Perturbation spectrum in inflation with a cutoff

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It has been pointed out that the perturbation spectrum predicted by inflation may be sensitive to a natural ultraviolet cutoff, thus potentially providing an experimentally accessible window to aspects of Planck scale physics. *A priori*, a natural ultraviolet cutoff could take any form, but a fairly general classification of possible Planck scale cutoffs has been given. One of those categorized cutoffs, also appearing in various studies of quantum gravity and string theory, has recently been implemented into the standard inflationary scenario. Here, we continue this approach by investigating its effects on the predicted perturbation spectrum. We find that the size of the effect depends sensitively on the scale separation between cutoff and horizon during inflation.

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I. INTRODUCTION

During the inflationary phase of the very early universe (see e.g. [1] for an overview and references) space-time is assumed to expand in a quasiexponential fashion. Quantum fluctuations of the inflaton field are continuously redshifted until their wavelength equals the physical horizon distance, whereupon they become ‘‘frozen’’ until they reenter the Hubble volume during the ensuing radiation or matter dominated epochs. These fluctuations are thought to be responsible for seeding the temperature fluctuations of the cosmic microwave background radiation (CMBR) and the gravitational clustering of matter, whose statistical properties may therefore provide a window into the realm of high-energy physics.

Crucially, in the case of a sufficiently long period of inflation, all of the scales of cosmological interest today correspond to wavelengths below the Planck length early on when the initial conditions are prescribed. Therefore, inspired by similar studies in the context of Hawking radiation [2–8], a series of papers [9–12] has investigated the sensitivity of the predictions of inflationary scenarios with respect to changes of trans-Planckian physics. Those studies encoded trans-Planckian physics in a simple way as nonlinearities of the dispersion relation of the Fourier mode functions (see also [13] for a different application of this ansatz).

Since linearity of the field and hence Gaussianity of the fluctuations remains unchanged, the potential consequences of such modifications are limited to a possible scale dependence of the power spectrum and a possible change in its overall amplitude. It was shown [12] that under rather general conditions on the dispersion relation no observable effects can be expected, although Ref. [9] reaches a somewhat different conclusion. However, those studies suffer from fundamental limitations. First of all, with the exception of Ref. [11] all of the employed dispersion relations were chosen *ad hoc* so as to provide bounds on the frequency, wavelength or both without reference to an underlying theory. More importantly, the question of mode generation, i.e. how each semiclassical quantum field degree of freedom emerges out of the Planck regime, has not been addressed.

In contrast, Ref. [14] proposes a scenario where the UV cutoff is provided by a modified quantum mechanical commutation relation that limits the experimentally attainable resolution of small spatial distances [23]. This UV cutoff is one of very few types of short-distance structures that appear in the classification presented in [15], which applies to all quantum gravity theories that effectively represent each dimension by a linear operator. Indeed, corresponding short-distance uncertainty relations of this kind have appeared in various studies of quantum gravity and string theory, see e.g. [16]. In Ref. [14] this short distance cutoff has been implemented into the theory of a minimally coupled scalar field living in an expanding Friedmann-Robertson-Walker (FRW) background and it has been shown how the decoupling degrees of freedom are continuously generated dynamically at the time of their ‘‘Planck scale crossing.’’ Here, we aim to extend the analysis of [14] by estimating the magnitude of any corrections to the standard predictions for the statistical distribution of inflationary perturbations arising from the modified short-distance behavior.

The approach in Ref. [14] which we will follow here utilizes that, as has been shown in [17], the quantum gravity and stringy uncertainty relation cutoff (see e.g. [16]) can be modeled by corrections to the commutation relations

$$[\mathbf{x},\mathbf{p}]=i(1+\beta\mathbf{p}^2) \tag{1}$$

and its higher dimensional generalizations. It is easy to check that such correction terms give rise to a lower bound $\Delta x_{min}=\sqrt{\beta}$ for distance measurements. The form of these correction terms is unique to first order in β. Correspondingly, the signature of the first order effects of this type of natural cutoff should be unique when moving up from low energies, see [18].

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Hilbert space representations of relations of the type of Eq. (1) are given by introducing an auxiliary variable ρ which is essentially the momentum variable p but differs from it at small distances, i.e. at distances close to Δx_{min}. While this is initially a quantum mechanical structure, it can be implemented into quantum field theory, see Ref. [14]. Within the scalar quantum field theory on an inflationary background as defined in Ref. [14] one finds that, interestingly, those variables, $\tilde{k}$, in which the mode equations decouple, no longer strictly coincide with the comoving momentum variables, k, although they do of course approximately coincide for small k, i.e. for large distances. Conversely, this means that the comoving momentum modes now decouple only when they have grown to large proper distances and that the comoving momentum modes do couple initially when they emerge from the cutoff scale. For the quantum theory of the actual mode creation mechanism, see Ref. [14]. Explicitly, one obtains within this framework the following mode equation for the decoupling $\tilde{k}$ modes:

$$\phi''_{\tilde{k}}+\frac{\nu'}{\nu}\phi'_{\tilde{k}}+\left(\mu-3\left(\frac{a'}{a}\right)'-9\left(\frac{a'}{a}\right)^2-\frac{3a'\nu'}{a\nu}\right)\phi_{\tilde{k}}=0. \tag{2}$$

Here, a is the scale factor of the FRW line element and we defined the functions

$$\mu(\eta,\tilde{k}):=-\frac{a^2\mathrm{plog}(-\beta\tilde{k}^2/a^2)}{\beta(1+\mathrm{plog}(-\beta\tilde{k}^2/a^2))^2} \tag{3}$$

$$\nu(\eta,\tilde{k}):=\frac{e^{-3/2\,\mathrm{plog}(-\beta\tilde{k}^2/a^2)}}{a^4(1+\mathrm{plog}(-\beta\tilde{k}^2/a^2))} \tag{4}$$

that utilize the "product log" plog, which is the inverse of the function $x\rightarrow xe^x$. The solutions are automatically defined only from a finite value of η, i.e. every mode possesses its "creation time." It is the time when, in terms of proper distances, the mode equals the size of the cutoff scale, i.e. it is the conformal time η_c defined implicitly by

$$a(\eta_c)=\tilde{k}\sqrt{e\beta}\sim\tilde{k}\,\Delta x_{min}. \tag{5}$$

At the creation time, the differential equation possesses what is called an irregular singular point. To see this, note that the function plog which enters the differential equation through the functions μ and ν is not analytic at the creation time. Below, we will further discuss possible implications for the choice of initial conditions and therefore for the choice — or possible uniqueness — of the initial vacuum.

All physical observables in our model universe can be expressed in terms of $\tilde{k}$ instead of the usual Fourier variable k. This argument applies also to the transfer function $T(t,\tilde{k})$ which relates today's observable perturbations to the horizon crossing amplitude of $\phi_{\tilde{k}}$, provided that the perturbation amplitudes are measured as a function of $\tilde{k}$. In practice, these measurements are carried out on cosmological scales where $\tilde{k}=k$ to extremely good accuracy, so we do not expect any consequences from the relabeling of physical observables such as the angular size distribution of cosmic microwave background radiation (CMBR) fluctuations. In other words, any statement about the scale dependence and Gaussianity of the horizon crossing amplitudes of $\phi_{\tilde{k}}$ translate into corresponding statements about CMBR fluctuations, at least to the same extent as in the standard theory. Let us note, however, that this would not be true if the cutoff had different properties for different fields, e.g. if linear metric fluctuations behave differently on small scales than the inflaton field. The following analysis assumes that this is not the case [24].

Equation (2) is linear in $\phi_{\tilde{k}}$ so that Gaussianity of the distribution of fluctuations in $\tilde{k}$-space is protected. Consequently, we expect no deviations from Gaussianity owing to the proposed modifications of the short-distance behavior. We can therefore restrict attention to examining possible new effects on the scale dependence and overall amplitude of the power spectrum.

II. ANALYSIS IN OSCILLATOR VARIABLES

It turns out to be very convenient to change from the field variables used in Ref. [14] to slightly new variables defined by

$$\varphi_{\tilde{k}}\equiv\nu^{1/2}\phi_{\tilde{k}}. \tag{6}$$

Indeed, while the mode equation Eq. (2) in terms of the original field ϕ is of the type of a harmonic oscillator with friction, there is no friction term in the mode equation when written in terms of the new variable φ:

$$\varphi''_{\tilde{k}}+\omega^2(\eta)\varphi_{\tilde{k}}=0 \tag{7}$$

where $\omega(\eta)$ obeys the time dependent, nonlinear dispersion relation

$$\omega^2(\eta)=\mu-6\left(\frac{a'}{a}\right)^2+\left(\frac{\nu'}{2\nu}\right)^2-\frac{3(a'\nu'+a''\nu)}{a\nu}-\frac{\nu''}{2\nu}. \tag{8}$$

The Wronskian condition from Ref. [14] now also simplifies to

$$\varphi_{\tilde{k}}\varphi^{*\prime}_{\tilde{k}}-\varphi'_{\tilde{k}}\varphi^*_{\tilde{k}}=i \tag{9}$$

as usual. Note also that if we denote the standard field mode with a vanishing minimum position uncertainty as $\chi_{\tilde{k}}=\varphi_{\tilde{k}}(\beta\rightarrow0)$, we obtain the usual equation of motion for the $\tilde{k}$ mode of a free, minimally coupled scalar field in an expanding FRW space-time, where χ is in the conventions, of e.g., [19,25]:

$$\chi''_{\tilde{k}}+\omega_0^2\chi_{\tilde{k}}=0 \tag{10}$$

with

$$\omega_0^2=\tilde{k}^2-\frac{a''(\eta)}{a(\eta)}. \tag{11}$$

Again, there is the question of initial conditions for $\varphi_{\tilde{k}}$ that determine the vacuum for $\hat{\phi}$. Ideally, regularity arguments at the irregular singular point of the mode equation, encountered at the creation time η_c for each mode, should fix the choice. We do not have a definite answer at this point, but asymptotic methods will shed some light on the situation. Some indications of vacuum fixation by regularity arguments are sketched in the Conclusions. Indeed, a solution of the singularity problem is not strictly necessary for the present analysis. It will be shown below that the evolution of $\varphi_{\tilde{k}}$ is essentially adiabatic from a certain time η_i onwards. The state of $\hat{\phi}$ at $\eta \geqslant \eta_i$ can be determined by consistency arguments to be the adiabatic vacuum (e.g., [19])

$$\varphi_{\tilde{k}}(\eta) = \frac{1}{\sqrt{2\omega(\eta)}} \exp\left(-i\int_{\eta_i}^{\eta} \omega(\tilde{\eta}) d\tilde{\eta}\right), \qquad (12)$$

where the normalization follows from Eq. (9). This is because, as argued in Refs. [12,20], any small deviation from the adiabatic vacuum close to the Planck scale would likely suppress inflation altogether due to back-reaction of the energy density contained in $\varphi_{\tilde{k}}$ on the cosmic expansion. In order to be consistent with the assumptions of Ref. [14] (i.e., negligible back-reaction), any admissible initial condition needs to converge to the adiabatic vacuum as soon as the latter is well defined.

III. ADIABATIC ANALYSIS

Equation (7) belongs to the class of harmonic oscillator equations featuring a dispersion relation that is asymptotically linear for small physical wave numbers but becomes nonlinear at high wave numbers (small wavelengths). In the context of cosmology, such systems were investigated in Refs. [9,10,12], and in the framework of Hawking radiation many times before (see [21] for references). Unlike in the above references, where the dispersion relation was typically tailored *ad hoc* to fit some desired shape, Eq. (8) followed directly from a general study of realistic short-distance structures of space-time and may therefore perhaps be considered more fundamental (see also Ref. [11] for a similar approach).

It is useful to express the separation between the cutoff scale (here parametrized by $\beta^{1/2}$) and the inflationary horizon scale in terms of the dimensionless parameter σ defined as

$$\sigma \equiv \beta^{1/2} H. \qquad (13)$$

If $\sqrt{\beta} \sim \Delta x_{min}$ is identified with the Planck length, the amplitude of temperature fluctuations of the cosmic microwave background indicates that $\sigma \sim 10^{-5}$ at the time when the presently observable scales left the horizon during inflation.

In order to generalize the notion of "horizon crossing" to our nonstandard equation of motion, we Taylor-expand Eq. (8) around $\sigma = 0$ and find that $\omega(\eta)^2 = \omega_0^2 + O(\sigma^2)$. Correspondingly, the usual definition of horizon crossing in terms of $\tilde{k} = aH$ is valid to within the same accuracy.

We are interested in sources of deviation from the standard (i.e., $\beta \to 0$) result for the scale dependence and overall normalization of the horizon crossing amplitude of $\phi_{\tilde{k}}$. Following Sec. II and recognizing that $\varphi_{\tilde{k}} = \phi_{\tilde{k}} + O(\sigma^2)$ at horizon crossing, we need to compare the amplitudes of $\varphi_{\tilde{k}}$ and $\chi_{\tilde{k}}$ at the horizon crossing time η_h, which is when $\tilde{k} \approx a(\eta_h) H$ [26].

One possible signature of the cutoff in the spectrum is due to nonadiabatic particle production during the evolution from η_i to η_h, which may give rise to a modulation of $\varphi_{\tilde{k}}(\eta_h)$ around the amplitude predicted for $\beta \to 0$ [12]. This may, in turn, be reflected by a breaking of scale invariance of the perturbation power spectrum. The *relative* magnitude of this effect, denoted in Ref. [12] as β_k, can be shown to be bounded by the maximum of the adiabaticity parameter

$$\mathcal{C}(\eta) = \left| \frac{\omega'(\eta)}{\omega^2(\eta)} \right|. \qquad (14)$$

If $\mathcal{C} \lesssim 1$, the usual notions of semiclassical quantum field theory in nonstationary space-times apply, with the adiabatic vacuum, Eq. (12), serving as a natural ground state. One finds numerically that $\mathcal{C} \approx 1$ for $\eta = \eta_i$ and drops to negligible levels afterwards, where

$$\eta_i \approx \eta_c (1 + \sigma^2) \qquad (15)$$

for σ ranging from 10^{-7} to 0.1. Perhaps not surprisingly, the beginning of adiabaticity approaches the initial singularity arbitrarily closely if cosmic expansion becomes sufficiently slow. Conversely, any bound on nonadiabatic particle production due to the cutoff derived from inflation is stronger than the equivalent bound from cosmic expansion today. Furthermore, if one defines a natural time scale for the evolution of $\varphi_{\tilde{k}}$ at the time η_i as $\tau(\eta_i) \sim 1/\omega(\eta_i)$, one can check numerically that $\eta_i \approx 0.75\tau$. In other words, the "non-adiabatic epoch" following the Planck scale crossing of each mode lasts about as long as the typical evolution time scale of the mode itself. Whatever physics determines this phase remains, for the time being, unknown. However, as argued in Sec. II, self-consistency demands the solution to converge onto the adiabatic vacuum as soon as it is well defined (i.e., as soon as $\mathcal{C} \ll 1$) [12,20] and this is the case for all $\eta \gtrsim \eta_i$.

Having shown that scale invariance is preserved if σ is small, we need to consider the overall amplitude of the power spectrum. Taking the adiabatic solution Eq. (12) as a reasonable approximation to the exact functions $\varphi_{\tilde{k}}(\eta)$ and $\chi_{\tilde{k}}(\eta)$ on length scales larger than the cutoff but smaller than the horizon scale (where expansion violates adiabaticity), i.e. for times $\eta_i \ll \eta \ll \eta_h$, one finds that

$$D(\eta) \equiv \frac{\varphi_{\tilde{k}}(\eta)}{\chi_{\tilde{k}}(\eta)} = \left(\frac{\tilde{k}}{\omega(\eta)} \right)^{1/2}. \qquad (16)$$

A good estimate for the impact of the nonlinear dispersion relation on the amplitude of the power spectrum is obtained by noting that this expression for $D(\eta)$ remains approximately valid until η_h, since cosmic expansion affects both solutions in roughly the same way. It is readily verified in this case that $D(\eta_h) = 1 + O(\sigma^2)$. Hence, the impact of the cutoff on the perturbation spectrum depends crucially on the separation between the cutoff and the Hubble scale, being negligible if $\sigma \ll 1$.

IV. SCALING ANALYSIS

The scaling behavior of the perturbation spectrum can also be investigated by studying the scaling behavior of the wave equation, Eq. (2). Let us begin by considering the case of an exactly de Sitter type expansion. In this case, we expect that time translation invariance is broken neither by our introduction of a cutoff nor by the background expansion. We therefore expect a scale invariant perturbation spectrum.

Indeed, we first observe that if $\phi_{\tilde{k}}(\eta)$ is a solution to the $\tilde{k}$ mode equation and r is any arbitrary positive number then $\phi_{\tilde{k}}(r\eta)$ is a solution of the mode equation for the mode $r\tilde{k}$. This is straightforward to verify and it is of course also true for the usual inflationary scenario without a cutoff.

The solutions $\phi_{r\tilde{k}}(\eta)$ that are obtained in this way by scaling the solution $\phi_{\tilde{k}}(\eta)$ all obey of course the same initial conditions. We can also conclude that if η is a special time for the solution $\phi_{\tilde{k}}$, then, correspondingly, η/r is a special time for the solution $\phi_{r\tilde{k}}$. For example, if we denote the creation and the horizon crossing times of the mode $\phi_{\tilde{k}}$ by η_c and η_h, then the mode $\phi_{r\tilde{k}}(\eta)$ possesses the creation and the horizon crossing times $\eta_c(r\tilde{k}) = \eta_c/r$ and $\eta_h(r\tilde{k}) = \eta_h/r$.

Let us further assume that the solution $\phi_{\tilde{k}}(\eta)$ is normalized with respect to the Wronskian condition. We also need that all the solutions $\phi_{r\tilde{k}}(\eta)$ are normalized with respect to the Wronskian condition for the respective $r\tilde{k}$ modes. As is straightforward to verify, the ansatz

$$\phi_{r\tilde{k}}(\eta) = N(r)\,\phi_{\tilde{k}}(r\eta) \tag{17}$$

yields

$$N(r) = r^{3/2} \tag{18}$$

so that $\phi_{r\tilde{k}}(\eta) = r^{3/2}\phi_{\tilde{k}}(r\eta)$, and therefore

$$\phi_{r\tilde{k}}(\eta/r) = r^{3/2}\phi_{\tilde{k}}(\eta). \tag{19}$$

Choosing for η the horizon crossing time of the $\tilde{k}$ mode we now obtain how the horizon crossing amplitude scales when scaling the decoupling momentum

$$\phi_{r\tilde{k}}(\eta_h(r\tilde{k})) = r^{3/2}\phi_{\tilde{k}}(\eta_h) \tag{20}$$

which means

$$\phi_{r\tilde{k}}(\text{horizon crossing}) \sim r^{3/2}. \tag{21}$$

In order to make contact with the conventions in the literature, let us now recall that, usually, field variables $\psi(\eta,k)$ in comoving momenta k are obtained by first scaling from proper position coordinates to comoving position coordinates and then, second, by Fourier transforming to the comoving momentum. In [14], however, we obtained fields $\phi(\eta,k)$ over comoving momenta k by first Fourier transforming from proper positions to proper momenta and then, second, by scaling to comoving momenta. However, scaling and Fourier transforming do not commute. As a consequence, as is readily verified,

$$\phi(\eta,k) = a^3\psi(\eta,k) \tag{22}$$

and in the de Sitter case

$$\phi(\eta,k) = -\frac{H^3}{\eta^3}\psi(\eta,k). \tag{23}$$

As far as present day observations of cosmological scales are concerned, the distinction between comoving and decoupling momenta does not matter and we therefore obtain from Eq. (20)

$$\psi(\eta_h/r, rk) = r^{-3/2}\psi(\eta_h, k). \tag{24}$$

We therefore finally obtain for the fields over comoving momenta as conventionally defined the scaling behavior of the horizon crossing amplitude

$$\psi(\text{horizon crossing}, rk) \sim r^{-3/2} \tag{25}$$

which yields indeed the usual scale invariant spectrum:

$$\langle 0|\psi^{\dagger}(\text{horizon crossing}, rk)\,\psi(\text{horizon crossing}, rk)|0\rangle \sim r^{-3}. \tag{26}$$

Indeed, this was to be expected because neither the background de Sitter space, nor our introduction of a cutoff, nor the choices of initial conditions (all solutions being obtained from another by mere scaling) broke time translation invariance.

On the other hand, in the case of a non–de Sitter background, the spectrum is of course not scale invariant. In the presence of our cutoff we will then obtain additional scale invariance breaking effects on the spectrum, because of the new cutoff dependent terms in the wave equation.

V. CONCLUSIONS

We investigated the signature of the cutoff in the perturbation spectrum from two perspectives, and in both cases we did not need to solve the mode equation explicitly. The adiabatic treatment in Sec. III is based on the fact that in order to be consistent with inflation, each mode needs to be in the adiabatic vacuum shortly after the mode is created, whereas the scaling analysis of Sec. IV utilizes that the wave equation scales trivially and that there is also no reason for the (still unknown) initial conditions to break the (almost) time-translation invariance of the background space-time. Both approaches show that the resulting fluctuation power spectrum is indeed scale invariant if the background space-time is de Sitter. The adiabatic analysis, in addition, shows that any corrections of the overall amplitude are at most of order σ^2, where σ is the ratio of the horizon scale and the minimum spatial resolution Δx_{min} admitted by the commutation relation Eq. (1).

While a detailed analysis in the framework of slow-roll inflation would be desirable, we expect the following first order effect in a more general inflationary space-time. For small σ, the dispersion relation, Eq. (8), can be approximated by

$$\omega^2 = \omega_0^2 + B\,\sigma^2 + O(\sigma^4), \tag{27}$$

with

$$B = \frac{3k^4}{a^2 H^2} + \frac{5k^2}{2H^2}\left(\frac{a''}{a^3} - H^2\right). \tag{28}$$

During inflation, the pressure is roughly equal to the negative energy density, so that the Friedmann equations yield $a''/a^3 \approx 2H^2$. Therefore, $B>0$ which implies that the dispersion relation is superluminal in the region of interest. Owing to the normalization in Eq. (12) (cf. Refs. [10,12]), the perturbation amplitude drops more quickly than usual as H declines with time, giving rise to additional reddening of the spectrum. Evidently, this effect vanishes as either the slow-roll parameter or σ go to zero.

The scale Δx_{min} at which a natural ultraviolet cutoff sets in could be as small as the $(3+1)$-dimensional Planck scale of 10^{-35} m, but the natural short distance cutoff scale may well be larger, as could be the case, e.g., in string theory and theories of large extra dimensions. Evidently, the signature of the cutoff in the CMBR would increase if the cutoff scale were larger than the Planck length during inflation. On the other hand, both the scale dependence and the amplitude of the power spectrum are very sensitively dependent on the details of the inflaton potential. Only after a concrete model for inflation has been specified one can derive an upper bound on Δx_{min} from observations.

In particular, if we assume conventional slow roll inflation with the inflaton coupling fine-tuned such as to obtain the observed amplitude of the CMBR perturbations, then a cutoff Δx_{min} at the Planck length suggests $\sigma \sim 10^{-5}$. The cutoff induced corrections to the perturbation spectrum would then be neglible, i.e., the conventional scenario with its parameters fine-tuned as usual is observationally consistent with a cutoff Δx_{min} at the Planck scale.

On the other hand, we may view Δx_{min} simply as a new free parameter in model building. For example, it might be possible to gain some more freedom in choosing the potential — for the prize of having to fine-tune Δx_{min}.

An interesting technical question remains: We have not shown how or even if the decoupling modes evolve into the adiabatic vacuum from some natural initial conditions at the singularity. Two possibilities can be imagined: either there exists a symmetry or regularity condition that uniquely specifies initial conditions that later evolve into the adiabatic solution. In this case the discussion in Sec. III applies.

Or, alternatively, the modes are generally created in a highly excited state as seen from the point of view of a comoving particle detector. This case would be inconsistent [12,20] with the assumption of slow-roll inflation made at the onset of Ref. [14], indicating that the combination of short-distance uncertainty of the kind described by Eq. (1) and inflation is not, in general, self-consistent.

We will conclude with some speculative ideas about the first possibility for the initial conditions at the singularity. Starting with the original equation of motion, Eq. (2), expanding the coefficients around $\eta = \eta_c$, and shifting the origin of the time coordinate to η_c, one obtains a differential equation of the form

$$\phi''_{\tilde{k}} - \frac{1}{2\eta}\phi'_{\tilde{k}} + \frac{A}{\eta}\phi_{\tilde{k}} = 0 \tag{29}$$

which can be solved analytically

$$\phi_{\tilde{k}}(\eta) = C_1 F(\eta) + C_2 F(\eta)^*, \tag{30}$$

where

$$F(\eta) = \left(\frac{\sqrt{A}}{2} + iA\sqrt{\eta}\right)\exp(-2i\sqrt{A\eta}). \tag{31}$$

The two constants can be specified in formal analogy with the standard procedure by picking the positive "frequency" branch and normalizing according to the Wronskian condition. The result is regular at $\eta = 0$. A preliminary analysis appears to indicate that there exists a unique solution for which $\phi^\dagger \phi$ is analytic at creation time and that it corresponds to this solution. If this solution indeed evolves into the later adiabatic vacuum solution then this would be a desirable intrinsic mechanism for fixing the vacuum [22,27].

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[23] The same approach was taken in the black hole context in Ref. [7].
[24] At horizon crossing of the mode $\tilde{k}$, i.e., when $\tilde{k}\approx aH$, we also note that $\tilde{k}^2$ and k^2 differ only by the constant factor $-\sigma^{-2}\mathrm{plog}(-\sigma^2)$, independent of k, where σ is defined in Eq. (13).
[25] Of course, also $\tilde{k}=k$ for $\beta\rightarrow 0$. However, for the reasons explained above we prefer to label everything in terms of $\tilde{k}$ in order to avoid discussing the $\tilde{k}\rightarrow k$-map.
[26] Equivalently, we could compare $\phi_{\tilde{k}}$ and $a^2\chi_{\tilde{k}}$, as $\nu\rightarrow a^{-4}$ for $\beta\rightarrow 0$.
[27] The authors of Ref. [22], which appeared after this work was first posted, use Eqs. (30),(31) as the leading term in the initial conditions of a numerical analysis. Setting $C_2=0$, they reproduce the same perturbation amplitude as the one predicted by starting in the adiabatic vacuum, providing some support for our conjecture.