Unitarity and interfering resonances in $\pi\pi$ scattering and in pion production $\pi N \to \pi\pi N$

M. Svec*
Physics Department, Dawson College, Montreal, Quebec, Canada H3Z 1A4
and Physics Department, McGill University, Montreal, Quebec, Canada H3A 2T8
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The additivity of Breit-Wigner phases has been proposed to describe interfering resonances in partial waves in $\pi\pi$ scattering. This assumption leads to an expression for partial wave amplitudes that involves products of Breit-Wigner amplitudes. We show that this expression is equivalent to a coherent sum of Breit-Wigner amplitudes with specific complex coefficients which depend on the resonance parameters of all contributing resonances. We use the analyticity of $\pi\pi$ partial wave amplitudes to show that they must have the form of a coherent sum of Breit-Wigner amplitudes with complex coefficients and a complex coherent background. The assumption of the additivity of Breit-Wigner phases is a new constraint on partial wave amplitudes independent of partial wave unitarity. It restricts the partial waves to analytical functions with a very specific form of residues of Breit-Wigner poles. Since there is no physical reason for such a restriction, we argue that the general form provided by the analyticity is more appropriate in fits to data to determine resonance parameters. The partial wave unitarity can be imposed using the modern methods of constrained optimization. We discuss the production amplitudes in $\pi N \to \pi\pi N$ reactions, and use analyticity in the dipion mass variable to justify the common practice of writing the production amplitudes in production processes as a coherent sum of Breit-Wigner amplitudes with free complex coefficients and a complex coherent background in fits to mass spectra with interfering resonances. The unitarity constraints on $\pi\pi$ partial wave amplitudes with resonances determined from fits to mass spectra of production amplitudes measured in $\pi N \to \pi\pi N$ reactions can be satisfied with an appropriate choice of complex residues of contributing Breit-Wigner poles.

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I. INTRODUCTION

In the 1930s, Breit and Wigner introduced [1,2] a parametrization of resonances observed in the energy dependence of integrated and differential cross sections of nuclear reactions. The original Breit-Wigner formula was only a one-resonance approximation and its justification was initially only phenomenological. A theoretical justification for the Breit-Wigner formula later emerged from quantum collision theory [3]. The evident existence of multiple and overlapping resonances in nuclear reactions led to two distinct generalizations of the Breit-Wigner formula for an isolated resonance to multiresonance description of the scattering process.

One generalization was undertaken by Feshbach [4,5], Humblet [6], and McVoy [7], who used the analyticity properties of the *S* matrix to show that the transition matrix can be written as a coherent sum of Breit-Wigner terms with complex coefficients and a coherent background. Since the transition matrix must satisfy unitarity, the parameters and coefficients of this multiresonance parametrization are not independent [5,8]. In principle it is possible to use the methods of nonlinear programming [9,10] and constrained optimization with computer programs such as MINOS developed at Stanford University [11] to impose the conditions of unitarity in fitting the experimental data.

Another approach to the multiresonance description of scattering processes was proposed by Hu in 1948 [12]. He observed that the Breit-Wigner contribution of an isolated resonance to the *S* matrix is unitary, and proposed describing the multiresonance contributions in the *S* matrix by the product of isolated Breit-Wigner contributions for each resonance. Since each term is unitary, the product also satisfies unitarity. The partial wave phase shift is then a sum of Breit-Wigner phases of contributing resonances and a background phase. As a result, the expressions for partial wave amplitudes involve products of Breit-Wigner amplitudes. This method was recently used by Bugg *et al.* [13] and Ishida *et al.* [14] in their analyses of $\pi\pi$ phase shift data.

Up to now the connection between these two descriptions of multiresonance contributions (interfering resonances) has not been clarified. In this work we show that the Hu description is a special case of a more general description based on analyticity. We show that the Hu method also leads to a coherent sum of Breit-Wigner amplitudes with complex coefficients and a complex coherent background for any partial wave, as expected from the analyticity of the *S* matrix. However, the complex coefficients have a very specific form in terms of resonance parameters of all contributing resonances. The assumption of the additivity of Breit-Wigner phases is a new constraint that restricts the partial waves to analytical functions with these specific residues of Breit-Wigner poles. Furthermore, we show that the additivity of Breit-Wigner phases is an assumption entirely independent of the unitarity property of partial wave amplitudes, which is a condition imposed on their inelasticity.

Since there is no physical reason why physical partial waves must have the form of a coherent sum of Breit-Wigner amplitudes with specific complex coefficients required by the additivity of Breit-Wigner phases, we conclude that the general form imposed by the analyticity is more appropriate for fits to data to determine resonance parameters. This conclu-

*Electronic address: milo@smetana.physics.mcgill.ca

sion is particularly relevent for analysis of interfering resonances in the mass spectra in production processes such as $\pi N \rightarrow \pi\pi N$ or $pp \rightarrow \pi\pi pp$. Using analyticity in the invariant mass variables, we justify the common practice of parametrizing the production amplitudes in terms of a coherent sum Breit-Wigner amplitudes with free complex coefficients and a complex coherent background [15–23].

The paper is organized as follows. In Sec. II we briefly review the unitarity and the problem of interfering resonances in potential scattering since it motivates the analysis in hadronic reactions. In Sec. III we review the two-body partial wave unitarity in $\pi\pi$ scattering and its relation to the general form of isospin partial waves. In Sec. IV, we introduce the assumption of additivity of Breit-Wigner phases in the $\pi\pi$ scattering, and show that it leads to partial waves in a form of a coherent sum of Breit-Wigner amplitudes with specific complex coefficients and a coherent background. In Sec. V we generalize dispersion relations for partial wave amplitudes in $\pi\pi$ scattering to Breit-Wigner poles, and show that the form obtained from the additivity of Breit-Wigner phases is a special case. In Sec. VI we focus the discussion of the two methods to a finite energy interval, and argue that the addition of Breit-Wigner phases imposes an unjustified constraint on fits to data. In Sec. VII we formulate unitarity for production amplitudes in $\pi^- p \rightarrow \pi^- \pi^+ n$ reaction and contrast it with partial wave unitarity in $\pi\pi$ scattering. In Sec. VIII we show that the method of addition of Breit-Wigner phases can be generalized to production amplitudes. We also use analyticity in the invariant mass to obtain a more general form for production amplitudes in terms of a coherent sum of Breit-Wigner amplitudes with free complex coefficients (pole residues) and a complex coherent background. We argue that this general form is more appropriate in fits to measured mass spectra. Although the discussion is confined to pion production amplitudes in $\pi N \rightarrow \pi\pi N$, the conclusions have a general validity. We also comment on determination of $\pi\pi$ partial wave amplitudes from resonance parameters determined in measurements of production amplitudes in $\pi N \rightarrow \pi\pi N$ reactions. The paper closes with a summary in Sec. IX.

II. UNITARITY AND INTERFERING RESONANCES IN POTENTIAL SCATTERING

A. Unitarity

We will consider the scattering of a spinless particle of mass m by a real, central potential $V(r)$ [24]. In the asymptotic form of the stationary scattering wave function, the outgoing wave is characterized by the scattering amplitude $f(k,\theta)$, where k is the wave number of the particle related to its energy by

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}, \tag{2.1}$$

and θ is the scattering angle. In units of $\hbar = 1$ the wave number k has the meaning of the momentum p. The scattering amplitude can be written in the form

$$f(k,\theta) = \sum_{l=0}^{\infty} (2l+1) T_l(k) P_l(\cos\theta). \tag{2.2}$$

The partial wave amplitudes T_l are given by

$$T_l = \frac{1}{2ik}[S_l(k) - 1], \tag{2.3}$$

where $S_l(k)$ is called the S matrix. For elastic scattering,

$$S_l = e^{2i\delta_l(k)}, \tag{2.4}$$

where the phase shifts δ_l describe the interaction and are related to the potential $V(r)$. For elastic scattering, $|S_l| = 1$, which is the condition of elastic unitarity.

When a particle collides with a target, nonelastic processes are possible, and particles are removed from the incident (elastic) channel. Since the interaction can alter only the outgoing part of the wave function, we require that the amplitude of the outgoing wave be reduced if nonelastic processes occur. The reduction of scattering amplitudes leads to conditions of inelastic unitarity:

$$|S_l| \leqslant 1. \tag{2.5}$$

This suggests that we write

$$S_l = \eta_l(k) e^{2i\delta_l(k)}, \tag{2.6}$$

where η_l is called the inelasticity, and has values

$$0 < \eta_l \leqslant 1. \tag{2.7}$$

The partial wave then has a general form

$$T_l = \frac{1}{2ik}[\eta_l e^{2i\delta_l} - 1]. \tag{2.8}$$

From Eq. (2.8) it follows that

$$\mathrm{Im}\, T_l = k|T_l|^2 + \frac{1}{4k}(1 - \eta_l^2). \tag{2.9}$$

This equation expresses the unitarity condition on the partial waves T_l.

B. Interfering resonances

In the following we will work with partial wave amplitudes

$$t_l = kT_l = \frac{1}{2i}[\eta_l e^{2i\delta_l} - 1], \tag{2.10}$$

and the energy E instead of k. A detailed study of the potential scattering [24] shows that the phase shift may be decomposed as $\delta_l = \xi_l + \rho_l$, where ξ_l is the background phase which does not depend on the shape and depth of the interaction potential $V(r)$ while the part ρ_l does depend on the details of the potential. Near resonant energy E_r,

$$\tan\rho_l \approx \frac{\Gamma(E)}{2(E_r - E)}, \tag{2.11}$$

where $\Gamma(E)$ is the width of the resonance. We introduce a Breit-Wigner resonance phase δ_l^r:

$$\delta_l^r = \tan^{-1}\left\{\frac{\Gamma(E)}{2(E_r - E)}\right\} = \arg\left[E_r - E + i\frac{1}{2}\Gamma(E)\right], \tag{2.12}$$

such that in the energy interval ΔE centered about E_r we have $\rho_l \approx \delta_l^r$ and

$$\delta_l \approx \xi_l + \delta_l^r. \tag{2.13}$$

From Eq. (2.12) it follows that

$$e^{2i\delta_l^r} = \frac{E - E_r - \frac{1}{2}i\Gamma}{E - E_r + \frac{1}{2}i\Gamma} = 1 + 2ia_l, \tag{2.14}$$

where

$$a_l = \frac{-\frac{1}{2}\Gamma}{E - E_r + \frac{1}{2}i\Gamma} \tag{2.15}$$

is the Breit-Wigner amplitude of the resonance r. For an isolated resonance we then obtain

$$t_l = \frac{1}{2i}(\eta_l e^{2i\xi_l} - 1) + \eta_l e^{2i\xi_l}\left(\frac{-\frac{1}{2}\Gamma}{E - E_r + \frac{1}{2}i\Gamma}\right). \tag{2.16}$$

If N resonances contribute over an interval ΔE then, following Hu [12] and Refs. [13,14], we can write

$$e^{2i\delta_l^r} = \prod_{n=1}^{N} \frac{E - E_r^{(n)} - \frac{1}{2}i\Gamma^{(n)}}{E - E_r^{(n)} + \frac{1}{2}i\Gamma^{(n)}} = \prod_{n=1}^{N}(1 + 2ia_l^{(n)}). \tag{2.17}$$

Prescription (2.17) clearly satisfies unitarity but seems to lead to a complicated expression for partial waves t_l in terms of Breit-Wigner amplitudes $a_l^{(n)}$.

On the other hand, the analyticity of the S matrix was used by Feshbach [4,5], Humblet [6] and McVoy [7] to derive a general form for t_l [5],

$$t_l = B_l(E) + \sum_{n=1}^{N} \frac{A_l^{(n)}(E)}{E - E_r^{(n)} + \frac{1}{2}i\Gamma^{(n)}(E)}, \tag{2.18}$$

where B_l is a background term and $A_l^{(n)}(E)$ are complex coefficients. The sum in Eq. (2.18) can be written as a coherent sum of Breit-Wigner amplitudes:

$$t_l(E) = B_l(E) + \sum_{n=1}^{N} R_l^{(n)}(E)\frac{-\frac{1}{2}\Gamma^{(n)}(E)}{E - E_r^{(n)} + \frac{1}{2}i\Gamma^{(n)}(E)}. \tag{2.19}$$

In Sec. IV we show that prescription (2.17) leads to an analytical form [Eq. (2.19)] with specific expressions for the coefficients $R_l^{(n)}$ and the background B_l.

III. ISOSPIN AMPLITUDES AND UNITARITY IN $\pi\pi \to \pi\pi$ SCATTERING

Hadron resonances have definite values of spin and isospin. It is therefore necessary to express the amplitudes for charged pion processes $\pi\pi \to \pi\pi$ in terms of isospin amplitudes $T^I(E,\theta)$, with a definite isospin $I = 0,1,2$, and work with partial wave amplitudes $T_l^I(E)$ [25]. At first we will work with the center-of-mass (c.m.) energy $E = \sqrt{s}$ to pursue the analogy with the potential scattering.

The partial wave amplitudes T_l^I satisfy partial wave unitarity equations [25,26]

$$\mathrm{Im}\, T_l^I = q|T_l^I|^2 + \Delta_l^I, \tag{3.1}$$

where Δ_l^I are contributions from inelastic channels, such as $\pi\pi \to \pi\pi\pi\pi$, $K\bar{K}$, $N\bar{N}$, and $q = \sqrt{\frac{1}{4}(s - 4\mu^2)}$ is c.m. momentum where μ is the pion mass.. Let us write Δ_l^I in the form

$$\Delta_l^I = \frac{1}{4q}[1 - (\eta_l^I)^2]. \tag{3.2}$$

Then the unitarity equation (3.1) has the same form as Eq. (2.9), and the partial waves T_l^I can be written as

$$T_l^I = \frac{1}{2iq}\{\eta_l^I e^{2i\delta_l^I} - 1\}, \tag{3.3}$$

where $\delta_l^I(E)$ are phase shifts and the inelasticity

$$\eta_l^I(E) = \sqrt{1 - 4q\Delta_l^I(E)} \tag{3.4}$$

is given by the inelastic unitarity contributions Δ_l^I. In analogy with potential scattering we expect that $0 < \eta_l^I \leqslant 1$. As we shall see later, the descriptions of interfering resonances in $\pi\pi$ scattering do not depend on the condition that $\eta_l^I \leqslant 1$.

The positivity of inelasticity η_l^I in Eq. (3.4) imposes a constraint

$$\Delta_l^I \leqslant \frac{1}{4q}. \tag{3.5}$$

We can now show that the unitarity equation (3.1) admits no solution for $\Delta_l^I > 1/4q$. If the inelastic unitarity contributions satisfy this condition, we can write

$$\Delta_l^I = \frac{1}{4q}[1 + (\eta_l^I)^2], \tag{3.6}$$

where $(\eta_l^I)^2 > 0$. Setting $T_l^I = V_l^I/2q$ from unitarity equation (3.1) we obtain

$$(\mathrm{Re}\, V_l^I)^2 + (\mathrm{Im}\, V_l^I - 1)^2 = -(\eta_l^I)^2, \tag{3.7}$$

which is not possible. Thus conditions (3.5) represent genuine constraints on the inelastic unitarity contributions Δ_l^I and the parametrization [Eq. (3.3)] of partial wave amplitudes with (3.4) is the most general solution of unitarity equation (3.1) for all E.

Since the values of inelastic terms Δ_l^I in partial wave unitarity equations like Eq. (3.1) are not known, we constrain the partial waves T_l^I by inequalities imposed by the unitarity. From the positivity of η_l^I and condition (3.5), we obtain

$$\mathrm{Im}\, T_l^I \leqslant q|T_l^I|^2 + \frac{1}{4q}. \tag{3.8}$$

If we add the requirement that $\eta_l^I \leqslant 1$, then $\Delta_l^I \geqslant 0$ from Eq. (3.2), and we also have the usual unitarity constraint

$$\mathrm{Im}\, T_l^I \geqslant q|T_l^I|^2. \tag{3.9}$$

Inequality (3.9) implies a positivity

$$\mathrm{Im}\, T_l^I \geqslant 0 \tag{3.10}$$

at all energies.

Finally we note the following observation. Let $f(z)$ be any complex function. Then $1+2if(z)$ is a complex function that can be written as

$$1+2if(z)=\eta(z)e^{2i\delta(z)}, \tag{3.11}$$

where $\eta > 0$ and δ is real. Thus any complex function can be written in the form

$$f(z)=\frac{1}{2i}(\eta e^{2i\delta}-1), \tag{3.12}$$

and satisfies the equation

$$\mathrm{Im}\, f=|f|^2+\frac{1}{4}(1-\eta^2). \tag{3.13}$$

We see that unitarity equations like Eq. (3.1) are a special case of Eq. (3.13) with η given by Eq. (3.4).

IV. INTERFERING RESONANCES IN $\pi\pi$ SCATTERING USING THE ADDITION OF BREIT-WIGNER PHASES

The general form of the phase shift parametrization of partial wave amplitudes T_l^I is

$$T_l^I=\frac{1}{2iq}[\eta_l^I e^{2i\delta_l^I}-1], \tag{4.1}$$

with the inelasticity η_l^I determined by unitarity via Eq. (3.4). In the following we will omit the indices l and I for simplicity. In analogy with potential scattering, we decompose the phase shifts δ into two parts:

$$\delta=\xi+\delta^r, \tag{4.2}$$

where ξ is the nonresonant background phase and δ^r is the phase due to physical particle resonances occurring in the partial wave T_l^I. The phase of a single isolated resonance is given by the Breit-Wigner formula

$$e^{2i\delta^r}=\frac{E-E_r-\frac{1}{2}i\Gamma(E)}{E-E_r+\frac{1}{2}i\Gamma(E)}. \tag{4.3}$$

Let us consider that N resonances contribute to the partial wave amplitude T_l^I. Following Refs. [12–14] we now assume that the resonant phase shift δ^r is given by the sum of the Breit-Wigner phases of the contributing resonances:

$$\delta^r=\sum_{n=1}^{N}\delta_n^r. \tag{4.4}$$

We assume that N is finite. Then

$$e^{2i\delta^r}=\prod_{n=1}^{N}e^{2i\delta_n^r}=\prod_{n=1}^{N}\frac{E-E_n-\frac{1}{2}i\Gamma_n}{E-E_n+\frac{1}{2}i\Gamma_n}. \tag{4.5}$$

For each Breit-Wigner phase we can write

$$e^{2i\delta_n^r}=1+2ia_n, \tag{4.6}$$

where a_n is the Breit-Wigner amplitude:

$$a_n=\frac{-\frac{1}{2}\Gamma_n}{E-E_n+\frac{1}{2}i\Gamma_n}. \tag{4.7}$$

Then we can write

$$e^{2i\delta^r}=1+2iT_{res}, \tag{4.8}$$

where T_{res} is given in terms of products of Breit-Wigner amplitudes a_n The partial wave amplitude T_l^I then has a general form

$$T=\frac{1}{2iq}(\eta e^{2i\xi}-1)+\frac{1}{q}e^{2i\xi}\eta T_{res}. \tag{4.9}$$

Let us consider the case $N=2$. Then the resonant part of the amplitude T_l^I is

$$T_{res}=a_1+a_2+2ia_1a_2, \tag{4.10}$$

where the interference term

$$a_1a_2=\frac{(-\frac{1}{2}\Gamma_1)(-\frac{1}{2}\Gamma_2)}{(E-E_1+\frac{1}{2}i\Gamma_1)(E-E_2+\frac{1}{2}i\Gamma_2)}. \tag{4.11}$$

With a notation

$$z_k=E_k-\frac{1}{2}i\Gamma_k, \tag{4.12}$$

we write

$$\frac{1}{(E-z_1)(E-z_2)}=\left[\frac{A}{E-z_1}+\frac{B}{E-z_2}\right]\frac{1}{C}. \quad (4.13)$$

The requirement that this equality holds leads to a relation

$$(A+B)E-(Az_2+Bz_1)=C. \quad (4.14)$$

Next we require that $A=-B$ to eliminate the E dependent term, and obtain $C=A(z_1-z_2)$. Then Eq. (4.13) has the form of a sum,

$$\frac{1}{(E-z_1)(E-z_2)}=\frac{1}{z_1-z_2}\left[\frac{1}{E-z_1}-\frac{1}{E-z_2}\right], \quad (4.15)$$

and we can write the resonant part [Eq. (4.10)] of the partial wave amplitude as the sum of two Breit-Wigner amplitudes,

$$T_{res}=C_1^{(2)}\frac{-\frac{1}{2}\Gamma_1}{E-E_1+\frac{1}{2}i\Gamma_1}+C_2^{(2)}\frac{-\frac{1}{2}\Gamma_2}{E-E_2+i\Gamma_2}, \quad (4.16)$$

where the complex coefficients

$$C_1^{(2)}=1-2i\frac{\frac{1}{2}\Gamma_2}{z_1-z_2},$$
$$C_2^{(2)}=1+2i\frac{\frac{1}{2}\Gamma_1}{z_1-z_2} \quad (4.17)$$

are exactly such that the unitarity condition

$$|e^{2i\delta^r}|=|1+2iT_{res}|=1 \quad (4.18)$$

is satisfied for all E. The energy dependence of the widths $\Gamma_n(E)$ introduces an energy dependence in $C_n(E)$, $n=1,2$.

Now consider the case of three interfering resonances $N=3$. Then

$$T_{res}=a_1+a_2+a_3+2i(a_1a_2+a_1a_3+a_2a_3)+(2i)^2a_1a_2a_3. \quad (4.19)$$

We can write the last term as a sum:

$$\frac{1}{(E-z_1)(E-z_2)(E-z_3)}=\left[\frac{A}{E-z_1}+\frac{B}{E-z_2}+\frac{C}{E-z_3}\right]\frac{1}{D}. \quad (4.20)$$

Requiring that terms proportional to E^2 and E in the numerator on the right-hand side of Eq. (4.20) vanish, we obtain a sum

$$\frac{K_1}{E-z_1}+\frac{K_2}{E-z_2}+\frac{K_3}{E-z_3}, \quad (4.21)$$

where

$$K_1=\frac{A}{D}=\frac{z_3-z_2}{X},$$
$$K_2=\frac{B}{D}=\frac{z_3-z_1}{X},$$
$$K_3=\frac{C}{D}=\frac{z_1-z_2}{X}, \quad (4.22)$$

with

$$X=z_1z_2(z_1-z_2)+z_3z_1(z_3-z_1)+z_3z_2(z_3-z_2). \quad (4.23)$$

The resonant part of the partial wave amplitude is again a coherent sum of the Breit-Wigner terms with complex coefficients

$$T_{res}=\sum_{n=1}^{3}C_n^{(3)}\frac{-\frac{1}{2}\Gamma_n}{E-E_n+\frac{1}{2}i\Gamma_n}, \quad (4.24)$$

where

$$C_1^{(3)}=1-2i\frac{\frac{1}{2}\Gamma_2}{z_1-z_2}-2i\frac{\frac{1}{2}\Gamma_3}{z_1-z_3}+(2i)^2\left(\frac{1}{2}\Gamma_2\right)\left(\frac{1}{2}\Gamma_3\right)K_1,$$
$$C_2^{(3)}=1+2i\frac{\frac{1}{2}\Gamma_1}{z_1-z_2}-2i\frac{\frac{1}{2}\Gamma_3}{z_2-z_3}+(2i)^2\left(\frac{1}{2}\Gamma_1\right)\left(\frac{1}{2}\Gamma_3\right)K_2, \quad (4.25)$$
$$C_3^{(3)}=1+2i\frac{\frac{1}{2}\Gamma_1}{z_1-z_3}+2i\frac{\frac{1}{2}\Gamma_2}{z_2-z_3}+(2i)^2\left(\frac{1}{2}\Gamma_1\right)\left(\frac{1}{2}\Gamma_2\right)K_3.$$

This procedure is general and valid for any finite N. Assuming that the resonant phase δ^r can be separated from the phase shift δ and is given by the sum of Breit-Wigner phases, we will always obtain the resonant part T_{res} of the partial wave amplitudes T_l^I in Eq. (4.9) as a sum of Breit-Wigner amplitudes:

$$T_{res}(E)=\sum_{n=1}^{N}C_n^{(N)}(E)\frac{\frac{1}{2}\Gamma_n(E)}{E-E_n+\frac{1}{2}i\Gamma_n(E)}. \quad (4.26)$$

In Eq. (4.26) the complex coefficients $C_n^{(N)}$ have an explicit form in terms of resonance parameters $E_n,\Gamma_n,n=1,\ldots,N$, such that T_{res} satisfies unitary condition (4.18). The form of coefficients $C_n^{(N)}$ depends on the number of resonances contributing to the partial wave T_l^I.

As the result of Eq. (4.26) we can conclude that the multiresonance parametrization of partial wave amplitudes, based on the additivity of Breit-Wigner phases, has the general form of a coherent sum of Breit-Wigner amplitudes a_n with complex coefficients and a complex coherent background,

$$T=\frac{1}{q}\left[B(E)+\sum_{n=1}^{N}R_n^{(N)}(E)\frac{-\frac{1}{2}\Gamma_n}{E-E_n+i\frac{1}{2}\Gamma_n}\right], \quad (4.27)$$

where

$$B=\frac{1}{2i}[\eta e^{2i\xi}-1],$$
$$R_n^{(N)}=e^{2i\xi}\eta C_n^{(N)}=(1+2iB)C_n^{(N)}. \quad (4.28)$$

Comparing Eq. (4.27) with expression (2.19), we see that the description of multiresonance contributions using the addition of Breit-Wigner phases leads to the same form of partial wave amplitudes as the analyticity of the S matrix. However, the complex background B and the complex coefficients $R_n^{(N)}$ in Eq. (4.27) have the explicit form [Eq. (4.28)] imposed by the additivity of Breit-Wigner phases.

Note that in the derivation of Eq. (4.26) for T_{res}, and in the resultant form [Eq. (4.27) with Eq. (4.28)], we have not needed or used the assumption that inelasticity $\eta\leqslant 1$. The Hu method is based on the unitarity of $1+2iT_{res}$, and is not related to the unitarity of the whole partial wave amplitude T_l^I.

Finally we give a relativistic form for the multiresonance description of partial wave amplitudes. The relativistic form of Breit-Wigner amplitudes [Eq. (4.7)] is given by

$$a_n=\frac{-m_n\Gamma_n(s)}{s^2-m_n^2+im_n\Gamma_n(s)}, \quad (4.29)$$

where we have used m_n instead of E_n to emphasize that E_n is the mass of the resonance. To obtain the corresponding coefficients $C_n^{(N)}$, we make replacements in Eq. (4.17) or (4.25):

$$\frac{1}{2}\Gamma_n\to m_n\Gamma_n, \quad (4.30)$$

$$z_n=E_n-\frac{1}{2}i\Gamma_n\to z_n=m_n^2-im_n\Gamma_n.$$

The partial wave amplitudes then have the relativistic form

$$T(s)=\frac{1}{q}\left[B(s)+\sum_{n=1}^{N}R_n^{(N)}(s)a_n(s)\right], \quad (4.31)$$

where B and $R_n^{(N)}$ are still given by Eq. (4.28) with replacements [such as Eq. (4.30)] to satisfy the unitarity of T_{res}.

V. GENERALIZED DISPERSION RELATIONS FOR PARTIAL WAVE AMPLITUDES AND INTERFERING RESONANCES IN $\pi\pi$ SCATTERING

In this section we shall relate the multiresonance parametrization [Eq. (4.31)] of partial wave amplitudes T_l^I with a multiresonance parametrization obtained from analyticity. To this end we shall use generalized dispersion relations for the amplitudes

$$t_l^I(s)=qT_l^I(s), \quad (5.1)$$

where s is the Mandelstam energy variable.

Our starting point is the well-known [27] dispersion respresentation of a complex function $f(z)$ with simple poles at z_n, $n=1,2,\dots,N$ in the complex plane z, a branch cut along a positive real axis from α to ∞ and with asymptotic property $|z|f(z)\to 0$ as $|z|\to\infty$. We shall also assume that the function $f(z)$ is a real function $f(z^*)=f^*(z)$. Using Cauchy's integral theorem and the process of contour deformation, it can be shown [27] that

$$f(z)=\sum_{n=1}^{N}\frac{R_n}{z-z_n}+\frac{1}{\pi}\int_{\alpha}^{\infty}\frac{\mathrm{Im}\,f(x')dx'}{x'-z}. \quad (5.2)$$

A remarkable feature of the proof of Eq. (5.2) is that it takes place for a fixed value of z [27]. As the result, dispersion relation (5.2) is also valid for moving poles for which $z_n=z_n(z)$. In such a case the residues R_n in Eq. (5.2) also depend on z, i.e., $R_n=R_n(z)$. Furthermore, dispersion relation (5.2) is easily generalized to include a left-hand cut and for functions that are not real. In the latter case $\mathrm{Im}f(x')$ in Eq. (5.2) is replaced by a discontinuity function along the cut(s).

In $\pi\pi$ scattering, the partial wave amplitudes $t_l^I(s)$ have a right-hand cut for $s\geqslant 4\mu^2$ (where μ is the pion mass), and a left-hand cut for $s\leqslant 0$ due to Mandelstam analyticity [28]. Let us assume that the amplitude t_l^I has a finite number N_l^I of complex poles,

$$s_n=m_n^2-im_n\Gamma(s), \quad n=1,\dots,N_l^I, \quad (5.3)$$

corresponding to the resonances in t_l^I. Note that the imaginary part of the poles depends on the energy variable s. In principle, the mass m_n could also depend on the energy s. This possibility has been recently considered by Pennington [29]. Omitting the indices I and l, the generalized dispersion relations for the partial wave amplitude t_l^I read

$$t(s)=I(s)+\sum_{n=1}^{N}\frac{R_n(s)}{s-s_n(s)}, \quad (5.4)$$

where $I(s)$ are the dispersion integrals over the left and right-hand cuts [28], and $R_n(s)$ are the pole residues. It is convenient to rewrite Eq. (5.4) in a form using Breit-Wigner amplitudes $a_n(s)$,

$$t(s)=I(s)+\sum_{n=1}^{N}R_n(s)a_n(s), \quad (5.5)$$

where we have redefined the pole residues with

$$a_n(s)=\frac{-m_n\Gamma_n(s)}{s-m_n^2+im_n\Gamma_n(s)}. \quad (5.6)$$

Representation (5.5) is valid for all $s\geqslant 4\mu^2$. Representation (5.5) of partial waves t_l^I coincides with parametrization (4.31), provided that

$$I(s) \equiv B(s) = \frac{1}{2i}[\eta(s)e^{2i\xi(s)} - 1],$$

$$R_n(s) \equiv R_n^{(N)}(s) = \eta(s)e^{2i\xi(s)}C_n^{(N)}(s), \tag{5.7}$$

$$= (1+2iB)C_n^{(N)}.$$

We see that the multiresonance parametrization, based on the additivity of Breit-Wigner phases like Eq. (4.4), imposes a special form on the dispersion integrals and pole residues given by Eq. (5.7).

In general, a partial wave t_l^I can be written in two forms:

$$t = \frac{1}{2i}[\eta e^{2i\delta} - 1] = I + \sum_{n=1}^{N} R_n a_n. \tag{5.8}$$

Apart from the partial wave unitarity equations (3.1) and (3.4) and the analyticity assumptions, there are no constraints on the partial waves. The assumption of additivity of Breit-Wigner phases [Eq. (4.4)] is a new constraint that restricts the partial waves to analytical functions that satisfy conditions (5.7). We find no physical justification for such a restriction, and no advantage in using it in phenomenological fits to data to determine resonance parameters.

VI. INTERFERING RESONANCES IN A FINITE ENERGY INTERVAL

In the previous two sections we have assumed that N is the total number of resonances contributing to a partial wave. Parametrizations (4.31) and (5.5) were valid for all energies $s \geq 4\mu^2$. In practice N is not known, and fits to data are done in a finite energy interval. Such is the case, e.g., in Refs. [13,14]. In this section we develop parametrizations of partial waves appropriate for analyses in a finite energy interval where only a few resonances contribute. The parametrizations will be based both on the additivity of Breit-Wigner phases and analyticity, and we shall compare their use in practical fits to data. The results will be used in Sec. VIII.

Let us consider an energy interval $4\mu^2 \leq s \leq s_M$, where M resonances contribute. In the framework of the assumption of additivity of Breit-Wigner phases, we will assume that the resonant phases of resonances outside of this energy interval are absorbed in the background phase. The total phase shift is then

$$\delta = \xi^{(M)} + \delta_M^r, \tag{6.1}$$

where

$$\delta_M^r = \sum_{n=1}^{M} \delta_n^r, \tag{6.2}$$

$$\xi^{(M)} = \xi + \sum_{n=M+1}^{N} \delta_n^r. \tag{6.3}$$

The partial wave then takes the form

$$t = \frac{1}{2i}(\eta e^{2i\xi^{(M)}} - 1) + \eta e^{2i\xi^{(M)}} T_{res}^{(M)}, \tag{6.4}$$

where the resonant part

$$T_{res}^{(M)} = \sum_{n=1}^{M} C_n^{(M)} a_n \tag{6.5}$$

is unitary:

$$e^{2i\delta_M^r} = 1 + 2iT_{res}^{(M)}. \tag{6.6}$$

Alternatively we can rewrite Eq. (4.31) in the form

$$t = B^{(MN)} + \sum_{n=1}^{M} R_n^{(N)} a_n, \tag{6.7}$$

where

$$B^{(MN)} = \frac{1}{2i}(\eta e^{2i\xi} - 1) + \sum_{m=M+1}^{N} R_m^{(N)} a_m \tag{6.8}$$

is the background term. Note that the sum in Eq. (6.7) is not unitary. We cannot compare the coefficients of Breit-Wigner amplitudes a_n, $n=1,\ldots,M$ in Eq. (6.4) with those in Eq. (6.7), since $\xi^{(M)}$ in Eq. (6.4) contains the terms a_m, $m = M+1,\ldots,N$ but the sum in Eq. (6.7) does not.

If we look at the general form [Eq. (5.5)] from analyticity, then for $s < s_M$ we can write

$$t = B^{(M)} + \sum_{n=1}^{M} R_n a_n, \tag{6.9}$$

where the background term

$$B^{(M)} = I + \sum_{m=M+1}^{N} R_m a_m. \tag{6.10}$$

In Eq. (6.9) the residues R_n are not constrained by conditions (5.7).

In fitting data using parametrization (6.4) we explicitly make use of the assumption of the additivity of Breit-Wigner phases. This is also the case when we use Eq. (6.7) if N is known and the coefficients $C_n^{(N)}$ in $R_n^{(N)}$ can be calculated. In general N is not known, and the background $B^{(MN)}$ and residues $R_n^{(N)}$ are free parameters. Then there is no difference in using Eq. (6.7) or the general form [Eq. (6.9)] from analyticity alone, since in Eq. (6.9) the background $B^{(M)}$ and residues R_n are not constrained except for unitarity. In all cases we use a constrained optimization of the χ^2 function. In the case of Eq. (6.4) we require that the inelasticity function $\eta \leq 1$. In the case of Eq. (6.7) or (6.9), we require that Im$t \geq |t|^2$, and use programs such as MINOS [11] for constrained optimization.

It is not obvious that the use of parametrization (6.4) from the additivity of Breit-Wigner phases, and parametrization (6.9) from analyticity alone, will lead to the same resonance parameters in both cases. The use of parametrization (6.4) confers no phenomenological or computational advantage over parametrization (6.9). The assumption of the additivity of Breit-Wigner phases restricts the background and the complex coefficients multiplying the Breit-Wigner amplitudes a_n, $n=1,\ldots,M$ in parametrization (6.4) to specific forms. Since there is no physical justification for such a restriction, and parametrization (6.9) is free from such constraints, we suggest that the use of parametrization (6.9) is more appropriate in determining resonance parameters in $\pi\pi$ scattering.

VII. UNITARITY IN PION PRODUCTION $\pi^- p \to \pi^- \pi^+ n$

It is a common misconception to identify partial wave production amplitudes in reaction $\pi^- p \to \pi^- \pi^+ n$ with partial waves T_l^I in $\pi\pi$ scattering and demand that the partial wave production amplitudes also satisfy the partial wave unitarity [Eq. (3.1)]. In this section we clarify the distinction between the two kinds of amplitudes and the associated unitarity relations.

The production process $\pi^- p \to \pi^- \pi^+ n$ is described by production amplitudes [25,30,31]

$$T_{\lambda_n,0\lambda_p}(s,t,m^2,\theta,\phi), \tag{7.1}$$

where λ_p and λ_n are proton and neutron helicities, s is the c.m.s. energy squared, t is the momentum transfer between the incident pion and the dipion system $(\pi^-\pi^+)$, m^2 is the dipion mass squared, and $\Omega=(\theta,\phi)$ is the solid angle of the final π^- pion in the dipion rest frame. dipion state does not have a definite spin. Production amplitudes like Eq. (7.1) can be expressed in terms of partial wave production amplitudes $M^J_{\lambda\lambda_n,0\lambda_p}(s,t,m^2)$ corresponding to definite dimeson spin J using the angular expansion [25,30,31],

$$T_{\lambda_n,0\lambda_p}=\sum_{j=0}^{\infty}\sum_{\lambda=-J}^{+J}(2J+1)^{1/2}M^J_{\lambda\lambda_n,0\lambda_p}(s,t,m^2)d^J_{\lambda 0}(\theta)e^{i\lambda\phi}, \tag{7.2}$$

where J is the spin and λ the helicity of the $(\pi^-\pi^+)$ dimeson system.

It is evident from Eq. (7.2) that the partial wave production amplitudes $M^J_{\lambda,\lambda_n,0\lambda_p}(s,t,m^2)$ cannot be identified with the $\pi\pi$ partial wave amplitudes $T^I_J(m^2)$. The amplitudes $M^J_{\lambda,\lambda_n,0\lambda_p}(s,t,m^2)$ can be thought of as two-body helicity amplitudes for a process $\pi^-+p\to M(J,m)+n$, where the "particle" $M(J,m)$ has spin J and mass m.

The production amplitudes $T_{\lambda_n,0\lambda_p}$ satisfy the unitarity condition [26]

$$-i(T_{\lambda_n,0\lambda_p}-T^*_{0\lambda_p,\lambda_n})=\sum_n\int T_{0\lambda_p,n}T^*_{\lambda_n,n}d\Phi_n, \tag{7.3}$$

where $d\Phi_n$ is the n-body Lorentz invariant phase space of the intermediate state n. Since the initial state in $\pi^- p \to \pi^-\pi^+ n$ is a two-body state and the final state is a three-body state, the amplitude $T_{\lambda_n,0\lambda_p}$ enters the unitarity integral only linearly. This occurs only when the intermediate state is $\pi^- p$ or $\pi^-\pi^+ n$. However, the three-body intermediate state involves a $3\to3$ amplitude and a three-body phase space integral. Separating the two-body intermediate states $\pi^- p$ and $\pi^0 n$, we can write Eq. (7.3) in the form

$$\begin{aligned}-i(T_{\lambda_n,0\lambda_p}-T^*_{0\lambda_p,\lambda_n})=&\sum_{\lambda_p'}\int T_{0\lambda_p,0\lambda_p'}T^*_{\lambda_n,0\lambda_p'}d\Phi_2\\&+\sum_{\lambda_n'}\int T_{0\lambda_p,0\lambda_n'}T^*_{\lambda_n,0\lambda_n'}d\Phi_2\\&+\Delta_{\lambda_n,0\lambda_p},\end{aligned} \tag{7.4}$$

where $T_{0\lambda_p,0\lambda_p'}$ and $T_{0\lambda_p,0\lambda_n'}$ are helicity amplitudes of reactions $\pi^- p\to\pi^- p$ and $\pi^- p\to\pi^0 n$, respectively. The amplitude $T^*_{\lambda_n,0\lambda_n'}$ corresponds to process $\pi^0 n\to\pi^-\pi^+ n$. The inelastic unitarity contribution $\Delta_{\lambda_n,0\lambda_p}(s,t,m^2,\theta,\phi)$ can be expanded in a form analogous to Eq. (7.2):

$$\Delta_{\lambda_n,0\lambda_p}=\sum_{J=0}^{\infty}\sum_{\lambda=-J}^{+J}(2J+1)^{1/2}\Delta^J_{\lambda\lambda_n,0\lambda_p}(s,t,m^2)d^J_{\lambda 0}(\theta)e^{i\lambda\phi}. \tag{7.5}$$

Using expansions (7.2) and (7.5) in Eq. (7.4), we obtain unitarity relations for partial wave production amplitudes

$$\begin{aligned}-i(M^J_{\lambda\lambda_n,0\lambda_p}-M^{J*}_{0\lambda_p,\lambda\lambda_n})=&\sum_{\lambda_p'}\int T_{0\lambda_p,0\lambda_p'}M^{J*}_{\lambda\lambda_n,0\lambda_p'}d\Phi_2\\&+\sum_{\lambda_n'}\int T_{0\lambda_p,0\lambda_n'}M^{J*}_{\lambda\lambda_n,0\lambda_n'}d\Phi_2\\&+\Delta^J_{\lambda_n,0\lambda_p}.\end{aligned} \tag{7.6}$$

Using time-reversal relations for two-body helicity amplitudes [44]

$$M^J_{0\lambda_p,\lambda\lambda_n}=(-1)^{\lambda_n-\lambda_p-\lambda}M^J_{\lambda\lambda_n,0\lambda_p}, \tag{7.7}$$

we see that the left hand side of the partial wave unitarity relation [Eq. (7.6)] does not simplify to $2\mathrm{Im}M^J_{\lambda\lambda_n,0\lambda_p}$ as is the case for the partial waves T^I_l in $\pi\pi$ scattering. The right hand side of Eq. (7.6) involves $M^J_{\lambda\lambda_n,0\lambda_p}$ only linearly and not quadratically, as is the case in $\pi\pi$ scattering. Futhermore, the right hand side of unitarity relation (7.6) includes (linearly) partial wave production amplitudes for the process $\pi^0 n\to\pi^-\pi^+ n$. We conclude that unitarity relations like Eq. (7.6) for partial wave production amplitudes $M^J_{\lambda\lambda_n,0\lambda_p}$ are complex relations that do not have the simple form

$$\mathrm{Im}\,T^I_l=q|T^I_l|^2+\Delta^I_l \tag{7.8}$$

of the partial wave unitarity relations in $\pi\pi$ scattering.

For brevity let us define $M^J_\Lambda \equiv M^J_{\lambda\lambda_n,0\lambda_p}$, where Λ stands for the helicities. The amplitude M^J_Λ is a complex function, and so is the function $1+2iqM^J_\Lambda$. In analogy with Eqs. (3.12) and (3.3) we can write

$$M^J_\Lambda = \frac{1}{2iq}(\eta^J_\Lambda e^{2i\delta^J_\Lambda}-1), \tag{7.9}$$

where $\eta^J_\Lambda(s,t,m^2)$ is the "inelasticity" and $\delta^J_\Lambda(s,t,m^2)$ is the "phase shift." The amplitude M^J_Λ satisfies a relation similar to Eq. (3.13):

$$\mathrm{Im}\, M^J_\Lambda = q|M^J_\Lambda|^2 + \frac{1}{4q}[1-(\eta^J_\Lambda)^2]. \tag{7.10}$$

Unlike in $\pi\pi$ scattering, the form of Eq. (7.10) does not coincide with the form of partial wave unitarity [Eq. (7.6)], and the "inelasticity" η^J_Λ cannot be related to the inelastic unitarity contributions Δ^J_Λ, in contrast to Eq. (3.4).

VIII. INTERFERING RESONANCES IN PRODUCTION PROCESSES

The amplitudes describing production processes such as $\pi N \to \pi\pi N$, $pp \to \pi\pi pp$ or $p\bar{p} \to 3\pi$ are far more complex than isospin amplitudes in the $\pi\pi$ scattering. As an example, consider pion production in $\pi^- p \to \pi^-\pi^+ n$. The angular distribution of the dipion $\pi^-\pi^+$ state is described by partial wave production amplitudes $M^J_{\lambda\lambda_n,0\lambda_p}(s,t,m^2)$ defined in Sec. VII with Eq. (7.2. The measurements of $\pi^- p \to \pi^-\pi^+ n$ on a polarized target actually determine the moduli of nucleon transversity amplitudes [30,31] which are linear combinations of helicity amplitudes $M^J_{\lambda\lambda_n,0\lambda_p}$. For masses $m \lesssim 1000$ MeV, the pion production is described by two S-wave $(J=0)$ nucleon transversity amplitudes S and $\bar{S}$, and by six P-wave $(J=1)$ nucleon transversity amplitudes $L,\bar{L},N,\bar{N},U,\bar{U}$ [30,31]. The amplitudes $\bar{A}=\bar{S},\bar{L},\bar{N},\bar{U}$ correspond to "up" nucleon transversity, while the amplitudes $A=S,L,N,U$ correspond to "down" nucleon transversity. The amplitudes L and $\bar{L}$ correspond to dimension helicity $\lambda=0$ while N and $\bar{N}$ and U and $\bar{U}$ are natural and unnatural parity amplitudes corresponding to combinations of $\lambda=\pm 1$.

The measurements of pion production on polarized targets enable one to advance hadron spectroscopy from the level of spin-averaged cross sections to the level of spin-dependent production amplitudes. These measurements determine the mass spectra $|A|^2$ and $|\bar{A}|^2$ of spin-dependent production amplitudes. Measurements of $\pi^- p \to \pi^-\pi^+ n$ on transversely polarized targets were done at CERN at 17.2 GeV/c [32–35] and at ITEP at 1.78 GeV/c [36]. Measurements of $\pi^+ n \to \pi^+\pi^- p$ [31,37–39] and $K^+ n \to K^+\pi^- p$ [40,41] at 5.98 and 11.85 GeV/c on transversely polarized deuteron target were also done at CERN. More recently it was shown that mass spectra of production amplitudes can be obtained in measurements of $\pi^- p \to \pi^0\pi^0 n$, $\pi^- p \to \eta\eta n$ [42] and $\pi^- p \to \eta\pi^- n$, $\pi^- p \to \eta\pi^0 n$ [43] on transversely polarized targets, allowing for amplitude spectroscopy of these interesting processes.

The analysis of mass spectra measured in production processes requires a parametrization of the production amplitudes in terms of the Breit-Wigner amplitudes to identify contributing resonances and to determine their parameters. Here we discuss two approaches, one based on the additivity of Breit-Wigner phases and the other on the analyticity of production amplitudes $A(s,t,m^2)$ in the mass variable at fixed s and t.

First we note that the unitarity equation (7.6) for the partial wave production amplitudes in $\pi N \to \pi\pi N$ is a complex relation and that the helicity amplitudes M^J_Λ or the transversity amplitudes A and $\bar{A}$ do not satisfy the two-body partial wave unitarity equation (3.1) with Eq. (3.4). Nevertheless, the experimentally measured production amplitudes $A(s,t,m^2)$ are complex functions, and as such can be written in the form

$$A(s,t,m^2) = \frac{1}{2i}[\eta_A e^{2i\delta_A}-1], \tag{8.1}$$

where the "inelasticity" $\eta_A = \eta_A(s,t,m^2)$ and "phase shift" $\delta_A = \delta_A(s,t,m^2)$ also depend on the helicities or transversities of the amplitude A. Obviously,

$$\mathrm{Im}\, A = |A|^2 + \frac{1}{4}(1-\eta_A^2). \tag{8.2}$$

However, there is no requirement now that $\eta_A \leqslant 1$ since η_A has no relation to unitarity as in Eq. (3.4).

We can pursue the analogy with the $\pi\pi$ scattering, and impose an assumption that the "phase shift"

$$\delta_A(s,t,m^2) = \xi_A(s,t,m^2) + \delta^r(m^2), \tag{8.3}$$

where δ^r is the sum of Breit-Wigner phases of the N resonances contributing to the amplitude A, and ξ_A is the "background" phase. If we restrict ourselves to a finite mass interval $4\mu^2 \leqslant m^2 \leqslant m_M^2$ with M resonances, we can write

$$\delta_A(s,t,m^2) = \xi_A^{(M)}(s,t,m^2) + \delta^r_M(m^2), \tag{8.4}$$

$$A(s,t,m^2) = \frac{1}{2i}(\eta_A e^{2i\xi_A^{(M)}}-1) + \eta_A e^{2i\xi_A^{(M)}} T^{(M)}_{res}(m^2), \tag{8.5}$$

in analogy with Eq. (6.4) for $\pi\pi$ scattering [44].

A more general approach is to use the analyticity of $A(s,t,m^2)$ in m^2 with s and t fixed. We can assume that kinematical singularities have been removed from the production amplitudes $A(s,t,m^2)$ [45]. Assuming that there are N Breit-Wigner poles in the amplitude $A(s,t,m^2)$ in the mass variable m^2, we can use the generalized dispersion relations for the variable m^2 with s and t fixed to obtain

$$A(s,t,m^2) = I(s,t,m^2) + \sum_{n=1}^{N} R_n(s,t,m^2) a_n(m^2) \tag{8.6}$$

where I is the contribution of dispersion integrals, R_n are complex pole residues, and a_n are the Breit-Wigner amplitudes [Eqs. (5.6)]. In a finite mass interval $4\mu^2 \leqslant m^2 \leqslant m_M^2$ with M resonances we can write

$$A(s,t,m^2) = B^{(M)}(s,t,m^2) + \sum_{n=1}^{M} R_n(s,t,m^2), a_n(m^2), \tag{8.7}$$

where the background

$$B^{(M)}(s,t,m^2) = I + \sum_{m=M+1}^{N} R_m a_m . \tag{8.8}$$

We note that for $M=N$ we reobtain constraints (5.7) with replacements $\eta \rightarrow \eta_A$ and $\xi \rightarrow \xi_A$. Again, the assumption of additivity of Breit-Wigner phases restricts the production amplitudes to analytical functions that satisfy constraints (5.7).

The measured mass spectra $|A|^2$ can now be fitted either with parametrization (8.5) or with the more general parametrization (8.7). There are no unitarity constraints to be imposed on the production amplitudes $|A|^2$ during the fits, since the right hand side of unitarity relation (7.6) is not known and the partial wave unitarity [Eq. (3.1) or (7.8)] for $\pi\pi$ amplitudes $T_l^I(s)$ does not apply to the production amplitudes $A(s,t,m^2)$. The unitarity constraint [Eq. (3.1) or (7.8)] can be imposed only in the analysis of data on the $\pi\pi \rightarrow \pi\pi$ reaction and below we discuss its effect on $\pi\pi$ amplitudes.

Since there is no physical justification for the assumption of additivity of Breit-Wigner phases in Eq. (8.3), and since the form of Eq. (8.5) confers no phenomenological or computational advantage over the more general analytical form [Eq. (8.7)], we conclude that the use of the form of Eq. (8.7) is more appropriate in fits to mass spectra in production processes to determine the resonance parameters of interfering resonances.

The parametrization of production amplitudes in terms of a coherent sum with complex coefficients and a complex coherent background as in Eq. (8.7) has been an accepted practice for a long time. Such parametrizations first appeared in connection with the possible double-pole character of the A_2 meson [15] and the splitting of the Q resonance in $K^+\pi^-\pi^+$ mass spectrum [16]. Recently such a parametrization was used in the study of $\sigma(750)-f_0(980)$ interference in S-wave production amplitudes in $\pi^- p \rightarrow \pi^-\pi^+ n$ measured on polarized target at CERN [17,18] and in the study of $\sigma - f_0(980)$ interference in the central collision $pp \rightarrow \pi^0\pi^0 pp$ [19]. More recently, an analysis of S-wave production amplitudes from threshold to 2 GeV in $pp \rightarrow \pi^0\pi^0 pp$ was made using three [20] and four [21] interfering Breit-Wigner amplitudes and a coherent background. The GAMS Collaboration used four interfering Breit-Wigner amplitudes and a coherent background in their fit of S-wave mass spectrum from threshold to 3 GeV in $\pi^- p \rightarrow \pi^0\pi^0 n$ measured at 100 GeV [22]. Also recently, the Fermilab E791 Collaboration used the form of Eq. (8.7) to fit the Dalitz plot of $D^+ \rightarrow \pi^-\pi^+\pi^+$ decays in their search for a scalar resonance σ [23].

Finally we comment on the determination of $\pi\pi$ partial wave amplitudes from measurements of $\pi N \rightarrow \pi\pi N$. The resonance parameters from the fits to mass spectra such as those measured in $\pi N \rightarrow \pi^+\pi^- N$ on polarized targets [17,18,39] or in $\pi^- p \rightarrow \pi^0\pi^0 n$ [20–22] must be the same in $\pi\pi$ partial waves. However, the $\pi\pi$ partial wave amplitudes are expected to satisfy partial wave unitarity constraints (3.1) and (3.4), or rather inequalities (3.8) and (3.9), which for the amplitudes t_l^I defined in Eq. (5.1) read

$$|t_l^I|^2 \leqslant \operatorname{Im} t_l^I \leqslant |t_l^I|^2 + \frac{1}{4}. \tag{8.9}$$

Unitarity conditions like Eq. (8.9) can always be satisfied by an appropriate choice of background and complex residues $R_n(s)$ in a general parametrization [Eq. (6.9)] based on analyticity. Although $\pi\pi$ partial waves and production amplitudes in $\pi N \rightarrow \pi\pi N$ with the same spin and isospin share the same Breit-Wigner poles, they are different analytical functions and thus the residues of the poles and the backgrounds are different. In particular, the residues in production amplitudes A depend on particle helicities and kinematic variables s and t. Accepting the resonance parameters obtained from the fits to the mass spectra $|A|^2$ measured in $\pi N \rightarrow \pi\pi N$ to describe the resonances in $\pi\pi \rightarrow \pi\pi$ scattering, the effect of unitarity conditions like Eq. (8.9) is to constrain the residues $R_n(s)$ and the background term in the general parametrization [Eq. (6.9)] of the $\pi\pi$ amplitudes.

It is also possible to use resonance parameters determined from measurements of $\pi N \rightarrow \pi\pi N$ to calculate the resonant part $T_{res}^{(M)}$ and to define the $\pi\pi$ partial waves using parametrization (6.4) with free background and inelasticity functions $\xi^{(M)}(s)$ and $\eta(s)$. The unitarity can be satisfied by imposing the condition $\eta \leqslant 1$.

Unitarity constraints like Eq. (8.9) may not uniquely determine the background and pole residues in the parametrization [Eq. (6.9)] from analyticity, and the use of parametrization (6.4) from additivity of Breit-Wigner phases is questionable. We conclude that the resonance parameters determined from mesurements of $\pi N \rightarrow \pi\pi N$ alone may not determine the $\pi\pi$ partial wave amplitudes without additional assumptions or direct measurements of $\pi\pi \rightarrow \pi\pi$ reactions.

IX. SUMMARY

We have shown that, in the case of $\pi\pi$ scattering, the assumption of the additivity of Breit-Wigner phases in a partial wave amplitude leads to a sum of Breit-Wigner amplitudes with complex coefficients and a coherent background [Eq. (4.31)]. The coefficients have a specific form [Eq. (4.28)] in terms of resonance parameters of all contributing resonances. The form of Eq. (4.31) is a special case of the general form [Eq. (5.5)] based on analyticity and it is not related to the unitarity property of partial waves [Eqs. (3.1) and (3.4)]. The claims [13,14] that the additivity of Breit-Wigner phases provides a correct description of interfering resonances in $\pi\pi$ scattering are not justified, since there is no physical reason why the Breit-Wigner poles must have the specific residues imposed by this assumption. We found that the Breit-Wigner phases of interfering resonances are not necessarily additive. We suggest that the general form [Eq. (5.5)] obtained from analyticity is more appropriate in fits to data. Unitarity conditions like Eq. (8.9), $|t_l^I|^2 \leqslant \mathrm{Im} t_l^I \leqslant |t_l^I|^2 + \frac{1}{4}$, can be effectively imposed using the modern methods of constrained optimization [9–11].

Mass spectra in production processes are described by production amplitudes. We used the case of a $\pi N \to \pi^+ \pi^- N$ reaction to illustrate the complexity of production amplitudes. Specifically, the production amplitudes do not satisfy the two-body partial wave unitarity equation (3.1); they depend on particle helicities and on several kinematic variables in addition to the invariant mass. We have used the analyticity of production amplitudes in the invariant mass variable to justify the common practice [15–23] of writing the production amplitudes as a coherent sum of Breit-Wigner amplitudes with free complex coefficients and a complex coherent background in fits to measured mass spectra to determine the resonance parameters of interfering resonances. Two-body unitarity constraints on $\pi\pi$ partial wave amplitudes with the same resonances can be satisfied by an appropriate choice of complex residues of the contributing Breit-Wigner poles. This reflects the fact that the $\pi\pi$ partial wave amplitudes and production amplitudes while sharing the same resonances are different analytical functions.

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