Check of QCD based on the τ-decay data analysis in the complex q^2 plane

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The thorough analysis of the ALEPH data on hadronic τ decay is performed in the framework of QCD. The perturbative calculations are performed in three- and four-loop approximations. The terms of the operator product expansion (OPE) are accounted for up to the dimension $D=8$. The value of the QCD coupling constant $\alpha_s(m_\tau^2)=0.355\pm 0.025$ is found from the hadronic branching ratio R_τ. The $V+A$ and V spectral functions are analyzed using the analytical properties of the polarization operators in the whole complex q^2 plane. Borel sum rules in the complex q^2 plane along the rays, starting from the origin, are used. It is demonstrated that QCD with OPE terms is in agreement with the data for a coupling constant close to the lower error edge $\alpha_s(m_\tau^2)=0.330$. The restriction on the value of the gluonic condensate was found to be $\langle(\alpha_s/\pi)G^2\rangle=0.006\pm 0.012$ GeV^2. The analytical perturbative QCD is compared with the data. It is demonstrated to be in strong contradiction with experiment. The restrictions on the renormalon contribution are found. The instanton contributions to the polarization operator are analyzed in various sum rules. In the Borel transformation they appear to be small, but not in the spectral moment sum rules.

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I. INTRODUCTION

The high-precision data on hadronic τ decay, obtained by the ALEPH [1], OPAL [2], and CLEO [3] Collaborations, namely, the measurements of the total hadronic branching ratio $R_\tau=B(\tau\to\nu_\tau+\text{hadrons})/B(\tau\to e\bar{\nu}_e\nu_\tau)$, vector V and axial vector A spectral functions, allow one to perform various tests of QCD at low energies to determine $\alpha_s(Q^2)$ at low Q^2, to check the operator product expansion (OPE) and to perform a search for other possible nonperturbative modifications of QCD — renormalons, analytical $\alpha_s(Q^2)$, instantons, etc. An early attempt to check OPE in QCD based on e^+e^- annihilation data was made by Eidelman *et al.* [4] but the accuracy of the data at that time was not good enough. Also the authors of Ref. [4] took for granted that the QCD coupling constant is rather small, the $\Lambda_{QCD}^{(3)}$ (for three flavors) is about 100 MeV and neglected the higher-order terms of the perturbative series. Now it is common belief that α_s is much larger and $\Lambda^{(3)}\sim 300-400$ MeV in the 2–3 loop approximation. Therefore the problem deserves reconsideration.

The goal of the investigation is to analyze the hadronic structure functions, found from τ decay, within the framework of QCD. We first use the standard QCD, by which we mean perturbative QCD and the terms of OPE with coefficient functions given by perturbation theory. It will be demonstrated that standard QCD is in agreement with the data at the values of the complex Borel parameter (Borel transform of Q^2) $M^2>0.8-1.0$ GeV^2 in the left complex half-plane with accuracy better than 2%. Then nonperturbative modifications of QCD will be studied and the restrictions on the parameters, characterizing those modifications, will be found (Secs. III, VI, and VII).

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In the previous paper by two of us (B.I. and K.Z.) [5] the difference of vector and axial current correlators was analyzed using ALEPH data on τ decay [1]. The analytical properties of the polarization operator in the whole complex q^2 plane were exploited and the vacuum expectation values of dimension six and eight operators (vacuum condensates) were found. Here we consider the $V+A$ correlator, where perturbative corrections are dominant.

Define the polarization operators of hadronic currents:

$$\Pi_{\mu\nu}^J(q)=i\int e^{iqx}\langle TJ_\mu(x)J_\nu(0)^\dagger\rangle dx$$
$$=(q_\mu q_\nu-g_{\mu\nu}q^2)\Pi_J^{(1)}(q^2)+q_\mu q_\nu\Pi_J^{(0)}(q^2), \quad (1)$$

where

$$J=V,A;\quad V_\mu=\bar{u}\gamma_\mu d,\quad A_\mu=\bar{u}\gamma_\mu\gamma_5 d.$$

The imaginary parts of the correlators are the so-called spectral functions ($s=q^2$),

$$v_1/a_1(s)=2\pi\,\mathrm{Im}\,\Pi_{V/A}^{(1)}(s+i0),$$
$$a_0(s)=2\pi\,\mathrm{Im}\,\Pi_A^{(0)}(s+i0), \quad (2)$$

which have been measured from hadronic τ decays for $0<s<m_\tau^2$.

The spin-1 parts $\Pi_V^{(1)}(q^2)$ and $\Pi_A^{(1)}(q^2)$ are analytical functions in the complex q^2 plane with a cut along the right semiaxes starting from the threshold of the lowest hadronic state: $4m_\pi^2$ for $\Pi_V^{(1)}$ and $9m_\pi^2$ for $\Pi_A^{(1)}$. The latter has a kinematical pole at $q^2=0$. This is a specific feature of QCD, which follows from the chiral symmetry in the limit of massless u,d quarks and its spontaneous violation. It can be easily shown [6] (see also Ref. [5]), that the kinematical pole arises from the pion contribution to $\Pi_{\mu\nu}^A$, which is given by

$$\Pi^A_{\mu\nu}(q)_\pi = -\frac{f_\pi^2}{q^2}(q_\mu q_\nu - g_{\mu\nu}q^2) - \frac{m_\pi^2}{q^2} q_\mu q_\nu \frac{f_\pi^2}{q^2-m_\pi^2}, \tag{3}$$

where f_π is the pion decay constant, $f_\pi = 130.7$ MeV [7].

II. HADRONIC BRANCHING RATIO AND THE VALUE OF $\alpha_s(m_\tau^2)$

The total hadronic branching ratio into final state with zero strangeness is given by a well-known expression, which can be written in the following form (see, e.g., Ref. [8]):

$$R_{\tau,V+A} = \frac{B(\tau \to \nu_\tau + \text{hadrons}_{S=0})}{B(\tau \to \nu_\tau e \overline{\nu_e})}$$
$$= 6|V_{ud}|^2 S_{EW} \int_0^{m_\tau^2} \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2 \times \left[\left(1 + 2\frac{s}{m_\tau^2}\right)(v_1 + a_1 + a_0)(s) - 2\frac{s}{m_\tau^2} a_0(s)\right], \tag{4}$$

where $|V_{ud}| = 0.9735 \pm 0.0008$ [7] is the Cabibbo-Kaboyashi-Maskawa matrix element, and $S_{EW} = 1.0194 \pm 0.0040$ includes electroweak corrections [9]. The spin-0 axial spectral function $a_0(s)$ is basically saturated by the $\tau \to \pi\nu_\tau$ channel and can be read off from Eq. (3): $a_0(s) = 2\pi^2 f_\pi^2 \delta(s - m_\pi^2)$. So the last term in Eq. (4) gives small correction

$$\Delta R_\tau^{(0)} = -24\pi^2 \frac{f_\pi^2 m_\pi^2}{m_\tau^4} = -0.008. \tag{5}$$

The rest of Eq. (4) contains only the imaginary part of $\Pi^{(1)}_{V+A}(s) + \Pi^{(0)}_A(s)$, for which the short notation $\Pi(s)$ will be used later on. As follows from Eq. (3), $\Pi^{(0)}_A(q^2)$ compensates the kinematical pole at $q^2 = 0$ in $\Pi^{(1)}_A(q^2)$. So the combination $\Pi(q^2)$ has no kinematical poles and is an analytical function of q^2 in the complex q^2 plane with a cut along the positive real axis.

The convenient way to calculate the R_τ in QCD or, turning the problem around, to find $\alpha_s(m_\tau^2)$ from experimentally known R_τ, is to transform the integral in Eq. (4) to the integral over the contour in the complex s plane going couterclockwise around the circle $|s| = m_\tau^2$ [10–13]:

$$R_{\tau,V+A} = 6\pi i |V_{ud}|^2 S_{EW} \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2 \times \left(1 + 2\frac{s}{m_\tau^2}\right)\Pi(s) + \Delta R_\tau^{(0)}. \tag{6}$$

The polarization operator is given by the sum of perturbative and nonperturbative terms. If we restrict ourselves by OPE terms, then

$$\Pi(s) = -\frac{1}{2\pi^2}\ln\frac{-s}{\mu^2} + \text{higher loops} + \sum_{n\geq 2} \frac{\langle O_{2n}\rangle}{(-s)^n}\left(1 + c_n \frac{\alpha_s}{\pi}\right). \tag{7}$$

Consider at first the perturbative part. For its calculation, it is convenient to use the Adler function, which is perturbatively constructed as an expansion in coupling constant

$$D(Q^2) \equiv -2\pi^2 \frac{d\Pi(Q^2)}{d\ln Q^2} = \sum_{n\geq 0} K_n a^n,$$
$$a \equiv \frac{\alpha_s}{\pi}, \quad Q^2 \equiv -s, \tag{8}$$

which is known up to the 4-loop term in the modified minimal subtraction ($\overline{\text{MS}}$) renormalization scheme: $K_0 = K_1 = 1$ and $K_2 = 1.64$ [14], $K_3 = 6.37$ [15] for 3 flavors. The renormalization-group equation for $a(Q^2)$ reads:

$$\frac{da}{d\ln Q^2} = -\beta(a) = -\sum_{n\geq 0} \beta_n a^{n+2}$$
$$\Rightarrow \ln\frac{Q^2}{\mu^2} = -\int_{a(\mu^2)}^{a(Q^2)} \frac{da}{\beta(a)}. \tag{9}$$

In the $\overline{\text{MS}}$ scheme for 3 flavors, $\beta_0 = 9/4$, $\beta_1 = 4$, $\beta_2 = 10.06$, and $\beta_3 = 47.23$ [16,17]. This allows us to get the perturbative contribution to the polarization operator explicitly at any order of perturbation theory:

$$\Pi(Q^2) - \Pi(\mu^2) = \frac{1}{2\pi^2}\int_{a(\mu^2)}^{a(Q^2)} D(a)\frac{da}{\beta(a)}. \tag{10}$$

Let us put $\mu^2 = m_\tau^2$ and choose some value $a(m_\tau^2)$. From Eq. (9) we can find $a(Q^2)$ for any Q^2 and by analytical continuation at any s. Computing the integral (10) it is possible to find the perturbative part of $\Pi(s)$ as a function of $a(s)$ in the whole complex s plane. The substitution of $\Pi(s)$ into Eq. (6) gives (up to the power corrections) the dependence of R_τ on $a(m_\tau^2)$. It must be stressed, that in this calculation, no expansion in inverse powers of $\ln Q^2$ is performed: only the validity of the expansion series in Eq. (8) and Eq. (9) is assumed.[1] Such representation has a serious advantage: on the right semiaxes, i.e., in the physical region, there is no expansion in $\pi/\ln(Q^2/\Lambda^2)$, which is not small at intermediate Q^2. For instance, in the next to leading order

$$2\pi \,\text{Im}\,\Pi(s + i0) = 1 + \frac{1}{\pi\beta_0}\left[\frac{\pi}{2} - \arctan\left(\frac{1}{\pi}\ln\frac{s}{\Lambda^2}\right)\right] \tag{11}$$

instead of

[1] Such manner of calculation in Ref. [1] was called contour-improved fixed-order perturbation theory.

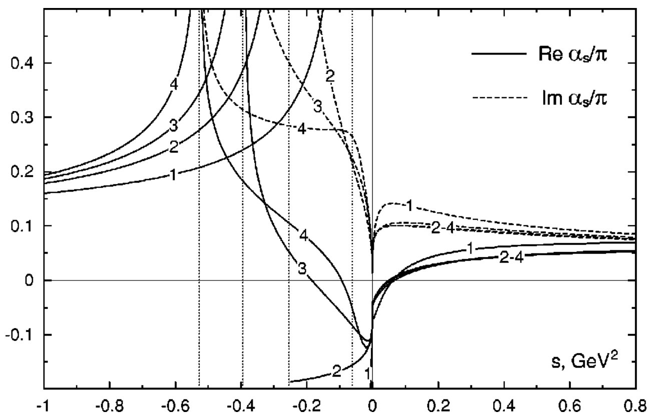

FIG. 1. Real and imaginary parts of $\alpha_{\overline{\mathrm{MS}}}(s)/\pi$ as exact numerical solution of RG equation (9) on real axes for different number of loops. The initial condition is chosen $\alpha_s = 0.355$ at $s = -m_\tau^2$, $N_f = 3$. Vertical dotted lines display the position of the unphysical singularity at $s = -Q_0^2$ for each approximation ($4 \to 1$ from left to right).

$$2\pi\,\mathrm{Im}\,\Pi(s+i0) = 1 + \frac{1}{\beta_0 \ln(s/\Lambda^2)},$$

which would follow in the case of small $\pi/\ln(s/\Lambda^2)$. [Equation (11) was first obtained in Ref. [18], the systematical method of analytical continuation from the spacelike to the timelike region with summation of π^2 terms, was suggested in Ref. [19] and developed in Ref. [20].] In the higher order, where $a(s)$ cannot be expressed via $\ln(s/\Lambda^2)$ in terms of elementary functions, this analysis is performed numerically.

It is well known, that in the one-loop approximation of the β function, the coupling $a(Q^2)$ has an infrared pole at some $Q^2 = Q_0^2$ (in some conventions coinciding with Λ^2). In the n-loop approximation ($n > 1$) instead of pole a branch cut appears with a singularity $\sim (1 - Q^2/Q_0^2)^{-1/n}$. The position of the singularity is given by

$$\ln\frac{Q_0^2}{\mu^2} = -\int_{a(\mu^2)}^{\infty} \frac{da}{\beta(a)}. \tag{12}$$

Near the singularity the last term in the expansion of $\beta(a)$ (9) dominates and gives the aforementioned behavior. To illustrate the behavior of the running coupling constant, we plotted the real and imaginary part of α_s/π for the $n = 1,2,3,4$-loop β function in Fig. 1. It demonstrates, that for real positive s, the difference between various approximations is almost unnoticable beyond the second loop and the expansion in inverse $|\ln(s/\Lambda^2) - i\pi|$ works well. At the same time the behavior in the unphysical cut strongly depends on the number of loops and cannot be described by some simple approximation. Only at $s < -1$ GeV^2, 2–4 loop calculations more or less coincide.

Let us turn now to OPE terms in Eq. (7). The contribution of the operators up to dimension 8 have been computed theoretically:

$$\sum_{n \geqslant 2} \frac{\langle O_{2n} \rangle}{(-s)^n}\left(1 + c_n \frac{\alpha_s}{\pi}\right)$$

$$= \frac{\alpha_s}{6\pi Q^4}\langle G^a_{\mu\nu} G^a_{\mu\nu}\rangle\left(1 + \frac{7}{6}\frac{\alpha_s}{\pi}\right) + \frac{128}{81 Q^6}\pi\alpha_s \langle \overline{q} q\rangle^2 \times\left[1 + \left(\frac{29}{24} + \frac{17}{18}\ln\frac{Q^2}{\mu^2}\right)\frac{\alpha_s}{\pi}\right] + \frac{\langle O_8\rangle}{Q^8}. \tag{13}$$

The contribution of the $D = 2$ operator, due to nonzero quark masses $m_{u,d}$, is negligible and omitted here. We have also neglected the $D = 4$ quark condensate $2(m_u + m_d)\langle \overline{q} q\rangle$, which is an order of magnitude less than the gluonic condensate. The coefficients in front of the $D = 4{,}6$ operators, have been computed by Shifman, Vainstein, and Zakharov (SVZ) [21], The α_s corrections to the $D = 4$ operator were found in Ref. [22]; α_s corrections to the $D = 6$ operator were calculated [23]; ambiguities among them were also discussed there.

A few comments about the operator O_6 are in order. In the nonfactorized form without α_s corrections, it looks as follows [21]:

$$\langle O_6\rangle = -2\pi\alpha_s\Big\langle (\overline{u}\gamma_\mu \lambda^a d)(\overline{d}\gamma_\mu \lambda^a u) + (\overline{u}\gamma_5\gamma_\mu \lambda^a d) \times(\overline{d}\gamma_5\gamma_\mu\lambda^a u) + \frac{2}{9}(\overline{u}\gamma_\mu\lambda^a u + \overline{d}\gamma_\mu\lambda^a d)\sum_{u,d,s}(\overline{q}\gamma_\mu\lambda^a q)\Big\rangle. \tag{14}$$

After factorization, three terms in Eq. (14) give the following contributions:

$$\langle O_6\rangle = 4\pi\alpha_s\langle\overline{q}q\rangle^2\left(1 - \frac{1}{N_c^2}\right)\left(1 - 1 + \frac{4}{9}\right), \tag{15}$$

where N_c is the number of colors. SVZ assumed that the accuracy of the factorization procedure is of order $N_c^{-2} \sim 10\%$, in the case of the V correlator, where the coefficient in the second bracket in Eq. (15) is equal to $-7/9$. Remember that in the $V-A$ correlator, the first term has opposite sign and the third term is absent, so the accuracy of the factorized operator O_6^{V-A} is at least, not worse, than in the V case. On the other hand in the $V+A$ correlator two comparatively large terms cancel each other under the factorization assumption in Eq. (15). Consequently the accuracy of the formula (15) for the operator O_6 is less, perhaps 20–30%. Large α_s corrections to all independent $D = 6$ operators [23] can only increase the errors.

The numerical value of the $D = 6$ operator can be estimated, for instance, with the help of our previous analysis of $V-A$ sum rules [5]:

$$\langle O_6^{V-A}\rangle = -\frac{64}{9}\pi\alpha_s\langle\bar{q}q\rangle^2\times 1.3$$
$$= -(6.8\pm 2.1)\times 10^{-3}\ \ \mathrm{GeV}^6. \quad (16)$$

The coefficient 1.3 stands for the α_s corrections. We find

$$\langle O_6\rangle = (1.3\pm 0.5)\times 10^{-3}\ \ \mathrm{GeV}^6. \quad (17)$$

The dimension 8 operators come from many different diagrams, which can be labeled by the number of quarks in vacuum. The purely gluonic condensates are suppressed by the loop factor $\sim\alpha_s/\pi$ and are neglected on this ground. The four-quark operators, computed in Refs. [24,25] and [5], vanish in the sum $V+A$ after factorization. The uncertainty of this cancellation can be estimated as $\sim 10\%$ of O_8^{V-A}, which is about 10^{-3} GeV^8. The two-quark operators have the same sign in the V and A correlators. They have been computed in Ref. [26] (we have performed the calculation independently to confirm this result) and can be written in the following form:

$$\langle O_8^{(2)}\rangle = \frac{2}{9}\Big\langle 2i\bar{u}\gamma^\alpha\{G^2_{(\alpha\beta)},D^\beta\}u - \bar{u}\gamma^\alpha\gamma^5\{(\tilde{G}G)_{\alpha\beta},D^\beta\}u$$
$$+\frac{i}{4}\bar{u}\gamma^\gamma[G^{\alpha\beta},(D_\gamma G_{\alpha\beta})]u + \frac{1}{2}\bar{u}(D^2\hat{J})u$$
$$- i\bar{u}\gamma^\alpha[G_{\alpha\beta},J^\beta]u\Big\rangle + (u\to d), \quad (18)$$

where $J_\mu = D_\nu G_{\mu\nu} = \pi\alpha_s\lambda^a\Sigma_q(\bar{q}\gamma_\mu\lambda^a q)$. The last two terms can be factorized and brought to the form $\pi\alpha_s\langle\bar{q}q\rangle\langle\bar{q}\hat{G}q\rangle$. However the leading terms in the number of colors N_c^0 cancel each other and only the terms $\sim N_c^{-2}$ are left. It has been shown in Ref. [5], that the factorization of the $D=8$ operators is not unambiguous at this level of accuracy. Taking the value of the operator $\langle\bar{q}\hat{G}q\rangle$ from Refs. [27,5], we may estimate the upper limit of the operator (18) as $|\langle O_8^{(2)}\rangle| < 10^{-4}$ GeV^8, which is tiny. So, for the upper limit of the total $D=8$ operator, we shall use the estimation $|\langle O_8\rangle| < 10^{-3}$ GeV^8.

It is worth mentioning that the $D=6,8$ operators in the $V+A$ polarization function are much smaller than in V or A separately.

We are now in a position to calculate $\alpha_s(m_\tau^2)$ from the experiment. We take the most recent data on the total hadronic decay ratio R_τ [7] and the ratio of decays with odd number of strange mesons $\tau^-\to X(S=-1)\nu_\tau$ [28,29]:

$$R_\tau = 3.636\pm 0.021,\quad R_{\tau,S} = 0.161\pm 0.007. \quad (19)$$

In our analysis we subtract $R_{\tau,S}$ to avoid the interference with additional parameters, in particular the mass of the s quark. One obtains

$$R_{\tau,V+A} = 3|V_{ud}|^2 S_{EW}(1+\delta'_{EW}+\delta^{(0)}+\delta^{(6)}_{V+A}) + \Delta R^{(0)}$$
$$= 3.475\pm 0.022, \quad (20)$$

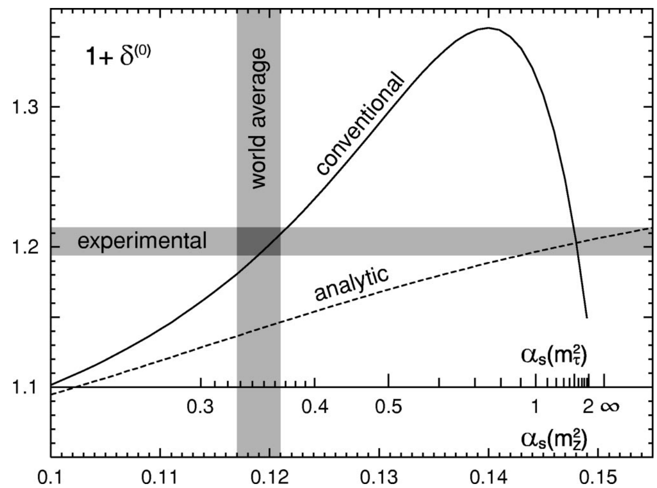

FIG. 2. Perturbative fractional correction $\delta^{(0)}$ versus $\alpha_s(m_\tau^2)$ and $\alpha_s(m_Z^2)$ in a conventional and analytical approach in a three-loop approximation. In the width of the experimental strip, the theoretical uncertainty of the operator $\langle O_6\rangle$ is included.

where $\Delta R_\tau^{(0)}$ is given by Eq. (5). We use conventional in τ-literature notations of fractional corrections δ. The electromagnetic correction is $\delta'_{EW} = (5/12\pi)\alpha_{em}(m_\tau^2) = 0.001$ [30], and the $D=6$ operator correction $\delta^{(6)}_{V+A} = -(5\pm 2)\times 10^{-3}$ as follows from our analysis, is in agreement with the estimation obtained in Ref. [12]. From Eq. (20) we separate out the perturbative correction:

$$1+\delta^{(0)} = 1.206\pm 0.010. \quad (21)$$

All errors in here are added in quadratures (perhaps, such a procedure underestimates the total error, maybe by a factor of 2).

The calculation of $\alpha_s(m_\tau^2)$ corresponding to $\delta^{(0)}$ as performed according to the method described above. The dependence of $1+\delta^{(0)}$ on $\alpha_s(m_\tau^2)$ [and on $\alpha(M_Z^2)$, to compare with the other data] for the three-loop β function and three-loop Adler function is shown in Fig. 2. It follows from Fig 2

$$\alpha_s(m_\tau^2) = 0.355\pm 0.025. \quad (22)$$

The estimation of the error in Eq. (22) was done with care. Because of the asymptotic character of the perturbative series (8) and (9) the higher loop contribution could be as large as the contribution of the last terms, namely, K_3a^3 and $\beta_2 a^4$. They result in the uncertainty $0.015-0.020$ in $\alpha_s(m_\tau^2)$, depending on its central value. Taking into account the uncertainty (21) in $\delta^{(0)}$, we obtain the error in Eq. (22). Furthermore, we have performed a two and four-loop calculation of $\alpha_s(m_\tau^2)$. The unknown four-loop coefficient in the Adler function (8) was taken equal to $K_4=50$ (cf. its estimations [31]). For each given $\delta^{(0)}$, the 4-loop $\alpha_s(m_\tau^2)$ is by 0.005 lower than three-loop value, while 2-loop $\alpha_s(m_\tau^2)$ is higher by 0.02. These results are within the error range (22). If some nonperturbative terms beyond OPE exist (e.g., instantons), they would also contribute to the error in Eq. (22). In Sec. IV it will be shown that the value $\alpha_s(m_\tau^2)$ close to the lower limit of Eq. (22) satisfies sum rules at low Q^2 much better.

III. $\alpha_s(m_\tau^2)$ AND ANALYTICAL QCD

Shirkov and Solovtsov [32] forwarded the idea of analytical QCD. According to it the coupling constant $\alpha_s(Q^2)$ is calculated by the renormalization group in the spacelike region $Q^2>0$. Then, by analytical continuation to $s=-Q^2>0$, $\alpha_s(s)$ was found, in particular, to its imaginary part $\mathrm{Im}\alpha_s(s)$ on the right semiaxes. It was assumed, that $\alpha_s(s)$ is an analytical function in the complex s plane with a cut along the right semiaxes $0\leqslant s<\infty$. The analytical $\alpha_s(s)_{\mathrm{an}}$ is then defined in the whole complex s plane by the dispersion relation

$$\alpha_s(s)_{\mathrm{an}}=\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,\alpha_s(s')}{s'-s}ds'. \tag{23}$$

Since the lower limit in this integral is put to zero, $\alpha_s(s)_{\mathrm{an}}$ indeed has no unphysical singularities (poles, cuts, etc.) at $Q^2>0$. The idea of analytical QCD has been developed in many papers, see, e.g., Ref. [33] and for review Ref. [34]. In particular, the calculations of $\alpha_s(m_\tau^2)_{\mathrm{an}}$ from τ-decay data, were performed within the framework of analytical QCD in Ref. [35].

A related approach was suggested by two of us (B.G. and B.I.) in Ref. [36]. We started from a well-known theorem, that the polarization operator for the e^+e^- annihilation $\Pi(s)$ is an analytical function of s in the complex s plane with a cut along positive semiaxes, and assumed that these analytical properties take place separately for perturbative and nonperturbative parts of $\Pi(s)$. In the first order of α_s, this hypothesis is equivalent to analytical QCD while in higher orders it may be more general.

Let us calculate $\alpha_s(m_\tau^2)_{\mathrm{an}}$ in the framework of analytical QCD from the same experimental data, i.e., $\delta^{(0)}$ given by Eq. (21). The only (but important) difference from the previous calculation is the following. The coupling $\alpha_s(s)_{\mathrm{an}}$ is an analytical function of s with a cut from $s=0$ to $s=\infty$. Consequently, the contour integral in Eq. (6) is now equal to the original integral (4) with $\mathrm{Im}\,\Pi(s)$ over the real positive axes. In the previous calculation, if such a transformation is performed, the integral would run from $s=-Q_0^2$ to m_τ^2. Qualitatively it leads to much smaller R_τ, in the analytical QCD than in the conventional approach with the same $\alpha_s(m_\tau^2)$, or vice versa, the same R_τ corresponds to much larger $\alpha_s(s)_{\mathrm{an}}$. Direct numerical calculation confirms this expectation. The dependence of $1+\delta^{(0)}$ versus $\alpha_s(s)_{\mathrm{an}}$ is also displayed in Fig. 2. It is seen, that in order to get the experimental value of $\delta^{(0)}$ in the analytical QCD, one should take $\alpha_s(m_\tau^2)_{\mathrm{an}}\approx 1.5$–$2.0$, which corresponds to $\alpha_s(m_Z^2)\approx 0.15$, in strong contradiction with the world average $\alpha_s(m_Z^2)=0.119\pm 0.002$ [7]. (The previous calculation of the τ decay [35], performed with less certainty, demonstrated the same trend; in particular $\Lambda^{(3)}=700-900$ MeV, much larger than in standard calculations.)

In the recent paper [37], an attempt was made to save the analytical QCD in the case of vector the polarization operator and to obtain the agreement with ALEPH data on the vector, Adler D function, by assuming large quark masses $m_u=m_d=250$ MeV and some form of Coulomblike quark-antiquark interaction. This hypothesis, however, is in strong contradiction with all the results following from the well-established partial conservation of axial current and chiral theory. For example, if the Gell-Mann–Oakes–Renner relation and the K/π mass ratios would be violated by an order of magnitude, the Goldberg-Treitman relation cannot be proved, etc. Also many sum rules for $V-A$ polarization operator would disagree with the data.

Therefore we come to the conclusion, that analytical QCD in any form, [32] or [36], is in strong contradiction with experiment and must be abandoned.

IV. CHECK OF QCD AT LOW Q^2 FOR $V+A$ CORRELATORS BY USING THE SUM RULES

Let us now turn to the study of the $V+A$ correlator in the domain of low Q^2, where the OPE terms play a much more essential role than in the determination of R_τ. A general remark is in order here. As was mentioned in Ref. [38] and stressed recently by Shifman [39], the condensates cannot be defined in a rigorous way, because there is some arbitrariness in the separation of their contributions from the perturbative part. Usually [38,39] they are defined by the introduction of some normalization point μ^2 with the magnitude of a few Λ^2. The integration over momenta in the domain below μ^2 is addressed to condensates, above μ^2 — to perturbation theory. In such formulation the condensates are μ dependent $\langle O_D\rangle=\langle O_D\rangle_\mu$ and, strictly speaking, they also depend on the way the infrared cutoff μ^2 is introduced. The problem becomes more severe when the perturbative expansion is performed up to higher-order terms and the calculation pretends on high precision. We mention, that this remark does not refer to chirality violating condensates, because perturbative terms do not contribute to chirality violating structures. For this reason, in principle, chirality violating condensates, e.g., $\langle 0|\bar{q}q|0\rangle$, can be determined with higher precission, than chirality conserving ones. Here we use the definition of condensates, which can be called n-loop condensates. As was formulated in Sec. II, we treat the renormalization-group equation (9) and the equation for the polarization operator (10) in the n-loop approximation, as exact ones; the expansion in inverse logarithms is not performed. Specific values of condensates are referred to such a procedure. Of course, their numerical values depend on the accounted number of loops; that is why the condensates, defined in this way, are called n-loop condensates.

Consider the polarization operator $\Pi=\Pi_{V+A}^{(1)}+\Pi_A^{(0)}$, defined in Eq. (1) and its imaginary part

$$\omega(s)=v_1(s)+a_1(s)+a_0(s)=2\pi\ \mathrm{Im}\,\Pi(s+i0). \tag{24}$$

In the parton model $\omega(s)\to 1$ at $s\to\infty$. Any sum rule can be written in the following form:

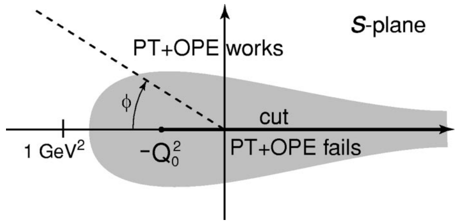

FIG. 3. Region of validity of the perturbation theory and OPE.

$$\int_0^{s_0} f(s)\,\omega_{\rm exp}(s)\,ds = i\pi \oint f(s)\,\Pi_{\rm theor}(s)\,ds, \qquad (25)$$

where $f(s)$ is some analytical in the integration region function. In what follows we use $\omega_{\rm exp}(s)$, obtained from the τ-decay invariant-mass spectra published in Ref. [1] for $0<s<m_\tau^2$ with step $ds=0.05$ GeV^2. The experimental error of the integral (25) is computed as the double integral with the covariance matrix $\overline{\omega(s)\omega(s')}-\overline{\omega}(s)\overline{\omega}(s')$, which also can be obtained from the data available in Ref. [1]. In the theoretical integral in Eq. (25), the contour goes from s_0+i0 to s_0-i0 counterclockwise around all poles and cuts off the theoretical correlator $\Pi(s)$, see Fig. 3. Because of the Cauchy theorem, the unphysical cut must be inside the integration contour.

The choice of the function $f(s)$ in Eq. (25) is actually a matter of taste. At first let us consider the usual Borel transformation:

$$B_{\rm exp}(M^2) = \int_0^{m_\tau^2} e^{-s/M^2}\omega_{\rm exp}(s)\frac{ds}{M^2}$$
$$= B_{\rm pt}(M^2) + 2\pi^2 \sum_n \frac{\langle O_{2n}\rangle}{(n-1)!\,M^{2n}}. \qquad (26)$$

We separated out the purely perturbative contribution $B_{\rm pt}$, which is computed numerically according to Eq. (25) and Eqs. (8)–(10). Remember that the Borel transformation improves the convergence of the OPE series because of the factors $1/(n-1)!$ in front of the operators and suppresses the contribution of the high-energy tail, where the experimental error is large. But it does not suppress the unphysical perturbative cut, the main source of the error in this approach, and even increases it since $e^{-s/M^2}>1$ for $s<0$. So the perturbative part $B_{\rm pt}(M^2)$ can be reliably calculated only for $M^2 \approx 0.8$–1 GeV^2 and higher; below this value the influence of the unphysical cut is out of control.

Both $B_{\rm exp}$ and $B_{\rm pt}$ in a 3-loop approximation for $\alpha_s(m_\tau^2)=0.355$ and 0.330 are shown in Fig. 4. The shaded areas display the theoretical error. They are taken equal to the contribution of the last term in the perturbative Adler function expansion K_3a^3 (8). We have also performed the calculation with the 4-loop β-function and $K_4=50\pm50$, but the result is very close to the 3-loop one, since positive contribution of the term K_4a^4 compensates for the small decrease in the coupling a. Since this result is observed by us in many other sum rules, we shall not give the 4-loop calculations later on,

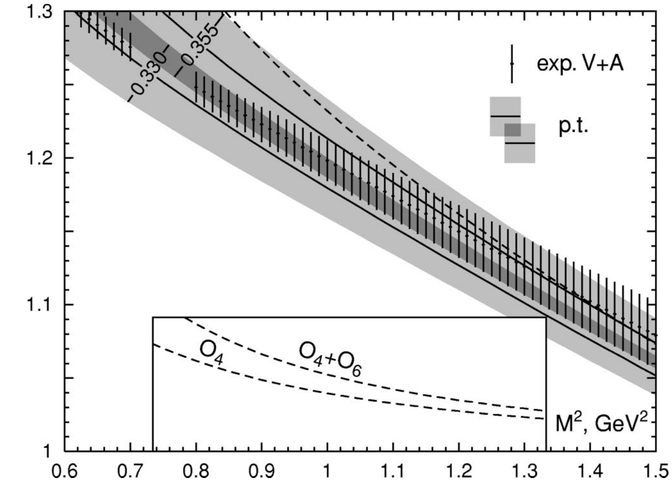

FIG. 4. Borel transformation (26) $B_{\rm exp}(M^2)$ and $B_{\rm pt}(M^2)$ for $\alpha_s(m_\tau^2)=0.355$ and 0.330. The dashed line displays the OPE contribution added to the 0.330-perturbative curve. The contribution of the operators $D=4$ (standard SVZ value) and $D=6$ [central value of (17)] with respect to 1, are shown separately in the box.

and instead estimate the theoretical error for any given $a(m_\tau^2)$ as the contribution of K_3a^3.

As follows from the analysis in Sec. II, for $M^2>1$ GeV^2, the contribution of $D=6,8$ operators to the Borel transform (26) is small in the $V+A$ channel, while the contribution of the $D=4$ condensate must be positive [we assume α_s corrections included in the operators $\langle O_{2n}\rangle$ in Eq. (26) and later]. So the theoretical curve must go below the experimental one. The result shown in Fig. 4 is in favor of the lower value of the coupling constant $\alpha_s(m_\tau^2)=0.33$. Literally the theoretical curve [perturbative at $\alpha_s(m_\tau^2)=0.33$ plus the contribution of O_4 and O_6 operators] agrees with experiment starting from $M^2=1.1$ GeV^2. If the uncertainties in perturbative contributions are taken into account (shaded area in Fig. 4) the agreement may start earlier, at $M^2=1$ GeV^2.

The Borel transformation in Fig. 4 includes the contributions of different operators. Although it is difficult to separate the perturbative part from the OPE one, the contributions of different operators can be separated from each other. One way is to differentiate the Borel transformation by M^2. This however leads to the certain loss in the accuracy of the experimental integral, since the growing power term $\sim s^n$ appears in the integral. So we apply the method used in [5] for $V-A$ sum rules, namely the Borel transformation in complex M^2 plane.

Let us consider the Borel transform $B(M^2)$ (26) at some complex $M^2=M_0^2e^{i\phi}$, $0<\phi<\pi/2$. If the phase ϕ is taken close to $\pi/2$, then the contribution of the high-energy tail becomes high. So we restrict ourselves by the values $\phi \leqslant \pi/4$ for the exponent to be decreasing enough. The real part of the Borel transform at $\phi=\pi/6$ does not contain the $D=6$ operator:

$$\mathrm{Re}\,B_{\rm exp}(M^2e^{i\pi/6}) = \mathrm{Re}\,B_{\rm pt}(M^2e^{i\pi/6}) + \pi^2\frac{\langle O_4\rangle}{M^4}. \qquad (27)$$

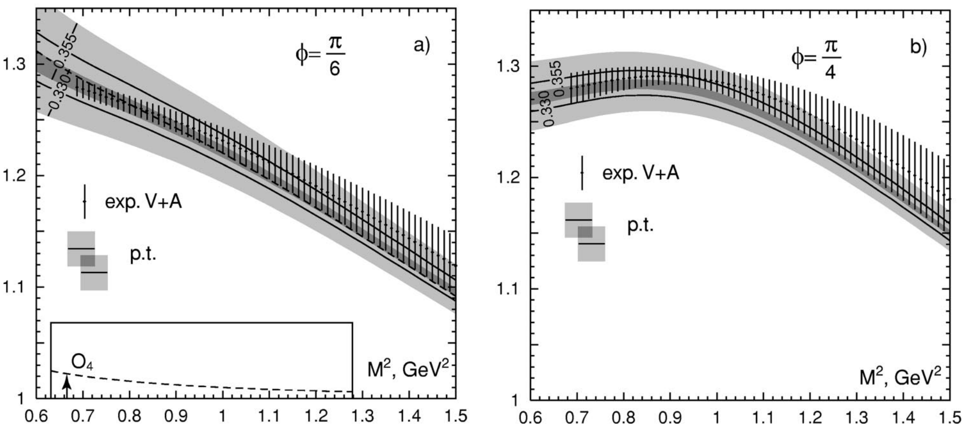

FIG. 5. Real part of the Borel transform (26) along the rays at the angles $\phi=\pi/6$ and $\pi/4$ to the real axes. The dash line corresponds to the gluonic condensate given by the central value of Eq. (28).

The contribution of $\langle O_8\rangle$ is less than 0.5% to the perturbative term and neglected here. The results are shown in Fig. 5(a). Again it is still difficult to accommodate the positive value of the gluonic condensate to the coupling $\alpha_s(m_\tau^2)=0.355$ and higher. If we accept the lower value of $\alpha_s(m_\tau^2)$, we get the following restriction on the value of the gluonic condensate:

$$\left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu} \right\rangle = 0.006 \pm 0.012 \ \ \mathrm{GeV}^4,$$

$$\alpha_s(m_\tau^2)=0.330 \ \text{ and } \ M^2>0.8 \ \ \mathrm{GeV}^2. \qquad (28)$$

The theoretical and experimental errors are added together in Eq. (28).

The real part of the Borel transform at $\phi=\pi/4$ does not contain the $D=4$ operator:

$$\mathrm{Re}\, B_{\mathrm{exp}}(M^2 e^{i\pi/4}) = \mathrm{Re}\, B_{\mathrm{pt}}(M^2 e^{i\pi/4}) - \pi^2 \frac{\langle O_6\rangle}{\sqrt{2} M^6}. \qquad (29)$$

The results are shown in Fig. 5(b). The perturbative curve at $\alpha_s=0.330$ is below the data. If we would take this curve as an exact one, without accounting for the perturbative errors, then from Eq. (29) we would conclude, that $\langle O_6\rangle<0$, which is in some contradiction with Eqs. (13) and (17). However, the account for the perturbative errors makes the situation different, but uncertain. Since the value of the $\langle O_6\rangle$ contribution to Eq. (29) is very small,

$$\pi^2 \frac{\langle O_6\rangle}{\sqrt{2} M^6} = (0.9 \pm 0.4) \times 10^{-2} \frac{\mathrm{GeV}^6}{M^6}, \qquad (30)$$

then by accounting for the perturbative errors, it is possible to satisfy the sum rule (29) at positive $\langle O_6\rangle$ starting from $M^2>0.8$ GeV^2. (In the narrow region near $M^2=0.9$ GeV^2, the theoretical curve goes out of the data on the $1.5-2$ experimental error, but we do not consider this a serious contradiction.) Unfortunately, no definite conclusion about the value of $\langle O_6\rangle$ can be drawn from the Fig. 5(b). The only statement is that its value cannot exceed Eq. (17) and probably is on the lower border of error.

V. CORRELATOR OF VECTOR CURRENTS

Previously we considered the $V+A$ correlators, where the power corrections are small. Instead one could take the pure vector current (vector spectral function was published by ALEPH in [40]). This does not give us any new information with the τ-decay data, since $V-A$ correlators have already been analyzed in Ref. [5]. Moreover the accuracy of the vector current spectral function is less than $V+A$, since both currents are mixed in some channels with K mesons and the number of events is twice as less.

However the analysis of the vector current correlator is important since it can also be performed with the experimental data on e^+e^- annihilation. The imaginary part of the electromagnetic current correlator, measured here, is related to the charged current correlator (1) by the isotopic symmetry. The statistical error in e^+e^- experiments is less than in τ decays because of significantly larger number of events. So it would be interesting to perform a similar analysis with e^+e^- data, which is a matter for separate research.

At first we consider the usual Borel transformation for the vector current correlator, since it was originally applied in Ref. [4] for the sum-rule analysis. It is defined as Eq. (26) with the experimental spectral function $\omega_{\mathrm{exp}}=2v_1$ instead of $v_1+a_1+a_0$ [the normalization is $v_1(s)\to 1/2$ at $s\to\infty$ in the parton model]. Respectively, on the right-hand side, one should take the vector operators $2O^V=O^{V+A}+O^{V-A}$, all O^{V-A} with $D\leqslant 8$ can be found in Ref. [5]. The numerical results are shown in Fig. 6. The perturbative theoretical curves are the same as in Fig. 4 the with $V+A$ correlator. The dashed lines display the contributions of the gluonic condensate given by Eq. (28), $2O_6^V=-5.5\times 10^{-3}$ GeV^6 and $2O_8^V=O_8^{V-A}=7\times 10^{-3}$ GeV^8, added to the 0.330-perturbative curve. The contribution of each condensate is shown in the box below. Notice that for such condensate values, the total OPE contribution is small, since positive O_4 and O_8 compensate negative O_6. The agreement is observed for $M^2>0.8$ GeV^2.

Now we apply the method of the Borel transformation along the rays to the vector polarization operator to separate the contribution of different operators from each other. The $D=8$ operator is important here, so we shall separate $O_{4,6}$ from O_8.

The Borel transformation at low M^2 exponentially sup-

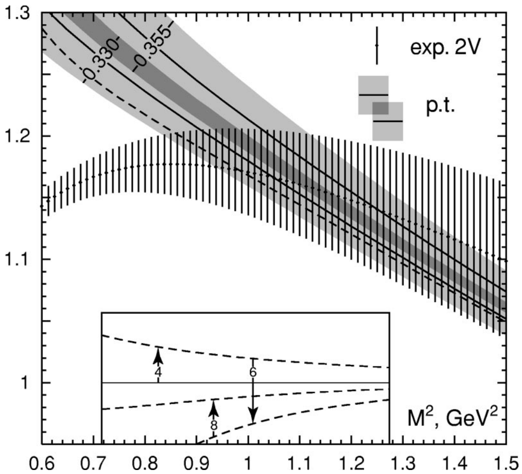

FIG. 6. Borel transformation for vector currents.

presses the contribution of the large s domain, where the experimental error is high. Besides this, we may use the oscillating behavior of the complex exponent to further suppress the high-error points near $s=m_\tau^2$. This would allow us to go to higher M^2. Here the real part of $B(M^2)$ has an obvious advantage since the function $\cos(\phi+(s/M^2)\cos\phi)$ has zero at $s=m_\tau^2$ and $\phi\sim\pi/4$ already at $M^2\approx 1\ \mathrm{GeV}^2$, while the largest (in M^2) zero of $\sin(\phi+(s/M^2)\cos\phi)$ in the imaginary part is twice as low. So let us take three different angles, say, $\phi=0,\ \pi/6$, and $\pi/4$. Solving the system of linear equations, we get

$$\mathrm{Re}\left[\frac{B(0)}{2-\sqrt{2}}-\sqrt{2}B(\pi/6)+\frac{B(\pi/4)}{\sqrt{2}-1}\right]=\mathrm{p.t.}+2\pi^2\frac{\langle O_4\rangle}{M^4}, \tag{31}$$

$$\mathrm{Re}\,\frac{-B(0)+2B(\pi/6)-2B(\pi/4)}{\sqrt{2}-1}=\mathrm{p.t.}+2\pi^2\frac{\langle O_6\rangle}{2M^6}. \tag{32}$$

For brevity we write $B(\phi)$ instead of $B_{\mathrm{exp}}(M^2e^{i\phi})$, and "p.t." stands for the perturbative contribution. The results for the Eqs. (31) and (32) are shown in Figs. 7(a) and 7(b), respectively.

Figure 7(a) demonstrates that the vector sum rule is satisfied at $\alpha_s(m_\tau^2)=0.330$ and gluonic condensate (28) (although higher values of the gluonic condensate, e.g., the SVZ value still does not contradict the data). Figure 7(b) shows, that the O_6^V contribution works in the right direction; its addition to the 0.330-perturbative curve shrinks the disagreement between the theory and experiment. However, some discrepancy (about 0.04, i.e., 0.1% in the worst case) still persists. It may be addressed either to the uncertainty in $\alpha_s(m_\tau^2)$ — a slightly higher value would be desirable, or to the underestimation of O_6^V (in absolute value, O_6^V is negative) by 20–30 %, or both. Remember that the numerical values of the condensates depend on the way, the infrared region is treated (O_6 is chirality conserving). We are considering here 3-loop condensates, defined in Sec. IV. The O_6^V value was taken equal to 7/18 of O_6^{V-A}, obtained from $V-A$ data analysis [5], where perturbative terms are absent, and some difference is not excluded.

VI. THE CHECK FOR RENORMALON-TYPE TERMS

In the asymptotic perturbative series a special part of terms — renormalons (infrared and ultraviolet) is often separated and the summation of them is performed (for a recent review see Ref. [41]). In such a sum, the term appears proportional to $1/Q^2$ at large Q^2, looking like a contribution of the $D=2$ operator. (In OPE the $D=2$ operator is proportional to m_q^2 and is very small.) Renormalons conserve chirality and may contribute to $V+A$ but not to $V-A$. Unfortunately, the coefficient in front of the $1/Q^2$ term of the renormalon origin cannot be calculated reliably. (In Ref. [42] it was claimed, that the renormalons are totally absent in the perturbative series asymptotics and therefore this coefficient is zero.) In a recent paper [43] the hypothesis was suggested, that infrared renormalons result in the substitution

$$\frac{\alpha_s}{\pi}\to\frac{\alpha_s}{\pi}\left(1-1.05\frac{\lambda^2}{Q^2}\right) \tag{33}$$

in the first α_s correction to polarization operator or Adler function (the Q^2 dependence of α_s was not accounted for in Ref. [43]). In Eq. (33), λ^2 is a tachyonic gluon mass, $\lambda^2<0$, and for its value the following estimation was found:

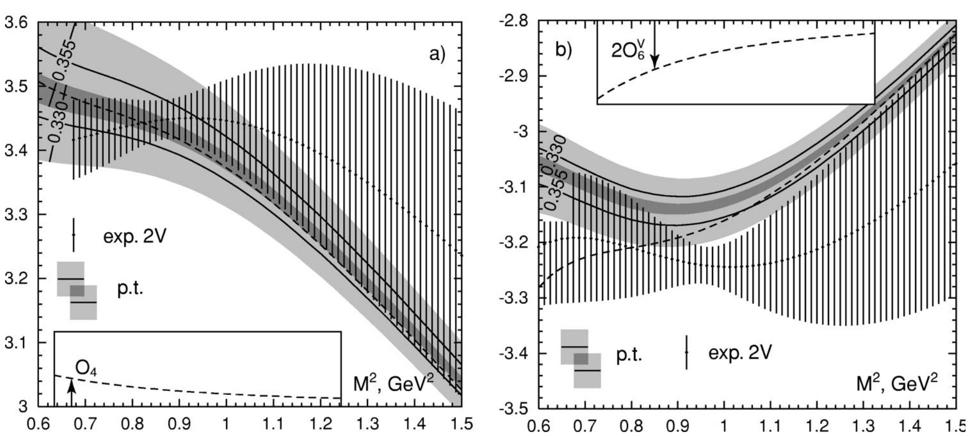

FIG. 7. The sum rules (a) (31) and (b) (32) for vector currents.

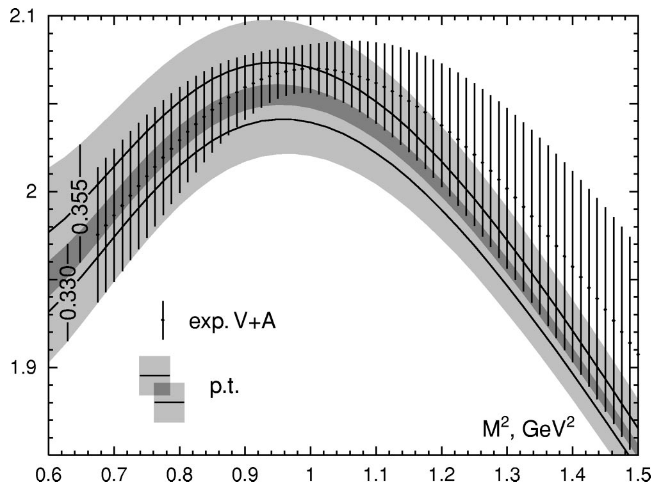

FIG. 8. Sum rule (35) with O_2 but without $O_{4,6}$.

$$-\lambda^2=(0.2-0.5)\ \ \mathrm{GeV}^2. \tag{34}$$

The authors of Ref. [43] could not discriminate even the highest value $\lambda^2=-0.5\ \ \mathrm{GeV}^2$.

Let us try to find the restriction on the O_2 operator from the sum rule for the $V+A$ correlator in the complex q^2 plane from ALEPH data. (For brevity we call it O_2, although it is not the $D=2$ operator, which stands in the OPE.) As we did in the previous section, for this purpose we take the real part of the Borel transform (26) $B(M^2e^{i\phi})$ at the angles $\phi=0,\ \pi/6$, and $\pi/4$ and separate the operator O_2 from $O_{4,6}$:

$$\mathrm{Re}\frac{B(0)-2B(\pi/6)+\sqrt{2}B(\pi/4)}{2-\sqrt{3}}=\mathrm{p.t.}+2\pi^2\frac{\langle O_2\rangle}{M^2}. \tag{35}$$

The experimental and perturbative parts of this combination are plotted in Fig. 8.

The sum rule (35) shown in Fig. 8, gives the following value of the dimension 2 operator:

$$\langle O_2\rangle=(1.0\pm1.5)\times10^{-3}\ \ \mathrm{GeV}^2,\quad \alpha_s(m_\tau^2)=0.33. \tag{36}$$

We got this estimation at $M^2=1\ \ \mathrm{GeV}^2$, where experimental error is minimal. In the model of Ref. [43],

$$\langle O_2\rangle=-1.05\frac{\alpha_s}{\pi}\frac{\lambda^2}{2\pi^2}. \tag{37}$$

At $\alpha_s(1\,\mathrm{GeV}^2)/\pi=0.18$, corresponding to $\alpha_s(m_\tau^2)=0.33$, there follows the restriction from Eq. (36):

$$-\lambda^2=(10\pm15)\times10^{-2}\ \ \mathrm{GeV}^2, \tag{38}$$

which is few times smaller than even the lower limit in Eq. (34). Notice, that similar restrictions on the value of the $D=2$ operator have been obtained in Ref. [44] from the other sum rules.

VII. INSTANTON CORRECTIONS

Some nonperturbative features of QCD may be described in the so-called instanton gas model (see Ref. [45] for an extensive review and the collection of related papers in Ref. [46]). Namely, one computes the correlators in the $SU(2)$-instanton field embedded in the $SU(3)$ color group. In particular, the 2-point correlator of the vector currents has been computed long ago [47]. Apart from the usual tree-level correlator $\sim\ln Q^2$, it has a correction that depends on the instanton position and radius ρ. In the instanton gas model, these parameters are integrated out. The radius is averaged over some concentration $n(\rho)$, for which one or another model is used. Concerning the two-point correlator of charged axial currents, the only difference from the vector case is that the term with zero modes must be taken with opposite sign. In coordinate representation, the answer can be expressed in terms of elementary functions, see Ref. [47]. An attempt to compare the instanton correlators with the ALEPH data in the coordinate space, has been undertaken in Ref. [48].

We shall work in momentum space. Here the instanton correction to the spin-J parts $\Pi^{(J)}$ of the correlator (1) can be written in the following form:

$$\Pi^{(1)}_{V,\ \mathrm{inst}}(q^2)=\int_0^\infty d\rho\, n(\rho)\left[-\frac{4}{3q^4}+\sqrt{\pi}\rho^4G^{30}_{13}\left(-\rho^2q^2\left|\begin{matrix}1/2\\0,0,-2\end{matrix}\right.\right)\right],$$

$$\Pi^{(0)}_{A,\ \mathrm{inst}}(q^2)=\int_0^\infty d\rho\, n(\rho)\left[-\frac{4}{q^4}-\frac{4\rho^2}{q^2}K_1^2(\rho\sqrt{-q^2})\right],$$

$$\Pi^{(1)}_{A,\mathrm{inst}}(q^2)=\Pi^{(1)}_{V,\mathrm{inst}}(q^2)-\Pi^{(0)}_{A,\mathrm{inst}}(q^2),\quad \Pi^{(0)}_{V,\mathrm{inst}}(q^2)=0. \tag{39}$$

Here K_1 is the modified Bessel function, and $G^{pq}_{mn}(z|\ \ldots)$ is the Meijer function. Definitions, properties, and approximations of Meijer functions can be found, for instance, in Ref. [49]. In particular, the function in Eq. (39) can be written as the following series:

$$\begin{aligned}\sqrt{\pi}G^{30}_{13}&\left(z\left|\begin{matrix}1/2\\0,0,-2\end{matrix}\right.\right)\\&=\frac{4}{3z^2}-\frac{2}{z}+\frac{1}{2\sqrt{\pi}}\sum_{k=0}^{\infty}z^k\frac{\Gamma(k+1/2)}{\Gamma^2(k+1)\Gamma(k+3)}\\&\quad\times\{[\ln z+\psi(k+1/2)-2\psi(k+1)-\psi(k+3)]^2\\&\quad+\psi'(k+1/2)-2\psi'(k+1)-\psi'(k+3)\},\end{aligned} \tag{40}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$. For large $|z|$ one can obtain its approximation by the saddle-point method:

$$G^{30}_{13}\left(z\left|\begin{matrix}1/2\\0,0,-2\end{matrix}\right.\right)\approx\sqrt{\pi}z^{-3/2}e^{-2\sqrt{z}},\quad |z|\gg1. \tag{41}$$

The formulas (39) should be treated in the following way. One adds Π_{inst} to the usual polarization operator (7) with

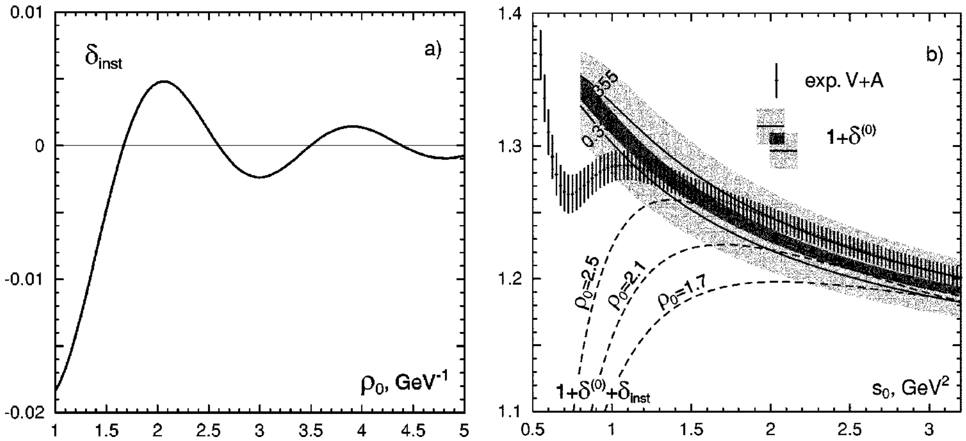

FIG. 9. The instanton correction to the τ decay ratio versus (a) ρ_0 and (b) "versus τ mass" for $n_0 = 1.5\times10^{-3}$ GeV^4.

perturbative and OPE terms. But the terms $\sim 1/q^4$ must be absorbed by the operator O_4 in Eq. (7), since the gluonic condensate $\langle G^2\rangle$ is averaged over all field configurations, including the instanton one. Notice the negative sign before $1/q^4$ in Eq. (39). It happens because the negative contribution of the quark condensate $\langle m\bar{q}q\rangle$ in the instanton field exceeds the positive contribution of the gluonic condensate $\langle G^2\rangle$. In the real world $\langle m\bar{q}q\rangle$ is negligible at $q^2 \sim 1$ GeV^2.

The correlators (39) possess appropriate analytical properties, they have a cut along the positive real axes:

$$\operatorname{Im}\Pi^{(1)}_{V,\rm inst}(q^2+i0) = \int_0^\infty d\rho\, n(\rho)\, \pi^{3/2}\rho^4 G^{20}_{13}\left(\rho^2 q^2 \middle| \begin{matrix} 1/2 \\ 0,0,-2 \end{matrix}\right), \tag{42}$$

$$\operatorname{Im}\Pi^{(0)}_{A,\rm inst}(q^2+i0) = -\int_0^\infty d\rho\, n(\rho)\, \frac{2\pi^2\rho^2}{q^2} \times J_1(\rho\sqrt{q^2})N_1(\rho\sqrt{q^2}). \tag{43}$$

We shall consider below the instanton concentration advocated by Shuryak (see Ref. [45] and references therein). It is a model with a fixed instanton radius (the RILM model in Ref. [45]):

$$n(\rho) = n_0\,\delta(\rho-\rho_0) \tag{44}$$

From Ref. [45] we take the numbers

$$\rho_0 = 1/3\ \text{fm} = 1.7\ \text{GeV}^{-1},$$
$$n_0 = 1\ \text{fm}^{-4} = 1.5\times10^{-3}\ \text{GeV}^4. \tag{45}$$

Now we consider the instanton contribution to the τ-decay branching ratio (4). Since the instanton correlator (39) has a $1/q^2$ singular term in the expansion near 0 [see Eq. (40)], the integrals must be taken over the circle, as in Ref. (6). In the instanton model, the function $a_0(s)$ differs from the experimental δ function, which gives the small correction (5). So we shall ignore the last term in Eq. (4) and consider the integral with $\Pi^{(1)}_{V+A}+\Pi^{(0)}_A$ in Eq. (6). Here we need the following formulas for the circle integrals, which can be rigorously obtained from the series representation of the Meijer function (40):

$$\frac{i}{2\pi}\oint_{|s|=s_0}\frac{ds}{s_0}\left(\frac{s}{s_0}\right)^k G^{30}_{13}\left(-\rho^2 s\middle| \begin{matrix} 1/2 \\ 0,0,-2\end{matrix}\right) = G^{21}_{24}\left(\rho^2 s_0 \middle| \begin{matrix} -k,1/2 \\ 0,0,-2,-k-1\end{matrix}\right), \quad k\geqslant 2$$

$$\frac{i}{2\pi}\oint_{|s|=s_0}\frac{ds}{s_0}\frac{s}{s_0} G^{30}_{13}\left(-\rho^2 s\middle| \begin{matrix} 1/2 \\ 0,0,-2\end{matrix}\right) = -\frac{4}{3\sqrt{\pi}\rho^4 s_0^2} + G^{21}_{24}\left(\rho^2 s_0 \middle| \begin{matrix} -1,1/2 \\ 0,0,-2,-2\end{matrix}\right),$$

$$\frac{i}{2\pi}\oint_{|s|=s_0}\frac{ds}{s_0} G^{30}_{13}\left(-\rho^2 s\middle| \begin{matrix} 1/2 \\ 0,0,-2\end{matrix}\right) = -G^{20}_{13}\left(\rho^2 s_0 \middle| \begin{matrix} 1/2 \\ 0,-1,-2\end{matrix}\right). \tag{46}$$

The first term on the right-hand side of the second equation looks like the contribution of the $D=4$ operator, but in fact it is not. Indeed, all the expressions on the right-hand side of Eq. (46) have the same LO term of the asymptotic expansion for large s_0, equal to $-\sin(2\rho\sqrt{s_0})/(\sqrt{\pi}\rho^4 s_0^2)$. However for $k\geqslant 3$, the accuracy of this approximation is bad and exact values of Meijer functions should be used for numerical evaluations.

With the help of Eq. (46), the instanton correction to the τ-decay branching ratio can be brought to the following form:

$$\delta_{\rm inst} = -48\pi^{5/2}\int_0^\infty d\rho\, n(\rho)\rho^4 G^{20}_{13}\left(\rho^2 m_\tau^2\middle| \begin{matrix} 1/2 \\ 0,-1,-4\end{matrix}\right) \approx \frac{48\pi^2 n_0}{\rho_0^2 m_\tau^6}\sin(2\rho_0 m_\tau). \tag{47}$$

Since the parameters (45) are determined quite approximately, we may explore the dependence of $\delta_{\rm inst}$ on them. The $\delta_{\rm inst}$ versus ρ_0 for fixed n_0 [Eq. (45)] is shown in Fig. 9(a).

As seen from Fig. 9(a), the instanton correction to the hadronic τ decay is extremely small except for the unreliably low value of the instanton radius $\rho_0 < 1.5\ \mathrm{GeV}^{-1}$. At the favorable value [45] $\rho_0 = 1.7\ \mathrm{GeV}^{-1}$ the instanton correction to R_τ is almost exactly zero. [Of course, smaller values of n_0 than Eq. (45) are also allowed.] This fact confirms our calculations of $\alpha_s(m_\tau^2)$ (Sec. II), where the instanton corrections were not taken into account.

The result (47) can be used in another way. Namely, the τ mass can be considered as the free parameter s_0. The dependence of the fractional corrections $\delta^{(0)}$ and $\delta^{(0)}_{0.330} + \delta_{\mathrm{inst}}$ on s_0 is shown in Fig. 9(b).[2] The result strongly depends on the instanton radius and rather essentially on the density n_0. For $\rho_0 = 1.7\ \mathrm{GeV}^{-1}$ and $n_0 = 1\ \mathrm{fm}^{-4}$ (45), the instanton curve is outside the errors already at $s_0 \sim 2\ \mathrm{GeV}^2$, where the perturbation theory is expected to work. Therefore Fig. 9(b) shows, that in the random instation liquid model (RILM), the instanton radius must be larger (say, $\rho_0 = 2.5\ \mathrm{GeV}^{-1}$) or the instanton density much (2–3 times) lower. The contribution of the $D=6$ operator $\delta^{(6)}_{V+A}$ is not shown in Fig. 9(b). It is equal to $\delta^{(6)}_{V+A} = -(5\pm 2)\times 10^{-3} m_\tau^6/s_0^3$ and quite large at $s_0 < 1.5\ \mathrm{GeV}^2$.

Consequently in this approach the perturbation theory + OPE + RILM (at not very large ρ_0) cannot satisfactory describe the data at $s_0 < 1.5\ \mathrm{GeV}^2$. Since the instanton contribution is large here, we disbelieve all the results obtained by the method of variable τ mass in this domain. (Perhaps the shadowed region in Fig. 3 is of importance in this method at low s_0.)

The τ decay ratio is not sensitive to the gluonic condensate. Let us consider now the sum rules that depend on it. The Borel transformation of the instanton part is

$$\mathcal{B}_{M^2}\Pi_{\mathrm{inst}} = 2\pi i \oint e^{-s/M^2}\Pi^{(1)}_{V,\mathrm{inst}}(s)\frac{ds}{M^2}$$
$$= 4\pi^2\int_0^\infty d\rho\, n(\rho)\Bigg[-\frac{4}{3M^4}$$
$$+\sqrt{\pi}\rho^4 G^{20}_{12}\left(\rho^2 M^2\middle|\begin{matrix}1/2\\0,-2\end{matrix}\right)\Bigg]. \quad (48)$$

The integration contour goes around the cut from $s = +\infty + i0$ to $s = +\infty - i0$. The term $\sim 1/M^4$ here comes from the term $\sim 1/q^4$ in Eq. (39); it must be included in the $\langle O_4\rangle$ contribution in Eq. (26). The Meijer function in Eq. (48) has the asymptotics

$$G^{20}_{12}\left(z\middle|\begin{matrix}1/2\\0,-2\end{matrix}\right)\approx z^{-5/2}e^{-z}, \quad |z|\gg 1$$

and are strongly suppressed at $M^2 > 0.8\ \mathrm{GeV}^2$. We calculated the instanton contribution to all Borel-like sum rules used here; it is indeed negligible compared to the errors. Consequently the results of previous sections remain unchanged.

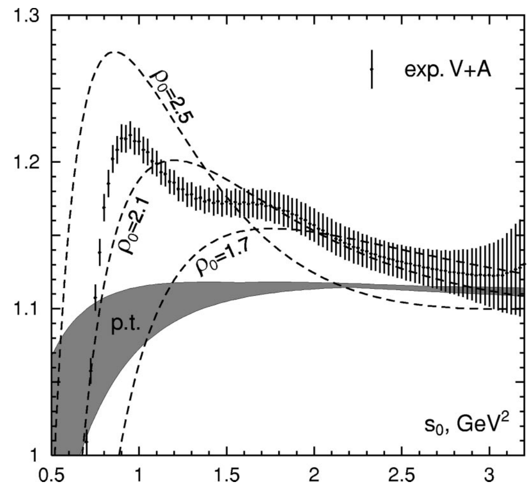

FIG. 10. Sum rule (49). Experimental, pure perturbative, and "perturbative + instanton" parts are shown. The O_4 contribution is not taken into account.

However, the spectral moment sum rules, often used in τ-decay data analysis [1], can be quite sensitive to the instanton corrections. Let us consider the following sum rule, constructed in this way:

$$4\int_0^{s_0}\frac{ds}{s_0}\frac{s}{s_0}\left(1-\frac{s^2}{s_0^2}\right)\omega_{\mathrm{exp}}(s)$$
$$= \mathrm{p.t.} - 8\pi^2\frac{\langle O_4\rangle}{s_0^2} + 16\pi^2\int_0^\infty d\rho\, n(\rho)\rho^4$$
$$\times\Bigg[-\frac{4}{3\rho^4 s_0^2} + \sqrt{\pi}G^{21}_{24}\left(\rho^2 s_0\middle|\begin{matrix}-1,1/2\\0,0,-2,-2\end{matrix}\right)$$
$$-\sqrt{\pi}G^{21}_{24}\left(\rho^2 s_0\middle|\begin{matrix}-3,1/2\\0,0,-2,-4\end{matrix}\right)\Bigg]. \quad (49)$$

The integral (49) is normalized to one in the parton model. It does not depend on the $D=6$ operator, and the factor $1 - s^2/s_0^2$ is introduced to suppress large experimental errors for large s_0. Remember our convention: the contribution of the term $\sim 1/q^4$ in Π_{inst} (39) is included in the operator $\langle O_4\rangle$ in Eq. (49). The contribution of different parts of Eq. (49) are plotted versus s_0 in Fig. 10. Since the weight function in the integral vanishes at $s=0$, the contribution of the unphysical cut is suppressed. So the theoretical errors are diminished here as well as the sensitivity on various perturbative parameters. The theoretical curve is shown as single shaded area, which includes both the uncertainty of $\alpha_s(m_\tau^2)$ and the error $\pm K_3 a^3$ for each $\alpha_s(m_\tau^2)$.

The operator $\langle O_4\rangle$ enters with negative sign in Eq. (49), so the theoretical curve must go above the experimental one. This is certainly not the case if the instanton corrections are not taken into account. For $\rho_0 = 2.1\ \mathrm{GeV}^2$, the theoretical and experimental results are in good agreement for $\langle O_4\rangle = 0$. By increasing the instanton density n_0, positive values

[2]Figure 9(b) can be compared with Fig. 15 in the ALEPH paper [1]. The discrepancy between theoretical curves at $s_0 < 1\ \mathrm{GeV}^2$ is explained by different approximations; we used 3-loop perturbation theory, while the authors of Ref. [1] used the 4-loop one with $K_4 = 50\pm 50$.

of $\langle O_4\rangle$ become possible. In this aspect the sum rules (47) with varying m_τ and Eq. (49) are not in agreement: Eq. (47) favors small n_0 while Eq. (49) prefers large n_0.

These results are, however, not convincing. The main conclusion, coming from consideration of spectral moment sum rules, is that they are not suitable for QCD analysis until we have a complete theory. (This statement surely refers also to the method, where the τ mass is considered as a free parameter.) The same situation took place for $V-A$ correlators; the spectral moments sum rules worked only at the circle radius $s_0>2\ \mathrm{GeV}^2$ [5].

VIII. CONCLUSION

The goal of this paper was to confront the recent precise experimental data on hadronic τ decay with QCD calculations at low Q^2 and to check the basic aspects of QCD: perturbative series, OPE as well as various nonperturbative QCD approaches. The data present the imaginary part of polarization operators $\mathrm{Im}\,\Pi_{V,A}(s)$, $s=q^2$ at $0<s<m_\tau^2$. If some procedure is applied to suppress or nullify the influence of the high-energy domain (Borel transformation, integration over closed circle in a complex s plane), then with the help of the dispersion relation, the values of $\Pi_{V,A}(s)$ in the whole complex s plane at low $|s|$ can be found from experiment. (By low $|s|$ we mean $|s|<2-3\ \mathrm{GeV}^2$.) These experimental values of $\Pi_{V,A}(s)$ can be compared with the theoretical calculations in the domain of the complex s plane, where QCD describes the data well enough, in order to find the values of the QCD parameters: α_s and condensates.

In Ref. [5], this program was realized for the Π_{V-A} polarization operator, and the values of dimension 6 and 8 condensates were found. In this paper, Π_{V+A} and Π_V polarization operators were studied, where the perturbative contribution is dominant (unlike Π_{V-A}, which is given entirely by condensates). It must be stressed, that the present situation has changed drastically in comparison with the earlier study of a similar problem [4]. In Ref. [4], the perturbative contribution was much less essential and the authors could restrict themselves to the LO term only. In this paper, the perturbative calculations were performed in the 3 and 4 loop approximation. The unphysical cut in the complex s plane in the perturbative part of the polarization operator, was taken into account and the calculations (at least partly) were performed in such a way, that allows one to minimize its influence (e.g., the Borel transformation along the rays, going from the origin at some angle). The terms of OPE were accounted for up to dimension $D=8$. It was shown that the $D=8$ contribution is very small in the case of $V+A$ correlator. The coincidence of theoretical and experimental values with accuracy better than 2% was required. Let us remember that usually the accuracy of the standard QCD sum-rule calculations are of order 10–15%.

The following results have been obtained.

(1) The value of QCD coupling constant $\alpha_s(m_\tau^2)=0.355\pm0.025$ was found from the hadronic branching ratio R_τ. It was shown that the sum rules at low $|s|$ favor the value close to the lower error edge $\alpha_s(m_\tau^2)=0.330$, corresponding to $\alpha_s(m_Z^2)=0.118$.

(2) It was demonstrated that QCD with inclusion of OPE terms, is in agreement with the data at the values of the complex Borel parameter $|M^2|>0.8-1.0\ \mathrm{GeV}^2$ in the left complex half-plane.

(3) The restriction on the value of the gluonic condensate was found $\langle(\alpha_s/\pi)G^2\rangle=(0.006\pm0.012)\ \mathrm{GeV}^4$ in comparison with the standard, SVZ value $0.012\ \mathrm{GeV}^4$.

(4) The value of the $D=6$ condensate found in Ref. [5] is in agreement with the $V+A$ and the V sum rules, but cannot be specified.

(5) The analytical perturbative QCD [32,34,36] was compared with the data and it was demonstrated that this approach is in strong contradiction with the experimental value of R_τ.

(6) The restrictions on the $1/Q^2$ term in the polarization operator of the renormalon origin were found to be much stronger than in the previous investigation [43].

(7) The instanton contributions to the polarization operator were analyzed and compared with the data in the framework of the RILM [45]. It was shown that the instanton contribution to R_τ is very small, and the same is true for Borel sum rules. However their contributions can be significant to the spectral moments sum rules, often used in the τ-decay data analysis.

(8) It was found that the method of spectral moments (integration over the circle with a polynomial) is less effective in the study of the polarization operators at low Q^2, than Borel sum rule because of larger contributions not given by OPE nonperturbative corrections (see Sec. VII and [5]).

We believe that the results of this paper will serve for improving the QCD sum-rules method.

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