\sqrt{3/2}$ (right).

With the latter choice, the critical line is at $\varphi=0$, $\theta=\pi/2$, or $z_1=z_2$. The zeros that we mentioned in Sec. IV A occur now for

$$\varphi=0, \quad \theta=\tfrac{1}{2}\pi, \quad r^2=\frac{2\alpha_1}{\alpha_1+\beta}. \tag{C9}$$

This point is only part of the domain if $\beta>\alpha_1$. The Killing vector is nonzero at such points. In this case, the nonzero component is

$$K_\psi=\beta-\alpha_1. \tag{C10}$$

The total prepotential can be written as

$$W^2=\frac{(\beta r^2)^2+2\alpha_1\beta(r^2-2)\zeta+(\alpha_1)^2(4-4r^2+\zeta^2)}{6(1-r^2)^2}, \tag{C11}$$

where we use

$$\zeta\equiv r^2\sin\theta\cos\varphi=z_1\bar{z}_2+\bar{z}_1 z_2. \tag{C12}$$

As W depends just on two real parameters, we can plot it in the plane for real z_1 and z_2 to see the whole picture. This leads to the contour plot in Fig. 4 for a typical case $\beta>\alpha_1$ (left figure). Observe that it is similar to Fig. 1, which represented the gauging in direction "3" in the other representation. The crucial line is the diagonal and along this line the potential is again the one of Fig. 2. We also clearly see that the line $\theta=0$ (horizontal line in the graph) does not have any special properties. The critical points that were found there in [12] came out of the analysis only due to the singular nature of the parametrization.

For the case $\beta=\alpha_1$ we have

$$W=\sqrt{\frac{2}{3}}\left|\beta\frac{1-\frac{1}{2}|z_1+z_2|^2}{1-|z_1|^2-|z_2|^2}\right|. \tag{C13}$$

The potential is then constant on the line $z_1=z_2$, and we have the right plot in Fig. 4. The culmination points of the lines are at $r=1$, i.e., they do not belong to the manifold.

This establishes the equivalence of the two parametrizations. In particular, only one critical point or connected set of critical points is possible.

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