ω dependence of the scalar field in Brans-Dicke theory

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This article examines the claim that the Brans-Dicke scalar field $\phi \to \phi_0 + O(1/\sqrt{\omega})$ for large ω when the matter field is traceless. It is argued that such a claim cannot be true in general.

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Brans-Dicke (BD) theory [1] is generally regarded as a viable alternative to Einstein's theory of general relativity. This theory has recently regained interest because, in the Einstein conformal frame, it turns out to be the low energy limit of many theories of quantum gravity, such as the supersymmetric string theory [2] or Kaluza-Klein theory [3]. The theory is relevant also in the extended inflationary scenario of cosmology [4]. The BD theory, which accommodates Mach's principle, describes gravitation through a spacetime metric ($g_{\mu\nu}$) and a massless scalar field (ϕ) that couples to both matter and spacetime geometry. The strength of the coupling is represented by a single dimensionless constant ω. In the Jordan conformal frame, the BD action takes the form

$$\mathcal{A} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \mathcal{L}_{matter}\right) \quad (1)$$

where $\mathcal{L}_{matter}$ is the Lagrangian density of ordinary matter. A variation of Eq. (1) with respect to $g^{\mu\nu}$ and ϕ gives, respectively, the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\phi^{,\sigma}\phi_{,\sigma}) + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi), \quad (2)$$

$$\Box\phi = \frac{8\pi T}{(2\omega+3)} \quad (3)$$

where R is the Ricci scalar, and $T = T^\mu_\mu$ is the trace of the matter energy momentum tensor.

In the weak field approximation, the metric tensor can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $\eta_{\mu\nu}$ is the Minkowskian metric tensor. Similarly $\phi = \phi_0 + \xi$, where ϕ_0 is a constant. Using these approximations in Eqs. (2) and (3), one concludes that [5]

$$\phi = \phi_0 + O\left(\frac{1}{\omega}\right) \quad (4)$$

and

$$R \sim O\left(\frac{1}{\omega}\right). \quad (5)$$

Thus it appears from above equations that the post-Newtonian expansion of BD theory reduces to general relativity in the infinite ω limit. But it was reported [6] that a number of exact solutions of BD theory do not go over to the corresponding solutions of general relativity in the limit $\omega \to \infty$. Recently, Banerjee and Sen [7] illuminated this point through the study of the BD field equations and pointed out that, when the trace (T) of the energy momentum tensor vanishes, the asymptotic behavior of ϕ is not represented by Eq. (4) but follows the relation

$$\phi = \phi_0 + O\left(\frac{1}{\sqrt{\omega}}\right). \quad (6)$$

Faraoni [8] also claimed to have found a similar ω dependence. As a result, the BD theory does not tend to general relativity in the $\omega \to \infty$ limit. This feature is significant because the lower limit of $\omega(\sim 500)$ for the solar system measurements is fixed using the $O(1/\omega)$ behavior in the standard parametrized post-Newtonian (PPN) approximation. It is therefore important to study the situation more closely, which we do here.

We noticed that Eq. (6) is not valid in general. In this Brief Report we will discuss some counterexamples to Eq. (6) in BD theory. We will also point out some assumptions inherent in [7] and [8] that led to Eq. (6).

When $T=0$, BD field equations yield

$$\Box\phi = 0 \quad (7)$$

and

$$R = \frac{\omega}{\phi^2}(\phi^{,\alpha}\phi_{,\alpha}). \quad (8)$$

From the above equation, Banerjee and Sen [7] argued that ϕ will exhibit the asymptotic behavior as given in Eq. (6). *But*

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such a conclusion holds only if R is assumed to be independent of ω. This is a strong condition which is not justified in general. Note that, Eq. (8) contains two unknown functions of ω: R and ϕ. Hence if one knows the dependence on ω of one of the functions, the same for the other could be obtained from Eq. (8). It is true that the ω independence of R leads to Eq. (6) but there is no way to know the functional behavior of R *a priori* unless one considers specific solutions. On the other hand, it is known that for a number of exact solutions of BD theory having traceless source, R *is* a function of ω.

To clarify the situation further, let us consider the static spherically symmetric vacuum solution of the BD theory given by Brans and Dicke [1]:

$$ds^2 = e^{2\alpha}dt^2 - e^{2\beta}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \tag{9}$$

where

$$e^{2\alpha} = \left(\frac{1 - B/r}{1 + B/r}\right)^{2/\lambda} \tag{10}$$

$$e^{2\beta} = \left(1 + \frac{B}{r}\right)^4 \left(\frac{1 - B/r}{1 + B/r}\right)^{2(\lambda - C - 1)/\lambda} \tag{11}$$

$$\phi = \phi_0 \left(\frac{1 - B/r}{1 + B/r}\right)^{C/\lambda} \tag{12}$$

with

$$\lambda = \left([C+1]^2 - C\left[1 - \frac{\omega C}{2}\right]\right)^{1/2}. \tag{13}$$

The Ricci scalar for the above metric is given by

$$R = \frac{4\omega C^2 B^2 r^4}{\lambda^2}(r+B)^{-4\{1 + [(C+1)/\lambda]\}} \times (r - B)^{-4\{1 - [(C+1)/\lambda]\}}. \tag{14}$$

To establish their claim, Banerjee and Sen leave C to be arbitrary but it is not clear to what extent it is so. Any choice of C arbitrarily dependent on ω will not render R to be ω independent. Only when either C is an arbitrary but fixed constant or $C(\omega) \propto 1/\sqrt{\omega}$ for large ω does R become effectively (but not exactly) independent of ω. Therefore, the arbitrariness of C as used by Banerjee and Sen is severely constrained. On the other hand, it is well known that to match the BD class I metric [1] with weak field post-Newtonian expansion of the BD field equations (which is a standard and probably unique way to fix unknown constants present in vacuum solutions), one must specify $C = -1/(2 + \omega)$. And under this choice R goes as $O(1/\omega)$ and so does ϕ. There are other examples, too. For instance, consider the stationary charged black hole solutions in BD theory recently obtained by Kim [9] given by

$$\begin{aligned} ds^2 = {} & \Delta^{-2/(2\omega+3)} \sin^{-4/(2\omega+3)}\theta \left[-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) dt^2 \right. \\ & - \frac{2a\sin^2\theta(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi \\ & \left. + \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 \right] \\ & + \Delta^{2/(2\omega+3)} \sin^{4/(2\omega+3)}\theta \left(\frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2\right) \end{aligned} \tag{15}$$

$$\Phi(r,\theta) = \Delta^{2/(2\omega+3)} \sin^{4/(2\omega+3)}\theta \tag{16}$$

$$A_\mu = \frac{er}{\Sigma}(\delta^t_\mu - a\sin^2\theta \delta^\phi_\mu) \tag{17}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2 + e^2$ with M, a, and e representing the Arnowitt-Deser-Misner (ADM) mass, angular momentum per unit mass and electric charge, respectively. In this case the source is electromagnetic field and hence traceless. The above solution reduces to the standard Kerr-Newman solution [10] in the limit $\omega \to \infty$. The curvature scalar for the metric is given by

$$\begin{aligned} R = {} & \omega\left(\frac{4}{2\omega+3}\right)^2 \frac{1}{\Sigma} \sin^{-4/(2\omega+3)}\theta [(r-M)^2 \Delta^{-(2\omega+5)/(2\omega+3)} \\ & + \cot^2\theta \Delta^{-2/(2\omega+3)}]. \end{aligned} \tag{18}$$

It is evident from Eqs. (16) and (18) that both the scalar field and scalar curvature go as $O(1/\omega)$ contradicting Eq. (6).

Faraoni [8] claimed to have deduced a rigorous behavior of ϕ in terms of ω supporting the result of [7]. But the above examples (BD class I and Kim's black hole solution) already contradict such a claim. The author [8] used the conformal invariance of BD theory under the transformations

$$\tilde{g}_{\mu\nu} = \phi^{2\alpha} g_{\mu\nu} \tag{19}$$

$$\tilde{\phi} = \phi^{1-2\alpha} \tag{20}$$

$$\tilde{\omega} = \frac{\omega - 6\alpha(\alpha - 1)}{(1 - 2\alpha)^2}, \tag{21}$$

$\alpha \neq \frac{1}{2}$. Starting with the fixed value of $\omega = 0$, Faraoni obtained from the above equations

$$\alpha = \frac{1}{2}\left(1 \pm \frac{\sqrt{3}}{\sqrt{3 + 2\tilde{\omega}}}\right) \tag{22}$$

which gives as $\alpha \to \frac{1}{2}$, $\tilde{\omega} \to \infty$. Under this limit, Eq. (20) gives

$$\tilde{\phi} \to \phi_0 + \frac{1}{\sqrt{\tilde{\omega}}} \ln \phi(\omega). \tag{23}$$

It was argued that the $\phi(\omega)$ corresponding to $\omega=0$ does not alter in the limit $\tilde{\omega}\to\infty$ and hence one ends up with a behavior similar to Eq. (6). But this argument should be taken with care as one may choose $\phi(\omega)$ to depend on the same parameter α that appears in the conformal transformation, so that $\phi(\omega)$ changes under transformation (21) even in the limit $\alpha\to 1/2$. A simple example will illustrate the point. Suppose $\phi(\omega)\sim 1+(1-2\alpha)/\sqrt{\omega-6\alpha(\alpha-1)}$. Under the transformation (21), $\phi(\omega)\to\chi(\tilde{\omega})\sim 1+(1/\sqrt{\tilde{\omega}})$. Then we have from Eq. (23) that $\tilde{\phi}\to\phi_0+O(1/\tilde{\omega})$. The inclusion of the parameter α in the specific choice of ϕ that we made in the above example is not unreasonable as the BD field equations admit an equivalence class of solutions for ϕ with a parameter α (see [8]), though α does not appear in the BD action. In this sense, α could be interpreted as some kind of a *gauge parameter*. Therefore, a solution corresponding to the choice of a particular *gauge*, namely, $\alpha=0$, does not have any special status and one is free to retain α in the expression for ϕ. The conclusion that $\tilde{\phi}\to\phi_0+O(1/\sqrt{\tilde{\omega}})$ for large $\tilde{\omega}$ thus does not *necessarily* follow from Eq. (23). However, if $\phi(\omega)$ is chosen not to depend on the parameter α that describes the conformal transformation, the transformed scalar field $\tilde{\phi}(\tilde{\omega})$ will truly behave like Eq. (6) in accordance with the claim of Ref. [8].

We argue that the functional ω independence of the Ricci scalar R is not *generally* true. Consequently, the dependence of the BD scalar field ϕ on the coupling constant ω essentially remains *arbitrary* when $T=0$ and not necessarily like the one expressed in Eq. (6). Usually one fixes the constants appearing in the exact solutions for ϕ using physical considerations: In the context of the Oppenheimer-Snyder collapse in the BD theory, this point is illustrated in Refs. [11,12]. Also, very recently, it has been discussed by Miyazaki [13] that the asymptotic behavior of ϕ could be fixed as $\phi\to O(1/\omega)$ due to the presence of cosmological matter distribution for which $T\neq 0$ although for local matter distribution T could be zero. This idea is perfectly consistent with the Machian nature of the Brans-Dicke theory.

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