Direct improvement of Hamiltonian lattice gauge theory

Jesse Carlsson* and Bruce H. J. McKellar†
School of Physics, The University of Melbourne, Parkville, Victoria 3052, Australia
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We demonstrate that a direct approach to improving Hamiltonian lattice gauge theory is possible. Our approach is to correct errors in the Kogut-Susskind Hamiltonian by incorporating additional gauge invariant terms. The coefficients of these terms are chosen so that the order a^2 classical errors vanish. We conclude with a brief discussion of tadpole improvement in Hamiltonian lattice gauge theory.

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I. INTRODUCTION

The idea of using the lattice as an ultraviolet regulator for quantum chromodynamics (QCD) was proposed by Wilson in 1974 in his action formulation of lattice gauge theory [1]. Soon after, Kogut and Susskind formulated the corresponding Hamiltonian version of lattice gauge theory [2]. Both approaches were developed by demanding the correct continuum limit be obtained in the limit of vanishing lattice spacing a. Creutz showed that the Kogut-Susskind Hamiltonian could be derived from the Wilson action using the transfer matrix method [3]. Later, Kogut demonstrated that the same could be done by taking the continuous time limit of the Wilson action and performing a canonical Legendre transformation [4].

To date, the majority of work in lattice QCD has been performed in the action formulation. An advantage of this approach is that it readily lends itself to Monte Carlo techniques. Working in the Hamiltonian approach brings a different intuition to the problem and serves as a check of universality. An advantage of Hamiltonian lattice gauge theory is in the applicability of techniques from many body physics [5]. Also, it appears that in finite density QCD a Hamiltonian approach is favorable due to the so-called complex action problem which rules out the use of standard Monte Carlo techniques in the action formulation [6].

Much work in the past decade has been devoted to improving lattice actions [7]. Initiated by Symanzik in 1983 [8], the aim of the improvement program is to reduce the deviation between lattice and continuum QCD at finite lattice spacing. For pure gauge theory on a lattice, the deviations between continuum and lattice theories start at order a^2. The motivation for improvement lies in the fact that the computational cost of a lattice QCD simulation is proportional to a^{-k}, where $6<k<7$. It is by far more efficient to build an improved theory than it is to work on finer lattices. The improvement program has allowed accurate calculations to be performed on relatively coarse lattices and brought the most complicated calculations within the reach of today's most powerful computers.

In contrast, the improvement of lattice Hamiltonians has only recently begun. Perhaps the most extensive treatment to date is due to Luo, Guo, Kröger and Schütte [9] who discussed the improvement of Hamiltonian lattice gauge theory for gluons. In their study it was discovered that deriving an improved Hamiltonian from a Symanzik improved action, whether by transfer matrix or canonical Legendre transformation, results in a kinetic Hamiltonian with an infinite number of terms coupling lattice sites which are arbitrarily far apart. To derive a local kinetic Hamiltonian coupling only nearest neighbor lattice sites it was found necessary to start with an improved action with an infinite number of terms, coupling distant lattice sites.

With this technique the order a^2 errors are removed from the Kogut-Susskind Hamiltonian. However, generating Hamiltonians with further improvement would seem exceedingly difficult. This is because one would need to start from a Lüscher-Weisz improved action with non-planar terms [10]. For this reason we propose a move to the Symanzik approach, as applied to the Hamiltonian, that is, in the spirit of the original Kogut-Susskind paper, to construct improved Hamiltonians directly by adding appropriate gauge invariant terms and fixing their coefficients so that errors are canceled.

To date we have implemented Symanzik improvement to order a^2 in the pure lattice gauge theory Hamiltonian (and to order a^4 in the kinetic part). We report those results here.[1] In addition, a brief discussion of tadpole improvement in Hamiltonian lattice gauge theory is given in Sec. III.

II. SYMANZIK IMPROVEMENT OF THE LATTICE HAMILTONIAN

A. Introduction

In this section we derive an improved Hamiltonian *directly* using the Symanzik approach of adding irrelevant terms and fixing their coefficients in order to cancel errors. As a first step we aim to correct the classical order a^2 errors arising in the lattice Hamiltonian for pure SU(N) gauge theory.

The Kogut-Susskind Hamiltonian for pure SU(N) gauge theory on the lattice is given by

$$H^{(0)}=K^{(0)}+V^{(0)}, \tag{1}$$

where the kinetic and potential terms are given respectively by

*Email address: j.carlsson@physics.unimelb.edu.au
†Email address: b.mckellar@physics.unimelb.edu.au

[1]The $O(a^2)$ results were given in preliminary form in [11].

$$K^{(0)}=\frac{a^3}{2}\sum_{x,i}\mathrm{Tr}\{E_i^L(x)E_i^L(x)\} \tag{2}$$

$$V^{(0)}=\frac{2N}{ag^2}\sum_{x,i<j}P_{ij}(x). \tag{3}$$

Here E^L is the lattice chromo-electric field, N is the dimension of the gauge group, and $P_{ij}(x)$ is the plaquette operator in the (i,j) plane,

$$P_{ij}(x)=1-\frac{1}{N}\mathrm{ReTr}\left\{\square_{\,i}^{\,j}\right\}. \tag{4}$$

We discuss the improvement of the kinetic and potential terms separately in what follows.

B. Improving the Potential Term

To improve the potential part of the Kogut-Susskind Hamiltonian, we follow the process of improving the Wilson action. By introducing the rectangle operator $R_{ij}(x)$ in the (i,j) plane (with the long side in the i direction),

$$R_{ij}(x)=1-\frac{1}{N}\mathrm{ReTr}\left\{\square\!\!\square_{\,i}^{\,j}\right\}, \tag{5}$$

and expanding in powers of a, we arrive at the order a^2 improved potential term:

$$V^{(1)}=\frac{2N}{ag^2}\sum_{x,i<j}\left[\frac{5}{3}P_{ij}(x)-\frac{1}{12}[R_{ij}(x)+R_{ji}(x)]\right]. \tag{6}$$

In principle, the next lowest order classical errors could be corrected by including additional, more complicated operators in the potential term. This has not been done because many additional diagrams are required to cancel the large number of contributions to the error of order a^4. Since these errors are swamped by order a^2g^2 quantum errors, addressing quantum corrections in the Hamiltonian approach would seem to be of more immediate importance.

C. Improving the kinetic term

Constructing a kinetic Hamiltonian with a finite number of terms has proven to be a nontrivial exercise. Luo, Guo, Kröger and Schütte demonstrated an interesting trade off when using either the transfer matrix or Legendre transformation methods to derive an improved Hamiltonian [9]. Both techniques require the starting point to be an improved action. When one starts from an improved action incorporating rectangular terms the resulting Hamiltonian has infinitely many terms and couples links which are arbitrarily far apart. To produce a Hamiltonian which couples only nearest neighbor links, it was found necessary to start from a carefully constructed highly non-local improved action.

Here we demonstrate an alternative approach, similar in nature to the Symanzik improvement of the Wilson action. One only needs to include additional gauge invariant terms with appropriate continuum behavior in the kinetic Hamiltonian. The coefficients of the additional terms are chosen so that the order a^2 errors vanish.

An important step in understanding the errors that arise in the kinetic Kogut-Susskind Hamiltonian involves making the distinction between *lattice* and *continuum* fields. We define the *lattice* gluon field $A_\mu^L(x')$ to be the average of the continuum gluon field A along the link joining x and $x+a\mu$:

$$A_\mu^L(x')=\frac{1}{a}\int_{\mathrm{Link}}dxA \quad\Rightarrow\quad U_\mu(x)=e^{igaA_\mu^L(x')}, \tag{7}$$

where x' is a point near the points x and $x+a\mu$. On the lattice the gluon field is defined at only one point along (or nearby) a link. This leads to interpolation errors in the integral in Eq. (7). For instance, by choosing to evaluate the gluon field at the midpoint of the link, the lattice and continuum fields are related by

$$A_\mu^L(x)=A_\mu(x)+\frac{a^2}{24}\partial_\mu^2A_\mu(x)+\frac{a^4}{1920}\partial_\mu^4A_\mu(x)+\ldots. \tag{8}$$

We see that the lattice gluon field reduces to its continuum counterpart in the continuum limit $(a\to0)$, but that they differ by interpolation errors of order a^2. From this we build the sequence of approximations to the lattice gluon field:

$$\begin{aligned}A_i^{(0)}(x)&=A_i(x)\\ A_i^{(1)}(x)&=A_i(x)+\frac{1}{24}a^2\partial_i^2A_i(x)\\ A_i^{(2)}(x)&=A_i(x)+\frac{1}{24}a^2\partial_i^2A_i(x)+\frac{1}{1920}a^4\partial_i^4A_i(x).\end{aligned} \tag{9}$$

Perhaps the most important property of the electric field is that it generates group transformations. Mathematically, this translates to the electric and gluon fields satisfying the commutation relations,

$$[E_i^\alpha(x),A_j^\beta(y)]=-\frac{i}{a^3}\delta_{xy}\delta_{ij}\delta_{\alpha\beta}. \tag{10}$$

It is desirable for this hold on the lattice for any degree of approximation. Let us consider what happens to these commutation relations on the lattice for the approximation labeled by the superscript (1) in Eqs. (9):

$$[E_i^{(1)\alpha}(x),A_j^{(1)\beta}(y)]=\left[E_i^{(1)\alpha}(x),A_j^\beta(y)+\frac{a^2}{24}\partial_j^2A_j^\beta(y)\right]. \tag{11}$$

We observe that if the lattice electric field is taken to be the continuum electric field, order a^2 errors arise in the commutation relations. To cancel this error we set

$$E_i^{(1)\alpha}(x)=E_i^\alpha(x)-\frac{a^2}{24}\partial_i^2E_i^\alpha(x). \tag{12}$$

We can take this to order a^4 by setting

$$E_i^{(2)\alpha}(x) = E_i^\alpha(x) - \frac{a^2}{24}\partial_i^2 E_i^\alpha(x) + \frac{7a^4}{5760}\partial_i^4 E_i^\alpha(x). \quad (13)$$

In this way a sequence of approximations to the *lattice* electric field E^L can be constructed.

Making use of these approximations we can analyze the classical errors arising in the kinetic Hamiltonian. To cancel these errors we take the approach of adding new terms and fixing their coefficients in order to cancel the order a^2 error. We have a great deal of freedom in choosing additional terms. They are restricted only by gauge invariance and the need for an appropriate continuum limit.

To understand the construction of gauge invariant kinetic terms it is important to recall that the electric field and link operator transform as follows under a local gauge transformation $\Lambda(x)$:

$$E_i(x) \to \Lambda(x) E_i(x) \Lambda^\dagger(x) \quad (14)$$

$$U_i(x) \to \Lambda(x) U_i(x) \Lambda^\dagger(x+ai). \quad (15)$$

Consequently, the next most complicated gauge invariant term we can construct (after $\mathrm{Tr}E^L E^L$) couples nearest neighbor electric fields:

$$\mathrm{Tr}\{E_i^L(x) U_i(x) E_i^L(x+ai) U_i^\dagger(x)\}. \quad (16)$$

More complicated gauge invariant terms are easily constructed. One only needs to couple electric fields on different links anywhere around a closed loop. Consequently, generating Hamiltonians with higher degrees of improvement would seem to be more readily achieved within this approach.

Incorporating nearest neighbor interactions leads to the simplest improved kinetic Hamiltonian:

$$K^{(1)} = \frac{a^3}{2}\sum_{x,i} \mathrm{Tr}\{X E_i^L(x) E_i^L(x) + Y E_i^L(x) U_i(x) E_i^L(x+ai) U_i^\dagger(x)\}. \quad (17)$$

We fix the coefficients X and Y to cancel the order a^2 error. To do this we expand the second term in a Taylor series in a. Ignoring $\mathcal{O}(g^2a^2)$ errors, we then substitute $E^L \approx E^{(1)}$ from Eq. (12). To cancel the order a^2 error we must set $X = 5/6$ and $Y = 1/6$. This results in the order a^2 improved kinetic Hamiltonian

$$K^{(1)} = \frac{a^3}{2}\sum_{x,i} \mathrm{Tr}\left\{\frac{5}{6}E_i^L(x) E_i^L(x) + \frac{1}{6}E_i^L(x) U_i(x) E_i^L(x+ai) U_i^\dagger(x)\right\}. \quad (18)$$

This is the result of Luo, Guo, Kröger and Schütte [9]. We can take this to order a^4 by including next nearest neighbor interactions. A similar calculation using $E^L \approx E^{(2)}$ from Eq. (13) gives

$$K^{(2)} = \frac{a^3}{2}\sum_{x,i} \mathrm{Tr}\left\{\frac{97}{120}E_i^L(x) E_i^L(x) + \frac{1}{5}E_i^L(x) U_i(x) E_i^L(x+ai) U_i^\dagger(x) - \frac{1}{120}E_i^L(x) U_i(x) U_i(x+ai) E_i^L(x+2ai) \times U_i^\dagger(x+ai) U_i^\dagger(x)\right\}. \quad (19)$$

III. TADPOLE IMPROVEMENT

Tadpole improvement, developed by Lepage and Mackenzie [12], is an important step in removing errors from lattice gauge theory. It is necessary for close agreement between lattice perturbation theory and Monte Carlo calculations on coarse lattices.

In the action formulation, tadpole improvement is handled by dividing all link operators by the mean link u_0. In the Hamiltonian approach two conflicting implementations have been suggested. The earliest starts from a tadpole improved action and carries factors of u_0 into the Hamiltonian [9]. More recently it was suggested that no tadpole improvement was necessary in the kinetic term of the improved Hamiltonian [13]. Here we present our own views on the correct implementation.

In the Hamiltonian approach the question of whether the electric field should be scaled arises. This question is easily answered by considering the commutation relations between the link operator and electric field:

$$[E_i^\alpha(x), U_j(y)] = \frac{g}{a^2}\delta_{ij}\delta_{xy}\lambda^\alpha U_i(x). \quad (20)$$

We see that if we divide all link operators by u_0 we have

$$\left[E_i^\alpha(x), \frac{1}{u_0}U_j(y)\right] = \frac{g}{a^2}\delta_{ij}\delta_{xy}\lambda^\alpha \frac{1}{u_0}U_i(x). \quad (21)$$

We observe that the electric field cannot be rescaled and still maintain the correct commutation relations. Thus under tadpole improvement the electric field cannot change. We must, however, divide the second of the kinetic terms by a factor of u_0^2. Tadpoles arise in this term because the electric and gluon fields do not commute.

Including tadpole improvement in Eqs. (6) and (18) leads to the order a^2 tadpole improved Hamiltonian:

$$H^{(1)} = K^{(1)} + V^{(1)} = \frac{a^3}{2}\sum_{x,i} \mathrm{Tr}\left\{\frac{5}{6}E_i^L(x) E_i^L(x) + \frac{1}{6u_0^2}E_i^L(x) U_i(x) E_i^L(x+ai) U_i^\dagger(x)\right\} - \frac{2N}{ag^2}\sum_{x,i<j}\left[\frac{5}{3u_0^4}P_{ij}(x) - \frac{1}{12u_0^6}[R_{ij}(x) + R_{ji}(x)]\right]. \quad (22)$$

IV. CONCLUSION

We have demonstrated that direct improvement of the Kogut-Susskind Hamiltonian by demanding the correct continuum limit is possible. The advantage of our direct approach is that it is easily extended to more complicated Hamiltonians. One simply needs to construct suitable gauge invariant terms to add to the kinetic Hamiltonian and fix the coefficients so that higher order errors are canceled.

Our next step is to perform variational and coupled cluster SU(3) calculations to determine precisely the level of improvement achieved by the improved Hamiltonians. Other groups have made progress with these calculations for U(1) [13] and SU(2) [14] with promising results.

In the near future we intend to extend the direct approach to the cancellation of quantum errors which have not yet been examined in Hamiltonian lattice gauge theory.

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