Why hyperbolic theories of dissipation cannot be ignored: Comment on a paper by Kostädt and Liu

Luis Herrera*

Area de Física Teórica, Facultad de Ciencias, Universidad de Salamanca, 37008 Salamanca, Spain

Diego Pavón[†]

Departamento de Física, Facultad de Ciencias, Edificio Cc, Universidad Autónoma de Barcelona, 08193 Bellaterra, Spain

Contrary to what is asserted in a recent paper by Kostädt and Liu, experiments can (and in fact do) tell hyperbolic theories apart from parabolic theories of dissipation. It is stressed that the existence of a non-negligible relaxation time does not imply that the system is out of the hydrodynamic regime.

As is well known, hyperbolic theories of fluid dissipation were formulated to get rid of some undesirable features of parabolic theories, such as acausality [1,2]. This was achieved at the price of extending the set of field variables by including the dissipative fluxes (heat current, nonequilibrium stresses, and so on) on the same footing as the old variables (energy densities, equilibrium pressures, etc.), thereby giving rise to more physically satisfactory but more involved theories from the mathematical point of view. A key quantity in these theories is the relaxation time τ of the corresponding dissipative process. This positive-definite quantity has a distinct physical meaning, namely, the time taken by the system to return spontaneously to the steady state (whether or not of thermodynamic equilibrium) after it has been suddenly removed from it. It is, however, somehow connected to the mean collision time t_c of the particles responsible for the dissipative process, oftentimes erroneously identified with it. In principle they are different since τ is (conceptually and many times in practice) macroscopic time, although in some instances it may correspond just to a few t_c. No general formula linking τ and t_c exists; the relationship between them depends in each case on the system under consideration. As mentioned above, it is therefore appropriate to interpret τ as the time taken by the corresponding dissipative flow to relax to its steady value.

Thus, it is well known that the classical Fourier law for heat current leads to a parabolic equation for temperature (diffusion equation), which does not forecast propagation of perturbations along characteristic causal cones (see [3–5] and references therein). In other words perturbations propagate with infinite speed. This noncausal behavior is easily visualized, by taking a look at the thermal conduction in an infinite medium (see [6]). The origin of this behavior is to be found in the parabolic character of Fourier's law, which implies that the heat flow starts (vanishes) simultaneously with the appearance (disappearance) of a temperature gradient. Although τ is very small for the phonon-electron and

phonon-phonon interactions at room temperature [$\mathcal{O}(10^{-11})$ and $\mathcal{O}(10^{-13})$ sec, respectively [7]], neglecting it is the source of difficulty, and in some cases a bad approximation, such as, for example, in superfluid helium [8] and degenerate stars where thermal conduction is dominated by electrons—see [3,4,9] for further examples.

In order to overcome this problem many researchers, starting with Cattaneo and Vernotte [10], generalized the Fourier law by introducing a relaxation time, thereby leading to a hyperbolic equation for the temperature.

Obviously, τ should not be neglected if one wishes to study transient regimes, i.e., the departure from an initial steady situation and the approach to a new one. In fact, leaving aside the problem of stability and the fact that parabolic theories are necessarily noncausal, it is obvious that, whenever the time scale of the problem under consideration is of the order of (or smaller than) the relaxation time, the latter cannot be ignored. It is obvious what is at stake here: neglecting the relaxation time amounts—in this situation—to disregarding the whole problem under consideration. Such a neglect literally means to throw the baby out with the bathwater.

In a recent paper by Kostädt and Liu [1], arguments have been put forward suggesting that parabolic theories of dissipation are good enough, and that hyperbolic (i.e., causal) theories are not necessary when dealing with dissipative fluid systems. In particular, these authors state: "In fact, recently, it has been shown by Geroch [11] and Lindblom [12] that the complicated dynamical structure which ensures causality is unobservable. The evolution of any physical fluid state according to any causal theory results in energy-momentum tensors and particle currents that are experimentally indistinguishable from the respective hydrodynamic expressions." We would like to stress that the quoted phrase is at variance with experimental evidence as a number of observations show unambiguously [9]. The aim of this Comment is to indicate the roots of the confusion leading to that erroneous view [13].

The basic assumption underlying the disposal of hyperbolic dissipative theories states that systems with relaxation times comparable to the characteristic time of the system are out of the hydrodynamic regime [14]. This can be valid only if the particles making up the fluid are the same ones that

*Also at UCV, Caracas, Venezuela. Email address: laherrera@telcel.net.ve

[†]Email address: diego@ulises.uab.es

transport the heat. However, this is (almost?) never the case. Specifically, for a neutron star, τ is of the order of the scattering time between electrons (which carry the heat) but this fact is not an obstacle (no matter how large the mean free path of these electrons may be) to considering the neutron star as formed by a Fermi fluid of degenerate neutrons. The same is true for the second sound in superfluid helium and solids, and for almost any ordinary fluid. In brief, the hydrodynamic regime refers to fluid particles that do not necessarily (and, as a matter of fact, almost never) transport the heat. Therefore large relaxation times (large mean free paths of particles involved in heat transport) do not imply a departure from the hydrodynamic regime (this fact has been stressed before [15], but is usually overlooked).

However, even in the case when the particles that make up the fluid are responsible for the dissipative process, it is not always valid to take for granted that τ and t_c are *always* of the same order, or, as comes to the same, that the dimensionless quantity $\Gamma \equiv (\tau c_s / L)^2$ is negligible in all instances [11,12]—here c_s stands for the adiabatic speed of sound in the fluid under consideration and L is the characteristic length of the system. That assumption would be right if τ were always comparable to t_c and L always "large," but there are important situations in which $\tau \gg t_c$ and L is "small," although still large enough to justify a macroscopic description. For tiny semiconductor pieces of about 10^{-4} cm in size, used in common electronic devices submitted to high electric fields, the above dimensionless combination (with $\tau \sim 10^{-10}$ sec, $c_s \sim 10^7$ cm/sec [16]) can easily be of the order of unity. In ultrasound propagation as well as light-scattering experiments in gases and neutron scattering in liquids, the relevant length is no longer the system size, but the wavelength λ, which is usually much smaller than L [17,18]. Because of this, hyperbolic theories may have some importance in the study of nanoparticles and quantum dots. Likewise, in polymeric fluids relaxation times are related to the internal configurational degrees of freedom and so much longer than t_c (in fact they are in the range of minutes), and $c_s \sim 10^5$ cm/sec, thereby $\Gamma \sim \mathcal{O}(1)$. In the degenerate core of aged stars the thermal relaxation time can be as high as 1 sec [19]. Assuming the radius of the core to be about 10^{-2} times the solar radius, one has $\Gamma \sim \mathcal{O}(1)$ again. Fully ionized plasmas exhibit a collisionless regime (Vlasov regime) for which the parabolic hydrodynamics predicts a plasmon dispersion relation at variance with the microscopic results; the latter agree, however, with the hyperbolic hydrodynamic approach [20]. Think, for instance, of some syrupy fluid flowing under imposed shear stress, and imagine that the shear is suddenly switched off. This liquid will come to rest only after a much longer time (τ) has elapsed than the collision time between its constituent particles. Many other examples could be added but we do not mean to be exhaustive.

Even in the steady regime the descriptions offered by causal and acausal theories do not necessarily coincide. The differences between them in such a situation arise from several causes. (i) τ is present in terms that couple the vorticity to the heat flux and shear stresses. These may be large even in steady states (e.g., rotating stars). There are also other acceleration coupling terms to bulk and shear stresses and heat flux. The coefficients for these vanish in parabolic theories, and they could be large even in the steady state. (ii) The convective parts of the time derivative contribute (which are not negligible in the presence of large spatial gradients). (iii) There are modifications in the equations of state due to the presence of dissipative fluxes [4].

However, it is precisely before the establishment of the steady regime that the two types of theory (hyperbolic and parabolic) differ more importantly. It is well known (see [3,4,9,21]) that a variety of physical processes take place on time scales of the order of (or even smaller than) the corresponding relaxation time, which as was stressed above does not imply that the system is out of the hydrodynamic regime. Therefore, if one wishes to study a dissipative process for times shorter than τ, it is mandatory to resort to a hyperbolic theory which is a more accurate macroscopic approximation to the underlying kinetic description.

Only for times longer than τ is it permissible to go to a parabolic theory, provided that the spatial gradients are not so large that the convective part of the time derivative becomes important, and that the fluxes and coupling terms remain safely small. But even in these cases, it should be kept in mind that the way a system leaves equilibrium may depend critically upon the relaxation time [21]. Therefore the future of the system at time scales much longer than the relaxation time (once the steady state is reached) may also critically depend on τ.

Thus, even though parabolic theories have proved very useful for many practical purposes, it appears that there are a number of well-known instances (such as transient regimes) where they fail hopelessly, but hyperbolic theories successfully predict the experimental results—i.e., they are distinguishable. Having said this, it is worth mentioning that at the moment it is rather uncertain which among the proposed hyperbolic theories [2] will eventually emerge as "the correct one." This discrimination seems to lie a long way ahead.

We hope this Comment will help to convince the reader that hyperbolic theories are indeed not of mere academic interest and it would not be wise to dispense with them.

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[13] It is not our purpose to challenge the proof offered by Kostädt and Liu on the stability of parabolic equations, only to focus on the unfortunate quoted phrase which is actually irrelevant to the stability proof.

[14] The concept of the hydrodynamic regime invokes the ratio between the mean free path of fluid particles and the characteristic length of the system. When this ratio is lower than unity, the fluid is within the hydrodynamic regime. When it is larger than unity, the regime becomes Knudsen's. In the latter case the fluid ceases to be a continuum and even hyperbolic theories are no longer reliable.

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