Evolution of cosmological perturbations in nonsingular string cosmologies

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In a class of nonsingular cosmologies derived from higher-order corrections to the low-energy bosonic string action, we derive evolution equations for the most general cosmological scalar, vector, and tensor perturbations. In the large scale limit, the evolutions of both scalar and tensor perturbations are characterized by conserved quantities, the usual curvature perturbation in the uniform-field gauge, and the tensor-type perturbed metric. The vector perturbation is not affected, being described by the conservation of the angular momentum of the fluid component in the absence of any additional dissipative process. For the scalar- and tensor-type perturbations, we show how, given a background evolution during kinetic driven inflation of the dilaton field, we can obtain the final power spectra generated from the vacuum quantum fluctuations of the metric and the dilaton field during the inflation.

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I. INTRODUCTION

Recent observations [1] of anisotropies in the cosmic microwave background radiation strongly support the paradigm of inflation as being the source of the primordial density fluctuations, producing an almost flat power spectrum [2]. The usual inflation models involve an accelerated period being driven by the energy density associated with the inflaton potential. However, there exist a class of models that lead to inflation simply because they are driven by the kinetic energy of a scalar field. There are a large class of such pole-like inflation models: those derived from induced gravity [3], scalar-tensor gravity [4] or the pre–big-bang scenario (PBB) [5] of string theory. Unfortunately, most of these models face the problem that they tend to yield blue spectra for both the scalar- and tensor-type perturbations with spectral indices $n_S \simeq 4$ and $n_T \simeq 3$, respectively, whereas the observationally supported Zel'dovich spectra correspond to $n_S \simeq 1$ and $n_T \simeq 0$ [6–9]. A possible resolution of this problem for the scalar perturbations was proposed in [10], where fluctuations of the axion field present in the low-energy string action can generate the observed spectral index. However, it remains the case that the tensor spectrum is difficult to reconcile with this class of kinetic driven inflation models.

Perhaps the biggest issue facing gravity models is how to tackle the initial curvature singularity. In the context of PBB, an interesting suggestion was made that the graceful exit problem could be resolved by including the quantum back reaction effect [11]. A different approach uses an expansion in terms of the inverse string tension (α') and coupling corrections. This has already met with some success [12,13]. Given that there exists a class of nonsingular cosmologies based on these higher-order corrections, it is natural to investigate the effect of these correction terms on the evolution of primordial fluctuations. In fact the impact of potential higher-order curvature corrections on the evolution of the perturbations has recently been studied in various inflationary models derived from string theory [14,15]. In particular, the classical evolution in the presence of a Gauss-Bonnet coupling term was considered in [16,17]. Effects of a string theory motivated axion coupling term $g(\phi)R\tilde{R}$ in the Lagrangian, where $R\tilde{R} \equiv \eta^{\mu\nu\sigma\tau}R_{\mu\nu}{}^{\epsilon\kappa}R_{\sigma\tau\epsilon\kappa}$, was investigated in [18]. The general feature that appears to emerge is that the higher-order curvature corrections generally flatten the spectral distributions of primordial vacuum fluctuations. Hence, one may imagine a scenario where the observationally relevant perturbations for the large scale structures leave the Hubble radius during a highly curved regime, leaving the power spectrum nearly scale invariant.

The aim of this paper is to calculate the vacuum fluctuations arising out of the most general nonsingular solutions formed to date from the first order curvature corrections, corrections which could arise in the context of the massless bosonic sector of the low-energy effective action of string theory.

The paper is organized as follows. In Sec. II we introduce a general action including possible curvature corrections up to fourth order in derivatives and we derive the corresponding field equations. Section III is devoted to studying the classical evolution of three types of perturbations. We derive closed form equations for both scalar- and tensor-type perturbations and determine the corresponding large-scale exact solutions. We also show that the vector-type perturbations are described by the conservation of the angular momentum of the additional fluid in the absence of dissipative processes. In Sec. IV we discuss the quantum generation of the scalar and tensor perturbations and we derive the corresponding power spectra. The general results are applied in Sec. V to the particular case of the pre–big-bang scenario of string cosmology. Finally, we discuss our main results and future applications in Sec. VI.

II. THE EVOLUTION EQUATIONS

To keep things as general as possible, we consider the following D-dimensional action:[1]

$$S=\int d^Dx\sqrt{-g}\left[\frac{1}{2}f(\phi,R)-\frac{1}{2}\omega(\phi)\phi^{;\mu}\phi_{;\mu}-V(\phi)+L^{(c)}+L_m\right], \tag{1}$$

where $f(\phi,R)$ is an algebraic function of a dimensionless scalar field ϕ and the scalar curvature R. $\omega(\phi)$ and $V(\phi)$ are general algebraic functions of ϕ. Through the Lagrangian $L^{(c)}$, we allow for the inclusion of terms with even higher numbers of derivatives such as contracted quadratic products of the curvature tensor, whereas L_m is the Lagrangian of additional matter fields (e.g. fluids, kinetic components, axions, moduli, etc.) with its associated energy-momentum tensor $T_{\mu\nu}$ defined as $\frac{1}{2}\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}\equiv\delta(\sqrt{-g}L_m)$. By choosing the appropriate parameters, the general action Eq. (1) includes the Brans-Dicke theory, non-minimally coupled scalar field, induced gravity, R^2 gravity, etc. [19]. For instance, Einstein gravity with a minimally coupled scalar field corresponds to the case $f=R$, $\omega=1$, and $L^{(c)}=0$.

In a string theory context (see [9] for a recent review and references therein), the low-energy effective action is obtained with $f=e^{-\phi}R$, $\omega=-e^{-\phi}$, $V=0$, and $L^{(c)}=0$. Higher-order corrections take the form of an infinite series expansion with expansion parameter $\alpha'=\lambda_s^2$, where λ_s is the fundamental string length scale. For the sake of simplicity, we shall restrict ourselves to the simplest extension of the lowest-order gravitational action ensuring that the equations of motion remain second order in the fields. In such a case, the most general Lagrangian density at the next to leading order in the Regge slope reads [20]:

$$L^{(c)}=-\frac{1}{2}\alpha'\lambda\xi(\phi)[c_1R^2_{GB}+c_2G^{\mu\nu}\phi_{;\mu}\phi_{;\nu}+c_3\Box\phi\phi^{;\mu}\phi_{;\mu}+c_4(\phi^{;\mu}\phi_{;\mu})^2], \tag{2}$$

where $G^{\mu\nu}\equiv R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R$ is the Einstein tensor and $R^2_{GB}\equiv R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-4R^{\mu\nu}R_{\mu\nu}+R^2$ is the well-known Gauss-Bonnet combination ensuring the ghost-free character of the theory. In fixing the coefficients c_i's, we require that the full action agrees with the three-graviton scattering amplitude [20], and thus impose the constraints, $\xi=-e^{-\phi}$, $c_3=-[c_2+2(c_1+c_4)]/2$, and $c_1=-1$, working in units $\alpha'=1$. λ is an additional parameter allowing for different species of string theories, $\lambda=-1/4,-1/8$ for the Bosonic and Heterotic, string respectively, and $\lambda=0$ for superstrings. The inclusion of such higher-order corrections for the background evolution has recently been investigated in the context of the pre–big-bang scenario of string cosmology, leading to a number of promising nonsingular cosmologies that smoothly interpolate between the growing to the decreasing curvature regime [12,13].

By varying Eq. (1) with respect to the metric and the scalar field we can derive the gravitational field equation and the equation of motion for the scalar field ϕ, respectively:

$$FG^\mu_\nu=\omega\left(\phi^{;\mu}\phi_{;\nu}-\frac{1}{2}\delta^\mu_\nu\phi^{;\rho}\phi_{;\rho}\right)-\frac{1}{2}\delta^\mu_\nu(FR-f+2V)+F^{;\mu}{}_{\nu}-\delta^\mu_\nu\Box F+T^{(c)\mu}{}_{\nu}+T^\mu_\nu, \tag{3}$$

$$\Box\phi+\frac{1}{2\omega}(f_{,\phi}+\omega_{,\phi}\phi^{;\mu}\phi_{;\mu}-2V_{,\phi}-T^{(c)}_\phi)=0, \tag{4}$$

where $F\equiv\partial f/\partial R$. $T^{(c)\mu}{}_{\nu}$ and $T^{(c)}_\phi$ represent the contributions derived from the next to leading order corrections given by Eq. (2),

$$\begin{aligned}T^{(c)\mu}{}_{\nu}\equiv{}&-\alpha'\lambda\Bigg(-4c_1[(R^\mu{}_{\sigma\nu\tau}+R_{\nu\sigma}\delta^\mu_\tau-\delta^\mu_\nu R_{\sigma\tau})\xi^{;\sigma\tau}+G^{\mu\sigma}\xi_{;\nu\sigma}-G^\mu_\nu\Box\xi]+c_1\xi\aleph^\mu_\nu+c_2\Big\{\xi[(\delta^\mu_\nu R_{\sigma\tau}-R^\mu{}_{\sigma\nu\tau})\phi^{;\sigma}\phi^{;\tau}\\&-R^\mu_\sigma\phi_{;\nu}\phi^{;\sigma}-R^\sigma_\nu\phi^{;\mu}\phi_{;\sigma}]+\frac{1}{2}\xi[G^\mu_\nu\phi^{;\sigma}\phi_{;\sigma}+R\phi^{;\mu}\phi_{;\nu}-2\phi^{;\mu\sigma}\phi_{;\nu\sigma}+2\Box\phi\phi^{;\mu}{}_{\nu}-\delta^\mu_\nu(\Box\phi)^2+\delta^\mu_\nu\phi^{;\sigma\tau}\phi_{;\sigma\tau}]\\&+\frac{1}{2}[\delta^\mu_\nu(\Box\xi\phi^{;\sigma}\phi_{;\sigma}-\xi^{;\sigma\tau}\phi_{;\sigma}\phi_{;\tau})-\Box\xi\phi^{;\mu}\phi_{;\nu}+\xi^{;\mu\sigma}\phi_{;\nu}\phi_{;\sigma}+\xi_{;\nu\sigma}\phi^{;\mu}\phi^{;\sigma}-\xi^{;\mu}{}_{\nu}\phi^{;\sigma}\phi_{;\sigma}]\\&+\frac{1}{2}[\Box\phi(\xi^{;\mu}\phi_{;\nu}+\xi_{;\nu}\phi^{;\mu})+2\xi^{;\sigma}\phi_{;\sigma}\phi^{;\mu}{}_{\nu}+2\delta^\mu_\nu(\xi^{;\sigma}\phi^{;\tau}\phi_{\sigma\tau}-\Box\phi\,\xi^{;\sigma}\phi_{;\sigma})-\xi_{;\sigma}\phi^{;\mu\sigma}\phi_{;\nu}]\\&-\frac{1}{2}[\xi^{;\sigma}\phi^{;\mu}\phi_{;\nu\sigma}+\xi^{;\mu}\phi_{;\nu\sigma}\phi^{;\sigma}+\xi_{;\nu}\phi^{;\mu\sigma}\phi_{;\sigma}]\Big\}+c_3\Big\{\xi[\phi^{;\mu\rho}\phi_{;\nu}\phi_{;\rho}+\phi^{;\mu}\phi_{;\nu}{}^{\rho}\phi_{;\rho}-\xi\delta^\mu_\nu\phi^{;\rho\sigma}\phi_{;\rho}\phi_{;\sigma}-\Box\phi\phi^{;\mu}\phi_{;\nu}]\\&+\frac{1}{2}\phi^{;\rho}\phi_{;\rho}[\xi^{;\mu}\phi_{;\nu}+\xi_{;\nu}\phi^{;\mu}-\delta^\mu_\nu\xi^{;\rho}\phi_{;\rho}]\Big\}+\frac{1}{2}c_4\xi\phi^{;\sigma}\phi_{;\sigma}[\delta^\mu_\nu\phi^{;\rho}\phi_{;\rho}-4\phi^{;\mu}\phi_{;\nu}]\Bigg),\end{aligned} \tag{5}$$

$$\aleph^\mu_\nu\equiv\frac{1}{2}\delta^\mu_\nu R^2_{GB}+4R^\mu{}_{\sigma\nu\tau}R^{\sigma\tau}-2RR^\mu_\nu-2R^\mu{}_{\rho\sigma\tau}R_\nu{}^{\rho\sigma\tau}+4R^\mu_\sigma R^\sigma_\nu, \tag{6}$$

[1]We adopt the convention $(-,+,\ldots,+)$, $R_{\nu\rho}=R^\lambda_{\nu\lambda\rho}$, $R^\mu_{\nu\lambda\rho}=\partial_\rho\Gamma^\mu_{\nu\lambda}+\ldots$, and set our units such that $\hbar=c=8\pi G=1$.

$$T^{(c)}_{\phi} \equiv \alpha' \lambda \{ c_1 \xi_{,\phi} R^2_{GB} + c_2 G^{\mu\nu}(\xi_{,\phi}\phi_{;\mu}\phi_{;\nu} - 2\xi_{;\mu}\phi_{;\nu} - 2\xi\phi_{;\mu\nu}) + c_3\{ (\xi_{,\phi}\Box\phi + \Box\xi)\phi^{;\mu}\phi_{;\mu} - 2\Box\phi\xi^{;\mu}\phi_{;\mu} + 4\xi^{;\mu}\phi^{;\nu}\phi_{;\mu\nu} + 2\xi[\phi^{;\mu\nu}\phi_{;\mu\nu} + R^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - (\Box\phi)^2]\} + c_4[\xi_{,\phi}(\phi^{;\mu}\phi_{;\mu})^2 - 4\xi^{;\mu}\phi_{;\mu}\phi^{;\nu}\phi_{;\nu} - 4\xi\Box\phi\phi^{;\mu}\phi_{;\mu} - 8\xi\phi^{;\mu}\phi^{;\nu}\phi_{;\mu\nu}]\}. \quad (7)$$

Equations (3)–(7) form a complete set of covariant generalized Einstein equations that include higher-order corrections. Hence they extend the domain of validity of any solutions into the highly-curved regimes included in Eq. (2). This system of equations provides the framework to study both the cosmological background and evolution of metric and field perturbations. Note that $\aleph^{\mu}_{\nu}$ corresponds to a quadratic expression which vanishes in four dimensions on account of the algebraic identities satisfied by the Riemann tensor [21].

III. PERTURBED FIELD EQUATIONS

From now on, we concentrate on the four-dimensional case, and consider the metric of a spatially homogeneous and isotropic model with the most general perturbations:

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2(\beta_{,i}+B_i)d\eta dx^i + a^2[g^{(3)}_{ij}(1+2\varphi) + 2\gamma_{,i|j} + 2C_{(i|j)} + 2C_{ij}]dx^i dx^j, \quad (8)$$

where $a(t)$ is the cosmic scale factor with $dt \equiv a d\eta$. Latin letters denote space indices. $\alpha(\mathbf{x},t)$, $\beta(\mathbf{x},t)$, $\varphi(\mathbf{x},t)$, and $\gamma(\mathbf{x},t)$ characterize the scalar-type perturbation, $B_i(\mathbf{x},t)$ and $C_i(\mathbf{x},t)$ are transverse $(B^{i}{}_{|i} = 0 = C^{i}{}_{|i})$ and represent the vector-type perturbation, whereas $C_{ij}(\mathbf{x},t)$ is transverse and tracefree $(C^{j}_{i|j} = 0 = C^{i}_{i})$, and corresponds to the tensor-type perturbation. Indices are based on $g^{(3)}_{ij}$ as the metric, and a vertical bar indicates a covariant derivative based on $g^{(3)}_{ij}$.

We decompose the energy-momentum tensor of the additional matter and the dilaton field into

$$T^{\mu}_{\nu}(\mathbf{x},t) = \overline{T}^{\mu}_{\nu}(t) + \delta T^{\mu}_{\nu}(\mathbf{x},t), \quad (9)$$

$$\phi(\mathbf{x},t) = \overline{\phi}(t) + \delta\phi(\mathbf{x},t). \quad (10)$$

An overbar indicates a background order quantity and will be omitted unless necessary. The three types of perturbations decouple from each other due to the symmetry in the background field equations and the fact that we are working to linear order in the perturbations. Thus, we can handle them individually and we find it convenient to separate their respective contributions in the perturbed energy-momentum tensor. We use the superscripts (s), (v), and (t) for the scalar, vector, and tensor parts, respectively, such that $\delta T^{\mu}_{\nu} \equiv \delta T^{(s)\mu}{}_{\nu} + \delta T^{(v)\mu}{}_{\nu} + \delta T^{(t)\mu}{}_{\nu}$. Using a normalized $(u^{\mu}u_{\mu} = -1)$ four-vector and its associated projection tensor $h_{\mu\nu} \equiv g_{\mu\nu} + u_{\mu}u_{\nu}$, a covariant decomposition of the (imperfect fluid) energy-momentum tensor into fluid quantities is given by [22]

$$T_{\mu\nu} = \rho u_{\mu}u_{\nu} + p h_{\mu\nu} + q_{\mu}u_{\nu} + q_{\nu}u_{\mu} + \pi_{\mu\nu}. \quad (11)$$

Here, we have defined $\rho \equiv T_{\mu\nu}u^{\mu}u^{\nu}$ as the energy density and $p \equiv \frac{1}{3}T_{\mu\nu}h^{\mu\nu}$ as the pressure. The energy flux and anisotropic pressure are, respectively, $q_{\mu} \equiv -T_{\nu\sigma}u^{\nu}h^{\sigma}_{\mu}$ and $\pi_{\mu\nu} \equiv T_{\sigma\tau}h^{\sigma}_{\mu}h^{\tau}_{\nu}$, and satisfy $q_{\mu}u^{\mu} = 0 = \pi_{\mu\nu}u^{\mu}$, $\pi_{\mu\nu} = \pi_{\nu\mu}$. Convenient decompositions for the fluid four-velocity and the energy flux are, respectively, $u_{\mu} \equiv -a(1+\alpha, \delta u_i)$ and $q_{\mu} \equiv a(0, \delta q_i)$. The content of the energy-momentum tensor to linear order then becomes

$$T^0_0 = -(\rho + \delta\rho), \quad (12)$$

$$T^0_i = -(\rho + p)\delta u_i + \delta q_i, \quad (13)$$

$$T^i_j = (p + \delta p)\delta^i_j + \pi^i_j, \quad (14)$$

where the background is assumed to be made up of a perfect fluid. In general, the decomposition of the flux in Eq. (11) suggests the apparition of additional degrees of freedom. For instance, we may choose to consider the normal frame $\delta u_i = 0$ or the energy frame $\delta q_i = 0$. In linear perturbation theory, however, δu_i and δq_i always appear together in a frame-independent combination of the from of Eq. (13). Hence, we can investigate the gauge transformation properties of such a combination in a frame independent way [19]. For later convenience, we split Eqs. (13) and (14) into the three types of perturbations as

$$T^0_i \equiv (\rho + p)(v^{(s)}_i + v^{(v)}_i), \quad (15)$$

$$T^i_j \equiv (p + \delta p)\delta^i_j + \pi^{(s)i}{}_j + \pi^{(v)i}{}_j + \pi^{(t)i}{}_j. \quad (16)$$

The background equations in Sec. III A are presented considering the general spatial curvature K of the background, whereas for the perturbation equations in Sec. III B–III D we assume a flat background (i.e. $K = 0$). For the scalar-type perturbation we set $T^{(s)\mu}{}_{\nu} = 0$, whereas for the vector- and tensor-type perturbations we keep the contribution from the fluid energy-momentum tensor.

A. Background field equations

From Eqs. (3),(4) we obtain a set of equations describing the background evolution of the homogeneous-isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) model in our generalized gravity model:

$$H^2 + \frac{K}{a^2} = \frac{1}{6F}(\omega\dot{\phi}^2 + RF - f + 2V - 6H\dot{F} - 2T^0_0 - 2T^{(c)0}{}_0), \quad (17)$$

$$\dot{H}-\frac{K}{a^2}=\frac{1}{2F}\left(-\omega\dot{\phi}^2+H\dot{F}-\ddot{F}+T^0_0 -\frac{1}{3}T^i_i+T^{(c)0}{}_0-\frac{1}{3}T^{(c)i}{}_i\right), \tag{18}$$

$$\ddot{\phi}+3H\dot{\phi}+\frac{1}{2\omega}(\omega_{,\phi}\dot{\phi}^2-f_{,\phi}+2V_{,\phi}+T^{(c)}_\phi)=0, \tag{19}$$

where we denote the Hubble parameter as $H\equiv\dot{a}/a$, the Ricci scalar $R=6(2H^2+\dot{H}+K/a^2)$, and the higher-order contributions are given by

$$T^{(c)0}{}_0=-\alpha'\lambda\left[12c_1\dot{\xi}H\left(H^2+\frac{K}{a^2}\right)-\frac{3}{2}c_2\xi\dot{\phi}^2\left(3H^2+\frac{K}{a^2}\right) +\frac{1}{2}c_3\dot{\phi}^3(\dot{\xi}-6\xi H)-\frac{3}{2}c_4\xi\dot{\phi}^4\right], \tag{20}$$

$$T^{(c)i}{}_i=-3\alpha'\lambda\left\{4c_1\left[\ddot{\xi}\left(H^2+\frac{K}{a^2}\right)+2\dot{\xi}H(\dot{H}+H^2)\right] -\frac{1}{2}c_2\dot{\phi}\left[\xi\dot{\phi}\left(2\dot{H}+3H^2-\frac{K}{a^2}\right)+4\xi\ddot{\phi}H+2\dot{\xi}\dot{\phi}H\right] -\frac{1}{2}c_3\dot{\phi}^2(2\xi\ddot{\phi}+\dot{\xi}\dot{\phi})+\frac{1}{2}c_4\xi\dot{\phi}^4\right\}, \tag{21}$$

$$T^{(c)}_\phi=\alpha'\lambda\left\{24c_1\xi_{,\phi}(\dot{H}+H^2)\left(H^2+\frac{K}{a^2}\right) +3c_2\left[-\left(H^2+\frac{K}{a^2}\right)(\dot{\xi}\dot{\phi}+2\xi\ddot{\phi}) -2\xi\dot{\phi}H\left(2\dot{H}+3H^2+\frac{K}{a^2}\right)\right]+c_3\dot{\phi}[\dot{\phi}\ddot{\xi}+3\dot{\xi}\ddot{\phi} -6\xi(\dot{\phi}\dot{H}+2\ddot{\phi}H+3\dot{\phi}H^2)] +c_4\dot{\phi}^2(-3\dot{\xi}\dot{\phi}-12\xi\ddot{\phi}-12\xi\dot{\phi}H)\right\}. \tag{22}$$

The solutions of the system of equations (17)–(22) provide the cosmological "graviton-scalar field" background in which we will study the propagation of scalar field and metric perturbations. Equations (17)–(19) are not independent. We can derive Eq. (18) from Eqs. (17) and (19), by using $\dot{T}^0_0+H(3T^0_0-T^i_i)=0$ which follows from energy-momentum conservation of the additional fluids or fields.

B. Scalar-type perturbations

To investigate the scalar-type perturbations there are many different temporal gauge conditions available for us to use. Due to the homogeneity of the background three-space the spatial gauge transformation is trivial [23]: although β and γ change under the spatial gauge transformation, they always appear together in a spatially gauge-invariant combination $\chi\equiv a(\beta+a\dot{\gamma})$, whereas α and φ are spatially gauge invariant. Except for the synchronous gauge condition, which happens to be the most widely used one, all the other fundamental temporal gauge conditions fix the gauge degrees of freedom completely, thus a variable in such a gauge condition uniquely corresponds to a gauge-invariant combination of the variable concerned and the variable used in fixing the gauge condition, see Eq. (23) and below. The most suitable gauge condition depends on the type of the problem and in general is not known *a priori*, hence the gauge-ready method proposed in [23,19] appears particularly convenient since it allows for a flexible use of the various fundamental gauge conditions.

In handling the perturbations involved with the scalar field in Einstein or generalized gravity theories [without the $L^{(c)}$ term in Eq. (2)], it is known that the field fluctuation in the uniform-curvature gauge or equivalently the curvature fluctuation in the uniform-field gauge (see below) allow the simplest analysis [24]. Following [17], we derive an equation for a gauge-invariant combination $\varphi_{\delta\phi}$ defined as

$$\varphi_{\delta\phi}\equiv\varphi-\frac{H}{\dot{\phi}}\delta\phi\equiv-\frac{H}{\dot{\phi}}\delta\phi_\varphi. \tag{23}$$

In the uniform-field gauge, which takes $\delta\phi\equiv0$, the gauge-invariant quantity $\varphi_{\delta\phi}$ is identified with φ. Similarly $\delta\phi_\varphi$ is the same as $\delta\phi$ in the uniform-curvature gauge which sets $\varphi=0$. The gauge invariant combination $\varphi_{\delta\phi}$ was first introduced by Lukash in [25,26]. If we have a solution in one gauge condition, the other solutions in the same gauge as well as the ones in other gauges can be derived from it as linear combinations. In the present case, choosing the uniform-field gauge also implies $\delta\xi=0$. We consider a case with $F=F(\phi)$, thus we have $\delta F=0$ as well. Ignoring the additional matter, that is we do not include any other components other than the scalar (dilaton) field, i.e. $T^{(s)\mu}{}_\nu=0$, and following the same steps described above Eq. (3) in [17], we can derive a closed form equation describing the classical evolution of the scalar-metric perturbation. Explicitly, we find

$$\frac{1}{a^3Q^{(s)}}(a^3Q^{(s)}\dot{\varphi}_{\delta\phi})^{\cdot}-s^{(s)}\frac{\Delta}{a^2}\varphi_{\delta\phi}=0, \tag{24}$$

where

$$Q^{(s)}\equiv\frac{\omega\dot{\phi}^2+3\dfrac{(\dot{F}+Q_a^{(s)})^2}{2F+Q_b^{(s)}}+Q_c^{(s)}}{\left(H+\dfrac{\dot{F}+Q_a^{(s)}}{2F+Q_b^{(s)}}\right)^2},$$

$$s^{(s)}\equiv1+\frac{Q_d^{(s)}+\dfrac{\dot{F}+Q_a^{(s)}}{2F+Q_b^{(s)}}Q_e^{(s)}+\left(\dfrac{\dot{F}+Q_a^{(s)}}{2F+Q_b^{(s)}}\right)^2Q_f^{(s)}}{\omega\dot{\phi}^2+3\dfrac{(\dot{F}+Q_a^{(s)})^2}{2F+Q_b^{(s)}}+Q_c^{(s)}}. \tag{25}$$

The contribution arising from the higher-order corrections is summarized in the quantities

$$Q_a^{(s)} \equiv \alpha' \lambda [-4c_1 \dot{\xi} H^2 + 2c_2 \xi \dot{\phi}^2 H + c_3 \xi \dot{\phi}^3],$$

$$Q_b^{(s)} \equiv \alpha' \lambda [-8c_1 \dot{\xi} H + c_2 \xi \dot{\phi}^2],$$

$$Q_c^{(s)} \equiv \alpha' \lambda \dot{\phi}^2 [-3c_2 \xi H^2 + 2c_3 \dot{\phi}(\dot{\xi} - 3\xi H) - 6c_4 \xi \dot{\phi}^2],$$

$$Q_d^{(s)} \equiv \alpha' \lambda \dot{\phi}^2 [-2c_2 \xi \dot{H} - 2c_3(\dot{\xi}\dot{\phi} + \xi \ddot{\phi} - \xi \dot{\phi} H) + 4c_4 \xi \dot{\phi}^2],$$

$$Q_e^{(s)} \equiv \alpha' \lambda [-16c_1 \dot{\xi} \dot{H} + 2c_2 \dot{\phi}(\dot{\xi}\dot{\phi} + 2\xi \ddot{\phi} - 2\xi \dot{\phi} H) - 4c_3 \xi \dot{\phi}^3],$$

$$Q_f^{(s)} \equiv \alpha' \lambda [8c_1(\ddot{\xi} - \dot{\xi} H) + 2c_2 \xi \dot{\phi}^2]. \qquad (26)$$

In units $\alpha' = 1$, our result reproduces those previously obtained for the particular case $c_1 = -1$, $c_2 = c_3 = c_4 = 0$ and $\lambda = -1/4$ [17].

C. Vector-type perturbations

Rotational perturbations do not enter the (0,0) components of the field equations, but they do contribute to the $(0,i)$ and (i,j) components. In fact, our scalar field perturbation does not couple directly with these vector-type perturbations. To investigate the influence of the scalar field, we consider an additional fluid present in the system. We can introduce the vector-type fluid quantities of the additional matter as

$$\delta T^{(v)0}{}_i \equiv (\rho + p) v^{(v)} Y_i^{(v)}, \quad \delta T^{(v)i}{}_j \equiv \pi^{(v)} Y^{(v)i}{}_j, \qquad (27)$$

where $Y_i^{(v)}$ and $Y^{(v)i}{}_j$ are vector-type harmonic functions [28]. $v^{(v)}$ is the velocity variable related to the vorticity and $\pi^{(v)}$ is the anisotropic stress. In the field equations, the vector perturbations always appear in a gauge-invariant combination $B_i + a\dot{C}_i \equiv \Psi^{(v)} Y_i^{(v)}$. From the $(0,i)$ and (i,j) components of the gravitational field equation we can derive[2]

$$\frac{k^2}{2a^2} Q^{(v)} \Psi^{(v)} = (\rho + p) v^{(v)}, \qquad (28)$$

$$\frac{1}{a^4} [a^4 (\rho + p) v^{(v)}]^{\cdot} = -\frac{k}{2a} \pi^{(v)}, \qquad (29)$$

where we have introduced

$$Q^{(v)} \equiv F - \frac{1}{2} \alpha' \lambda (8c_1 \dot{\xi} H - c_2 \xi \dot{\phi}^2). \qquad (30)$$

If we ignore the anisotropic stress of the fluid, $\pi^{(v)}$, Eq. (29) implies the conservation of the angular momentum of the fluid, as

$$a^3 (\rho + p) \cdot a \cdot v^{(v)}(\mathbf{x}, t) = L(\mathbf{x}). \qquad (31)$$

Notice that this result is independent of the generalized nature of our gravity model. The evolution of vorticity depends on our generalized gravity indirectly through the background evolution. The generalized nature, however, appears directly in connecting the vorticity to the metric perturbation in Eq. (28). Equation (29), in fact, follows from the 0-component of the conservation of the fluid part of the energy-momentum tensor ($T^{0;\nu}_{\nu} = 0$), hence is naturally independent of the field equation. Conservation of angular momentum forces the rotational perturbation to decay in an expanding medium, which renders it generally cosmologically uninteresting in the context of linear perturbations in an expanding universe.

D. Tensor-type perturbations

From the (i,j) component of the field equation we can derive the linearized equation for the gravitational wave perturbation

$$\frac{1}{a^3 Q^{(t)}} (a^3 Q^{(t)} \dot{C}^i_j)^{\cdot} - s^{(t)} \frac{\Delta}{a^2} C^i_j = \frac{1}{Q^{(t)}} \delta T^{(t)i}{}_j, \qquad (32)$$

where

$$Q^{(t)} \equiv F - \frac{1}{2} \alpha' \lambda (8c_1 \dot{\xi} H - c_2 \xi \dot{\phi}^2),$$

$$s^{(t)} \equiv \frac{1}{Q^{(t)}} \left[F - \frac{1}{2} \alpha' \lambda (8c_1 \ddot{\xi} + c_2 \xi \dot{\phi}^2) \right]. \qquad (33)$$

$\delta T^{(t)i}{}_j$ includes contributions to the tensor-type energy-momentum tensor, i.e., transverse-tracefree anisotropic stresses which can possibly arise if we include additional imperfect fluids. Since the gauge transformation does not affect the transverse-trace free parts, C^i_j and $\delta T^{(t)i}{}_j$ are naturally gauge invariant. In the context of string cosmology, $F = -\omega = -\xi = e^{-\phi}$ and $V = 0$, we note that one easily recovers the α' corrections obtained in [29]. Whereas we assumed $F = F(\phi)$ for the scalar-type perturbation, the above results for the vector- and tensor-type perturbations are valid for the general action in Eq. (1) with general $f(\phi, R)$.

IV. QUANTUM GENERATION AND EVOLUTION OF PERTURBATIONS

A. Classical evolution

Assuming that there is no additional matter, $T^\mu_\nu = 0$, we have been able to derive a closed form equation for the linearized classical evolution of both scalar and tensor metric perturbations. In a unified formalism [27], the dynamics of a perturbed variable Φ is typically governed by the wave equation

[2]Compared with [28] we have $v^{(v)} \equiv v_c$, $\pi^{(v)} \equiv p \pi_T^{(1)}$ and $\Psi^{(v)} = \Psi$.

$$\frac{1}{a^3Q}(a^3Q\dot{\Phi})^{\cdot}-s\frac{\Delta}{a^2}\Phi=0, \qquad (34)$$

which may be readily derived from the second-order perturbed action

$$\delta^{(2)}S=\frac{1}{2}\int a^3Q\left(\dot{\Phi}^2-\frac{s}{a^2}\Phi^{|i}\Phi_{,i}\right)d^3x\,dt, \qquad (35)$$

where $(Q,s,\Phi)\in\{(Q^{(s)},s^{(s)},\varphi_{\delta\phi});(Q^{(t)},s^{(t)},C^i_j)\}$. Introducing the quantities $\psi\equiv z\Phi$ with $z\equiv a\sqrt{Q}$, the linearized wave equation (34) can be written for each Fourier mode, $\psi_k\equiv z\Phi_k$, in terms of the eigenstates of the Laplace-Beltrami operator, $\Delta\psi_k=-k^2\psi_k$. Explicitly, Eq. (34) becomes

$$\psi_k''+[sk^2-V(\eta)]\psi_k=0, \quad V(\eta)=\frac{z''}{z}, \qquad (36)$$

where $'\equiv d/d\eta$. The "pump" field z accounts for the parametric amplification of the metric fluctuations, whereas s may be interpreted as an effective shift in the frequency. We note that the wave equation (36) can be deduced from the second-order perturbed action

$$\delta^{(2)}\mathcal{S}=\frac{1}{2}\int\left[\psi'^2+\frac{z''}{z}\psi^2+s\psi\Delta\psi\right]d^3x\,d\eta, \qquad (37)$$

where the kinetic term is diagonal, emphasising the canonical nature of the variable ψ. To linear order, the decomposition in Fourier modes implies that each comoving wave number k evolves independently of other comoving modes, satisfying Eq. (36) for all times. In a general relativity context and in the spatially flat FLRW manifold, we recall that Eq. (36) reduces to the equation of a minimally coupled massless scalar field for tensor perturbations [30], whereas a time-dependent effective mass is present in the scalar perturbation case [31]. In the present context of generalized gravity theory, however, the wave equation differs from the Klein-Gordon equation for two reasons. On the one hand, both scalar and tensor perturbations are coupled not only to the background scale factor, but potentially also to the scalar field background through the algebraic function $F=\partial f(\phi,R)/\partial R$. Hence, the growth of the comoving amplitude of metric perturbations rises because of the joint contribution of the metric and the scalar field background to the pump field $z=z(a,\phi)$. On the other hand, the higher-order corrections act as an additional source for the parametric amplification of the metric perturbations. These deviations from the general relativity case manifest themselves through the unique function Q, which encompasses all sources of modification.

In the case where the background evolution undergoes power-law-type inflation, the external potential responsible for the parametric amplification process [32] behaves as $V(\eta)\sim|\eta|^{-2}$, where we have set $z\sim|\eta|^{\gamma}$. For each comoving wave number, the wave equation (36) then reduces to

$$\psi_k''+k^2\left[\pm\tilde{s}-\frac{\nu^2-1/4}{|k\eta|^2}\right]\psi_k=0, \qquad (38)$$

where we have introduced $\tilde{s}=|s|$ and $\nu\equiv\frac{1}{2}|1-2\gamma|$. At lowest order, i.e. neglecting the frequency shift and corrections in the pump field, the difference between scalar and tensor metric perturbations is encoded in the argument $\nu\in\{\nu_s;\nu_t\}$, reflecting the background dynamics through the exponent $\gamma\in\{\gamma_s;\gamma_t\}$ of the pump field. The dynamical behavior of the canonical variable, given by Eq. (38), can be divided into two asymptotic regimes. On the one hand, the power-law behavior of the external potential implies that it vanishes in the asymptotic past $(|k\eta|\gg 1)$. Hence, Eq. (38) yields two oscillating solutions, corresponding to negative and positive frequency modes, respectively. On the other hand, the monotonic growth of the effective potential of the perturbation ensures that it will soon drive the evolution of the metric perturbation (parametric amplification process). In the large wavelength $(|k\eta|\ll 1)$ limit, which corresponds to scales larger than the Hubble radius, Eq. (38) yields the solution

$$\begin{aligned}\Phi_k(\eta,\mathbf{x})&=C_k(\mathbf{x})+D_k(\mathbf{x})\int^{\eta}\frac{d\eta'}{z^2}\\&=C_k(\mathbf{x})+D_k(\mathbf{x})|\eta|^{1-2\gamma},\end{aligned} \qquad (39)$$

where $C_k(\mathbf{x})$ and $D_k(\mathbf{x})$ are constants of integration. For $\gamma<\frac{1}{2}$, the metric perturbation Φ_k approaches a constant for $\eta\rightarrow 0_-$. For $\gamma\geqslant\frac{1}{2}$, however, the second term with D_k grows in time, with an additional logarithmic factor appearing at $\gamma=\frac{1}{2}$. More generally, for both scalar and tensor metric perturbations in the large-scale limit, $sk^2\ll z''/z$, thus ignoring the Laplacian term in Eq. (36), the exact solution is given by

$$\Phi_k(\eta,\mathbf{x})=C_k(\mathbf{x})+D_k(\mathbf{x})\int^{\eta}\frac{d\eta'}{a^2Q}, \qquad (40)$$

implying the conservation of Φ_k in the large-scale limit. Since the coefficient $C_k(\mathbf{x})$ does not depend explicitly on $V(\phi)$, $f(\phi,R)$ or $\omega(\phi)$, the perturbations are conserved independently of general changes in the background equation of state and remain *a priori* conserved even under changes of the underlying gravity. Since we have not specified a cosmological scenario yet, these results remain valid for any four-dimensional FLRW background model that may be derived from the general action equation (1); we have assumed $F=F(\phi)$ for the scalar-type perturbations. For modes with physical sizes of order of the Hubble radius, however, we stress that the conservation of the perturbed variable Φ_k may be delayed in a situation where the background evolution becomes increasingly sensitive to the effects of the higher-order corrections. Indeed, for both scalar and tensor metric perturbations, the time-dependent variables Q and s may eventually be altered by the presence of the curvature corrections. In that case, the source Q of the effective potential for the perturbation may no longer grow monotonically and may even decrease at the onset or during a high-curvature regime. In the context of standard inflation models, a similar effect has been associated in [33] with an enhancement of the amplitude of scalar metric perturbations just after the scales cross the Hubble radius in some special situations. But it remains from Eq. (40) that the evolution of the perturbed variables should be frozen out for scales far beyond the Hubble radius. Apparently, there is also no restriction on the sign of the frequency shift occurring when the curvature (α') corrections dominate the background dynamics: s may become negative or infinite depending on its particular form. One interpretation of the denominator of the frequency shift [Eq. (25) and Eq. (33) for scalar and tensor metric perturbations, respectively] is that as the α' correction terms become comparable to the lowest-order terms; it marks the breakdown of the approximation we have adopted, in that we are entering a regime where we should include all the higher-order corrections. Although we expect these modifications to be dependent on the type of gravity chosen for the background evolution, we now proceed to investigate further the evolution of the perturbation variables inside the Hubble radius, for we are interested in obtaining the spectrum as the variables leave the Hubble radius during inflation.

B. Quantum generation of vacuum fluctuations

In the semiclassical approximation [34], the perturbed parts of the fields and metric are regarded as quantum mechanical operators, whilst the background parts are considered as classical. In the Heisenberg representation, where the quantum operators (denoted with an overhat) carry the time dependence, we have, for instance,

$$\phi(\mathbf{x},t)=\overline{\phi}(t)+\delta\hat{\phi}(\mathbf{x},t), \quad \varphi(\mathbf{x},t)\to\hat{\varphi}(\mathbf{x},t),$$

$$\hat{\varphi}_{\delta\phi}\equiv\hat{\varphi}-\frac{H}{\dot{\phi}}\delta\hat{\phi}, \quad \delta\hat{\phi}_{\varphi}\equiv\delta\hat{\phi}-\frac{\dot{\phi}}{H}\hat{\varphi}, \quad \text{etc.} \tag{41}$$

Tensor metric perturbations, leading to the formation of a cosmological stochastic background of gravitational waves, have two polarization states, whereas scalar metric perturbations have a single component. For conciseness, we shall restrict the description of the quantum generation to tensor-type perturbations; being without the polarization states the scalar-type perturbation is simpler, and can be read from the tensor-type analyses by ignoring the tensor and the polarization indices. It proves convenient to introduce a decomposition based on the two polarization states which, for a flat three-space background, is given by [35,36]

$$\hat{\Phi}_{ij}(\mathbf{x},t)\equiv\int\frac{d^3k}{(2\pi)^{3/2}}\hat{\Phi}_{ij}(\mathbf{x},t;\mathbf{k})$$
$$\equiv\int\frac{d^3k}{(2\pi)^{3/2}}\left[\sum_l e^{i\mathbf{k}\cdot\mathbf{x}}\Phi_{l\mathbf{k}}(t)\hat{a}_{l\mathbf{k}}e_{ij}^{(l)}(\mathbf{k})+\text{H.c.}\right]. \tag{42}$$

Here, H.c. denotes the Hermitian conjugate, $l=+,\times$ represent the two polarization states and $\Phi_{l\mathbf{k}}(t)$ is the mode function for the tensor metric perturbation. The $e_{ij}^{(l)}(\mathbf{k})$ form a base of polarization states that yields $e_{ij}^{(l)}(\mathbf{k})e^{(l')ij}(\mathbf{k})=2\delta_{ll'}$. The annihilation and creation operators $\hat{a}_{l\mathbf{k}}$ and $\hat{a}_{l\mathbf{k}}^{\dagger}$ of each polarization state satisfy the standard commutation relations on constant time hypersurfaces

$$[\hat{a}_{l\mathbf{k}},\hat{a}_{l'\mathbf{k}'}]=0=[\hat{a}_{l\mathbf{k}}^{\dagger},\hat{a}_{l'\mathbf{k}'}^{\dagger}],$$
$$[\hat{a}_{l\mathbf{k}},\hat{a}_{l'\mathbf{k}'}^{\dagger}]=\delta_{ll'}\delta^3(\mathbf{k}-\mathbf{k}'). \tag{43}$$

By projecting $\hat{\Phi}_{ij}(\mathbf{x},t)$ on the basis of polarization states, we easily obtain the mode expansion for each polarization state of the gravitational wave:

$$\hat{\Phi}_l(\mathbf{x},t)\equiv\frac{1}{2}\int\frac{d^3k}{(2\pi)^{3/2}}\hat{\Phi}_{ij}(\mathbf{x},t;\mathbf{k})e^{(l)ij}(\mathbf{k})$$
$$=\int\frac{d^3k}{(2\pi)^{3/2}}[e^{i\mathbf{k}\cdot\mathbf{x}}\Phi_{l\mathbf{k}}(t)\hat{a}_{l\mathbf{k}}+\text{H.c.}]. \tag{44}$$

The mode function $\Phi_{l\mathbf{k}}(t)$ is a complex solution of the classical mode evolution equation. Replacing the classical variable Φ with the Hilbert operator $\hat{\Phi}$ in the second-order perturbed action Eq. (35) leads to an equation for the mode function $\Phi_{l\mathbf{k}}$ that satisfies essentially the same form as the one obeyed by the classical variable, namely Eq. (34). From the quadratic effective action in the perturbed variable, we then derive the momentum $\hat{\pi}_{\Phi_l}$ canonically conjugate to $\hat{\Phi}_l$,

$$\hat{\pi}_{\Phi_l}(\mathbf{x},t)\equiv\frac{\partial\mathcal{L}}{\partial\dot{\hat{\Phi}}_l}=a^3Q\dot{\hat{\Phi}}_l(\mathbf{x},t). \tag{45}$$

This implies that the equal-time commutation relation $[\hat{\Phi}_l(\mathbf{x},t),\hat{\pi}_{\Phi_l}(\mathbf{x}',t)]=i\delta^3(\mathbf{x}-\mathbf{x}')$ leads to

$$[\hat{\Phi}_l(\mathbf{x},t),\dot{\hat{\Phi}}_l(\mathbf{x}',t)]=\frac{i}{a^3Q}\delta^3(\mathbf{x}-\mathbf{x}'). \tag{46}$$

Requiring agreement between Eq. (43) and Eq. (46), the Wronskian of the mode function $\Phi_{l\mathbf{k}}(t)$ must satisfy

$$\mathcal{W}_t\{\Phi_{l\mathbf{k}},\Phi_{l\mathbf{k}}^*\}\equiv\Phi_{l\mathbf{k}}\dot{\Phi}_{l\mathbf{k}}^*-\Phi_{l\mathbf{k}}^*\dot{\Phi}_{l\mathbf{k}}=\frac{i}{a^3Q}, \tag{47}$$

where the index t means that the derivatives in the Wronskian are with respect to the cosmic time coordinate. The power spectrum based on the vacuum expectation value ($\hat{a}_{l\mathbf{k}}|0\rangle\equiv0$, $\forall\mathbf{k}$) is then

$$\mathcal{P}_{\hat{\Phi}_l}(k,t)\equiv\frac{k^3}{2\pi^2}\int\langle0|\hat{\Phi}_l(\mathbf{x}+\mathbf{r},t)\hat{\Phi}_l(\mathbf{x},t)|0\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r$$
$$=\frac{k^3}{2\pi^2}|\Phi_{lk}(t)|^2. \tag{48}$$

In other words, the power spectrum corresponds to the Fourier transform of the two-point correlation function of the metric fluctuations. In a space without any preferred direction, the two polarization states of the gravitational waves are expected to contribute equally. Hence, we have

$$\mathcal{P}_{\hat{C}_{ij}}(\mathbf{k},t)=2\sum_l \mathcal{P}_{\hat{C}_l}(\mathbf{k},t)=2\sum_l \frac{k^3}{2\pi^2}|\hat{C}_{l\mathbf{k}}(t)|^2. \quad (49)$$

The power spectrum of scalar-metric perturbations can be derived in a similar manner, although no polarization decomposition is required [37]. We find

$$\mathcal{P}_{\hat{\varphi}_{\delta\phi}}(\mathbf{k},t)=\frac{k^3}{2\pi^2}|\hat{\varphi}_{\delta\phi\,\mathbf{k}}(t)|^2. \quad (50)$$

We now turn our attention to the spectral distributions $|\hat{C}_{l\mathbf{k}}(t)|$ and $|\hat{\varphi}_{\delta\phi\,\mathbf{k}}(t)|$.

Depending on the sign of the frequency shift $s=\pm\tilde{s}$ (which we shall consider constant in a first approximation), Eq. (38) becomes either a Bessel equation (for positive s) or a modified Bessel equation (for negative s). For positive s, the normalized mode function solution of Eq. (38) corresponds to a superposition of Hankel functions of the first and second kind,

$$\Phi_k=\frac{\sqrt{\pi}}{2}\frac{\sqrt{|\eta|}}{a\sqrt{Q}}[c_1(k)H_\nu^{(1)}(x)+c_2(k)H_\nu^{(2)}(x)], \quad (51)$$

with $x\equiv\sqrt{\tilde{s}}k|\eta|$ from which the usual condition between the coefficients $|c_2|^2-|c_1|^2=1$ is obtained from Eq. (47). In the large-scale limit, $\sqrt{\tilde{s}}k|\eta|\ll 1$, we have, for $\nu\neq 0$ and $\nu=0$, respectively,

$$\mathcal{P}_{\hat{\Phi}}^{1/2}=\frac{1}{\sqrt{Q}}\frac{H}{2\pi}\frac{\tilde{s}^{-\nu/2}}{aH|\eta|}\frac{\Gamma(\nu)}{\Gamma(3/2)}\left(\frac{k|\eta|}{2}\right)^{3/2-\nu}, \quad (52)$$

$$\mathcal{P}_{\hat{\Phi}}^{1/2}=\frac{2\sqrt{|\eta|}}{a\sqrt{Q}}\left(\frac{k}{2\pi}\right)^{3/2}\ln(\sqrt{|s|}k|\eta|). \quad (53)$$

Here we have considered the conventional choice of vacuum $c_2=1$ and $c_1=0$, corresponding to positive frequency in the asymptotic flat space time for $\eta\to-\infty$. An additional $\sqrt{2}$ factor appears for the gravitational wave power spectrum $\mathcal{P}_{\hat{C}_{ij}}^{1/2}$ that follows from proper consideration of the two polarization states; see Eqs. (49) and (50).

For the case of negative s, we have the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ as the two independent solutions of Eq. (38), that is

$$\Phi_k=\frac{\sqrt{|\eta|}}{a\sqrt{Q}}[c_1(k)I_\nu(x)+c_2(k)K_\nu(x)], \quad (54)$$

with $x\equiv\sqrt{\tilde{s}}k|\eta|$ and an unusual condition upon the coefficients $c_1^*c_2-c_1c_2^*=i$ arising from Eq. (47). In the large scale limit, the power spectrum exhibits similar dependence on the wave number as in Eqs. (52) and (53). However, modes evolving inside the Hubble radius face an instability since

$$I_\nu(x)\sim\frac{1}{\sqrt{2\pi x}}e^x, \quad K_\nu(x)\sim\sqrt{\frac{\pi}{2x}}e^{-x}, \quad (55)$$

which may lead to invalidate our assumption of small perturbations, hence the use of linear approach to cosmological perturbations. This is the condition we alluded to earlier, as marking a breakdown in the underlying action, requiring us to replace it with even higher-order corrections to enable us to deal properly with a high curvature epoch. Fortunately, a number of features emerge in the weak coupling regime that allows us to proceed further.

V. AN EXAMPLE: THE PRE–BIG-BANG SCENARIO

The observationally relevant scales are thought to have exited the Hubble radius within about 60 e-folds before the end of inflation. We have in mind a scenario where inflation occurs in our generalized gravity. This is then followed by a smooth transition to the ordinary radiation dominated era of the standard FLRW model. In the large scale limit, the perturbation variable Φ is conserved independently of changes in the underlying gravity theory, and the power-spectra based on the quantum vacuum expectation value can be identified at a later epoch as the classical power spectra based on the classical volume average. Thus, Eqs. (52) and (53) are now considered valid for the classical power spectra and the spectral index of the scalar and tensor-type perturbations become [27]

$$n_S-1=3-2\nu_s, \quad n_T=3-2\nu_t. \quad (56)$$

Here, ν_s and ν_t are to be evaluated at the time of the Hubble radius crossing during inflation, and will thus directly reflect the kinematics of the background when the perturbations exit the Hubble radius.

As an application of our general results, we may consider in particular the pre–big-bang scenario of string cosmology [5]. In our formalism, it corresponds to the case $f=e^{-\phi}R$, $\xi=\omega=-e^{-\phi}$ and $V=0$. Examples of regular backgrounds have recently been obtained by supplementing the low-energy effective action of string theory by the kind of higher-order corrections given in Eq. (2) [12,13]. Figure 1 is a typical example of nonsingular evolution: when the curvature scale is of the order of the inverse string scale, the α' corrections may eventually stabilize the background into a de Sitter–like regime of constant Hubble parameter $H\sim\mathcal{O}(\lambda_s^{-1})$ and linearly growing (in cosmic time) dilaton. Then, quantum loop corrections (based on the string coupling expansion) or a nonperturbative potential (yet to be determined) may trigger an exit to the FLRW radiation-dominated phase, with constant dilaton field.

It has been known for some time that this cosmological model derived from string theory predicts a very blue spectra

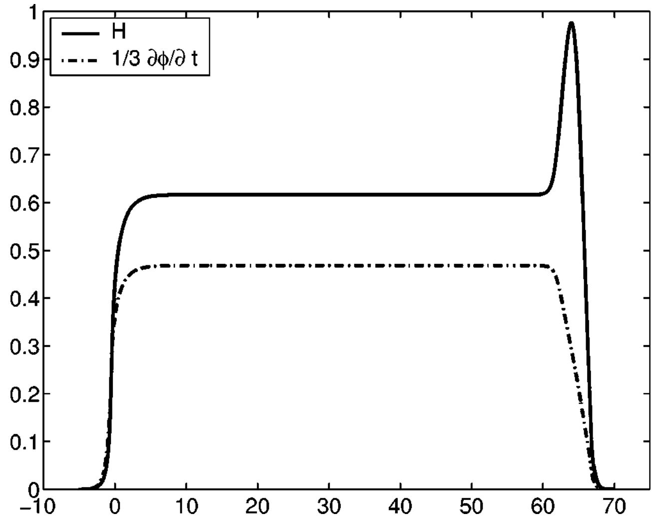

FIG. 1. Here we reproduce (from [29]) a nonsingular evolution for the Hubble parameter $H=\dot{a}/a$ and for $\dot{\phi}/3$, as a function of the number of e-folds, $N=\ln a$. The low-energy, dilaton-driven phase takes place approximately for $-\infty<N\leqslant-3$. After a short transition, this initial period is followed by a string phase with a nearly constant Hubble parameter and linearly growing dilaton, for $2\leqslant N\leqslant 55$. After a successful exit triggered by loop corrections, the background evolution enters the FLRW radiation-dominated phase at $N\simeq 68$.

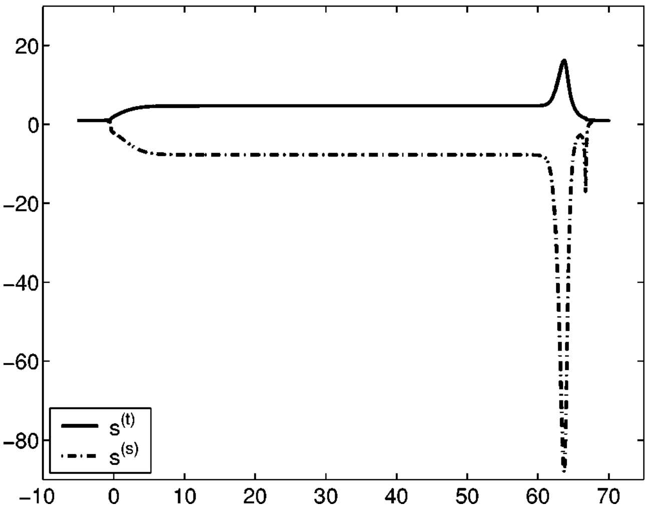

FIG. 2. Here we compare the frequency shifts for scalar- and tensor-type perturbations, $s^{(s)}$ and $s^{(t)}$, respectively, in the regular background of Fig. 1. The frequency shift for tensor-type perturbations is found to be always positive, whereas α' corrections are responsible for the sign change of the frequency shift of scalar-type perturbations. The latter may imply an exponential instability for the perturbation amplitude of the comoving mode whose physical wavelength is smaller than the size of the Hubble radius during the stringy high-curvature regime.

for both the scalar- and tensor-type perturbations [6–9]. Recalling the (conformal time) expression of the tree-level solutions [38],

$$a(\eta)=(-\eta)^{(1/2)(1-\sqrt{3})}, \quad \phi(\eta)=-\sqrt{3}\ln(-\eta), \quad (57)$$

the exponents of the scalar and tensor pump fields coincide, $\gamma_s=\gamma_t=1/2$, hence $\nu_s=\nu_t=0$. Using Eq. (56), we then obtain

$$n_S-1=n_T=3, \quad (58)$$

whereas the observationally favored Harrison-Zel'dovich type spectra correspond to $n_S\simeq 1$ and $n_T\simeq 0$. Hence, the pole-like acceleration expansion (in the string frame) of the pre–big-bang scenario cannot, in principle, be considered as the source for the observed fluctuations in the cosmic microwave background radiation (CMBR) and also the observed large scale structures. A possible resolution of this problem for the scalar perturbations was proposed in [10], where fluctuations of the axion field present in the low-energy string action can generate the observed spectral index. There also remains a possibility that the high-curvature regime can last long enough so that all the relevant scales that we observe today actually exit the Hubble radius during this era.

To determine whether a high-curvature regime may or may not flatten the spectral slope, we have to follow the evolution of the large frequency modes, associated with small physical wavelengths. We start by recalling that during the high-curvature regime, both the Hubble rate and the dilaton field become time independent, $H\equiv\tilde{H}$ and $\dot{\phi}\equiv\tilde{\phi}$. Hence, for both scalar- and tensor-type perturbations, the potential reduces to its tree-level (TL) form and we can express the pump field in the convenient form, $z=\text{const}\times z_{TL}$. Leaving aside the frequency shift, i.e. setting $s\equiv 1$ in Eq. (36), it is already well known [39] that the spectral slope of the high-frequency modes, crossing the Hubble radius in the high curvature, stringy regime, and re-entering in the radiation era, is fully determined by the fixed point values $\tilde{\phi},\tilde{H}$. Indeed, during such a string phase, the scale factor undergoes the usual de Sitter exponential expansion, while the logarithmic evolution of the dilaton, in conformal time, is weighted by the ratio $\tilde{\phi}/\tilde{H}$, i.e. $\phi(\eta)\sim-(\tilde{\phi}/\tilde{H})\log(-\eta)+\text{const}$ [40]. By introducing the convenient shifted variable $\tilde{\bar{\phi}}\equiv\tilde{\phi}-3\tilde{H}$ and the ratio $\vartheta\equiv\tilde{\bar{\phi}}/\tilde{H}$, and referring the spectrum to a fixed point allowing a subsequent (loop catalyzed) exit [13], i.e. $-1\leqslant\vartheta\leqslant 0$, one easily finds that the exponents of the pump fields satisfy $0\leqslant\gamma_s=\gamma_t=(1/2)(1+\vartheta)\leqslant 1/2$. Hence, the spectral indices for the high-frequency modes, leaving the Hubble radius during the de Sitter–like era and reentering into the radiation dominance epoch is given by

$$2\leqslant n_S-1=n_T\leqslant 3. \quad (59)$$

Although still not satisfying the observational bounds, they provide an indication of a possible resolution for suitable higher-order corrections.

In Fig. 2, we illustrate the evolution of the frequency shifts given by Eq. (25) and Eq. (33) in the regular background of Fig. 1. In the asymptotic past, the curvature corrections are negligible, i.e. $\alpha'\to 0$ and we have $s^{(s)}=s^{(t)}=1$. At the onset of the high-curvature regime, the behavior of the frequency shift for scalar- and tensor-type perturbations is strongly affected by the α' corrections, $s^{(t)}$ is attracted to a positive value, while $s^{(s)}$ decreases and becomes even negative. Then, when the universe reaches the FLRW radiation-dominated era with a constant dilaton field, the higher-order corrections identically vanish and we recover $s^{(s)}=s^{(t)}=1$.

For tensor-type perturbations, the influence of the (all-time positive) frequency shift has been recently investigated in [29]. Its contribution has been found to amount to an overall rescaling, by a numerical factor of order unity, of the total energy density of the background, but does not alter the spectral index of the high-frequency modes in Eq. (59). For a scalar-type perturbation, however, the situation is more complex, due to the sign change of the effective frequency shift during the high-curvature regime. For a comoving mode evolving inside the Hubble radius at the onset of the high-curvature phase, the sign change is *a priori* responsible for turning the oscillatory behavior of the perturbation variable into exponential growth or exponential suppression. In the latter case, this may imply an abrupt end point in the spectral distribution, where the highest frequency mode to get amplified corresponds to the one that left the Hubble radius just before the onset of the string high-curvature phase. In the former case, on the contrary, the growth of the amplitude of the perturbation may be exponential until it exits the Hubble radius. Neither case seems satisfactory, and the change of sign is really an indication that the scalar fluctuations should be considered in the light of the full string action in this regime. For example, including higher-order curvature contributions to probe the highly curved regime, we have focused on the next to leading order in the infinite series of the inverse string tension expansion. Although this represents a significant improvement compared to previous studies, it still places a limit to our ability to analyze the results. It is natural to ask how sensitive our results are with respect to the inclusion of even higher-order derivative terms, such as R^3 or $R_{\mu\nu;\rho}R^{\mu\nu;\rho}$. To estimate the impact of such terms, while conserving equations of motion to second-order in the fields, we have looked at the relative contribution of $(\partial_\mu\phi)^4$ and $(\partial_\mu\phi)^6$. Indeed this very preliminary analysis indicates that the frequency shift may no longer be negative by including the infinite series expansion. The tensor contributions are not affected by this kind of higher order terms.

VI. CONCLUSION

In this paper, we have for the first time presented a complete set of equations describing the classical evolution of scalar, vector and tensor perturbations in the context of a nonsingular cosmology arising out of higher-order corrections to the low-energy string action. From these we have obtained the large-scale exact solutions for both the scalar and tensor perturbations, the associated large-scale conserved quantities and shown how there is general conservation of the angular momentum for the vector perturbations in the absence of dissipative processes. The perturbed actions for the scalar- and tensor-type perturbations have been obtained, thus laying ground work for future applications to the quantum generation processes. In particular we have calculated the final spectra of the scalar- and tensor-type perturbations generated from the vacuum quantum fluctuations of the metric and the field, given the nonsingular background evolution during the kinetic driven inflation.

Our results are valid for the general action of Eq. (1), and for the general perturbed FLRW metric in Eq. (8). Moreover, our results up to Sec. IV that describe the classical evolution of the background and the perturbations are generally valid without assuming any cosmological scenario, like the pre–big-bang or other types of inflation models. In order to calculate the initial conditions for the seed structures generated from the quantum fluctuations, we need to specify a cosmological scenario for the background. As an example, we have applied our general results to the case of the pre–big-bang scenario of string cosmology. Our results confirm previous expectations, that the low frequency modes, crossing the Hubble radius in the low-curvature regime, are unaffected by higher-order corrections. However, the higher-order corrections generally reduces the spectral indices for the high-frequency modes which are leaving the Hubble radius during the de Sitter–like era. Although still not satisfying the observational bounds, they provide an indication of a possible resolution for suitable higher-order corrections.

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