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Massless neutrino oscillations

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Quantum dynamical semigroups provide a general framework for studying the evolution of open systems. Neutrino propagation both in vacuum and in matter can be analyzed using these techniques: They allow a consistent treatment of nonstandard, dissipative effects that can alter the pattern of neutrino oscillations. In particular, initially massless neutrinos can give rise to a nonvanishing flavor transition probability, involving in addition the Majorana *CP*-violating mixing phase.

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I. INTRODUCTION

Elementary particle systems are usually treated as isolated quantum systems: Their dynamics can be modeled by means of effective field theories, allowing a coherent interpretation of the experimental results. Although very general, this framework cannot accommodate all phenomena involving elementary particles; in particular, those leading to irreversibility and dissipation are clearly excluded. Indeed, a more general treatment is needed to properly describe these effects: It can be physically motivated in the framework of open quantum systems [1–3].

These systems can be thought of as being subsystems in interaction with large environments. The time evolution of the total system is unitary and follows the rules of ordinary quantum mechanics; nevertheless, the dynamics of the subsystem alone, obtained by eliminating the environment degrees of freedom, shows in general irreversibility and decoherence.

When there are no initial correlations between subsystem and environment and their mutual interaction is weak, the subdynamics can be described in a mathematically precise way in terms of quantum dynamical semigroups. These are linear evolution maps satisfying general properties that assure the consistent physical interpretation of the dynamics: They include the condition of entropy increase (irreversibility), forward-in-time composition law (semigroup property), complete positivity. This framework is very general and can be applied to model irreversibility and dissipation in very different physical situations [1–11]; in particular, it can be used to study the evolution of elementary particle systems, treated now as open systems [12–14,15–19].

The possibility that decoherence phenomena might affect the physics of elementary particles is supported by recent studies on the fundamental dynamics of extended objects (strings and branes) [20]; indeed, time evolutions described by quantum dynamical semigroups can be the result of the interaction with a gas of quanta obeying infinite statistics (e.g., a gas of $D0$ branes) [21]. In other terms, the dynamics of fundamental extended objects could effectively generate at low energies a weakly coupled environment.

Similar phenomena have also been described in the framework of quantum gravity: Due to the quantum fluctuation of the gravitational field and the appearance of virtual black holes, space-time loses its continuum aspect at distances of the order of Planck's scale and assumes a foamlike behavior [22]. As a consequence, new, nonstandard phenomena can arise, leading to loss of quantum coherence [23–28].

Unfortunately, our present knowledge of string theory does not allow us to estimate precisely the magnitude of the nonstandard, dissipative effects induced on elementary particle systems; they are nevertheless expected to be very small, being suppressed by at least one inverse power of the Planck mass, as rough dimensional analysis suggests. In spite of this, the new effects can affect interference phenomena and turn out to be in the reach of future, planned experiments. Indeed, detailed investigations of neutral meson systems, neutron interferometry, and photon propagation using quantum dynamical semigroups have already allowed deriving order of magnitude limits on some of the phenomenological constants parametrizing the new effects, using available experimental data [17,19,29–31].

In the present work, we shall discuss in detail how nonstandard, dissipative phenomena can affect neutrino propagation, and in particular neutrino oscillations. We shall limit our considerations to the oscillations of two species of neutrinos; in this case, the possible dissipative effects can be described in terms of six phenomenological parameters. A preliminary investigation, limited to vacuum oscillations, has been reported in Ref. [18]. There, it has been shown that the dissipative phenomena modify the transition probability $\mathcal{P}$ among the two neutrino flavors, introducing in particular exponential damping factors. In a simplified situation, limits on one of the dissipative parameters have subsequently been obtained using recent SuperKamiokande data [32].

In the following, a much more complete discussion will be presented, with detailed analysis of oscillation phenomena in presence of irreversibility, both in vacuum and in matter. Dissipation affects both situations; in particular, the resonance condition for neutrino propagation in matter turns out to be modified, leading to distinctive observable effects. Various approximate expressions for the transition probability $\mathcal{P}$ will be given: They can be useful in fitting experimental data. A discussion on a possible physical mechanism that could give rise to the nonstandard effects will also be presented, although much of the technical analysis will be relegated to the Appendix.

As a final remark, let us stress that the presence of nonstandard, dissipative phenomena modify neutrino physics in two important aspects. First, they give the neutrinos an effective mass, so that oscillations are possible even for massless neutrinos. Further, contrary to the standard case, the expression of the transition probability $\mathcal{P}$ depends in general on the *CP*-violating phase that is present in the mixing matrix for Majorana neutrinos. This allows us, at least in principle, to distinguish between Dirac versus Majorana neutrinos in oscillation experiments. We find this possibility as one of the most intriguing outcomes of our investigation.

II. NEUTRINOS AS OPEN QUANTUM SYSTEMS

The familiar description of neutrino oscillations involves the study of the evolution of neutrinos created in a given flavor by the weak interactions and subsequently detected at a later time. The traveling neutrinos are usually assumed to be ultrarelativistic, so that the analysis of the transition probability for the original tagged neutrinos to be found in a different flavor can be performed using an effective description [33–37].

For sake of simplicity, in the following we shall limit our considerations to the mixing of two neutrino species.[1] In this case, the neutrino system can be effectively modeled by means of a two-dimensional Hilbert space; the two neutrino mass eigenstates will be henceforth fixed as the basis in this space. In the presence of dissipation, the physical neutrino states cannot be described in terms of elements of the Hilbert space: A more general formalism is needed that makes use of density matrices. These are Hermitian, positive operators (i.e., with non-negative eigenvalues), normalized to have unit trace.

With respect to the fixed basis, the two flavor states, which we shall conventionally call ν_e and ν_μ, are represented by the following 2×2 matrices:

$$\rho_{\nu_e}=\begin{pmatrix} \cos^2\theta & e^{-i\varphi}\cos\theta\sin\theta \\ e^{i\varphi}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix}. \tag{2.1a}$$

$$\rho_{\nu_\mu}=\begin{pmatrix} \sin^2\theta & -e^{-i\varphi}\cos\theta\sin\theta \\ -e^{i\varphi}\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}\equiv 1-\rho_{\nu_e}, \tag{2.1b}$$

where θ is the "vacuum" mixing angle, while the additional phase φ can be nonvanishing for neutrinos of Majorana type. That this extra phase cannot be eliminated by a simple basis redefinition is a well-known consequence of the reality condition for the Majorana neutrinos, and, at least in principle, its presence can be experimentally probed [38,39]. Nevertheless, in the usual approach, this cannot happen via the analysis of oscillation phenomena alone [40]. As we shall see, the situation is different in the presence of dissipative effects, so that in the following we shall keep φ nonvanishing unless explicitly stated.

As explained in the introductory remarks, the evolution in time of any neutrino state ρ will be described by means of linear maps, $\Gamma_t:\rho(0)\mapsto\rho(t)$, that generalize the standard quantum mechanics unitary evolution. Not all generalized maps Γ_t turn out to be physically acceptable: They need to satisfy very general physical requirements. First, the maps Γ_t should transform neutrino states into neutrino states, and therefore should map any initial density matrix $\rho(0)$ into a density matrix $\rho(t)\equiv\Gamma_t[\rho(0)]$, for any t. Furthermore, they should have the property of obeying the semigroup composition law, $\Gamma_t[\rho(t')]=\rho(t+t')$, for $t,t'\geqslant 0$, of increasing the (von Neumann) entropy, $S=-\mathrm{Tr}[\rho(t)\ln\rho(t)]$, of being completely positive.

It has been proved long ago that evolution maps Γ_t satisfying these properties are generated by equations of the following form [1–3]:

$$\frac{\partial\rho(t)}{\partial t}=-iH_{\rm eff}\,\rho(t)+i\rho(t)H_{\rm eff}+L[\rho(t)]. \tag{2.2a}$$

The first two pieces on the right hand side (RHS) represent the standard quantum mechanical contributions: They give rise to the traditional description of neutrino oscillations in terms of the effective (time-independent) Hamiltonian $H_{\rm eff}$. We shall neglect effects due to possible neutrino instability: $H_{\rm eff}$ can then be taken to be Hermitian. The additional piece $L[\rho]$ is a linear map that encodes possible dissipative, nonstandard effects. It can be written as,

$$L[\rho]=-\frac{1}{2}\sum_j\,(A_j^\dagger A_j\rho+\rho A_j^\dagger A_j)+\sum_j\,A_j\rho A_j^\dagger, \tag{2.2b}$$

where the operators A_j must be such that $\Sigma_j A_j^\dagger A_j$ is a well-defined 2×2 matrix (entropy increase can be easily implemented by taking the A_j to be Hermitian). In its absence, pure states (i.e., states of the form $|\psi\rangle\langle\psi|$) would be transformed by Γ_t into pure states. Only when the extra piece $L[\rho]$ is also present does $\rho(t)$ become less ordered in time due to a mixing-enhancing mechanism; it produces irreversibility and possible loss of quantum coherence.

In the case of two neutrino flavors, $L[\rho]$ can be fully parametrized in terms of six, real phenomenological constants, a, b, c, α, β, and γ, with a, α, and γ non-negative, satisfying the following inequalities [1,15,16]:

[1]The discussion can be generalized to the case of three or more neutrinos; however, the explicit formulas for the transition probabilities would become much more involved and the discussion less transparent.

$$2R \equiv \alpha+\gamma-a \geqslant 0, \qquad U \equiv RS-b^2 \geqslant 0,$$

$$2S \equiv a+\gamma-\alpha \geqslant 0, \qquad V \equiv RT-c^2 \geqslant 0,$$

$$2T \equiv a+\alpha-\gamma \geqslant 0, \qquad Z \equiv ST-\beta^2 \geqslant 0, \tag{2.3}$$

$$X \equiv RST-2bc\beta-R\beta^2-Sc^2-Tb^2 \geqslant 0.$$

They are direct a consequence of the property of complete positivity. In order for the 2×2 matrix $\rho(t)$ to represent a neutrino state, its eigenvalues should be positive for any time t; this is crucial for the physical consistency of the whole formalism. The eigenvalues of $\rho(t)$ are in fact interpreted as probabilities. The property of complete positivity precisely ensures that this holds true in any possible condition. (For a complete discussion, see Ref. [41].)

The one-parameter family of finite evolution Γ_t generated by Eq. (2.2) are called quantum dynamical semigroups; they will be the basis of the phenomenological treatment of the dissipative effects in the neutrino system. The description of irreversible, nonstandard phenomena by means of equations of the form (2.2) is actually very general and can be applied to the study of very different physical systems. Originally developed in the framework of quantum optics [5–7], it has also been successfully used in the analysis of statistical models [1–3], the interaction of a microsystem with a measuring apparatus [8–11], the study of dissipative effects in systems involving elementary particles, and in particular neutral mesons [15,16,29–31]. Although essentially phenomenological in nature, all these analysis can be supported by physical considerations.

A general picture in which the quantum dynamical semigroup description of dissipative effects naturally emerges is provided by open systems, i.e., by systems in weak interactions with a large environment. In the case of elementary particles, these effects are likely to originate from the fundamental dynamics of strings or branes, which is, in general, rather complex. Nevertheless, an effective description of the environment that encodes some of the properties of the underlying fundamental dynamics turns out to be adequate for a more physical discussion of evolution equations of type (2.2).

Quite in general, the total Hamiltonian of a system $\mathcal{S}$ in interaction with an environment $\mathcal{E}$ can be decomposed as

$$H_{\rm tot}=H\otimes \mathbf{1}+\mathbf{1}\otimes H_{\mathcal{E}}+gH', \tag{2.4}$$

where H is the system Hamiltonian in the absence of $\mathcal{E}$, while $H_{\mathcal{E}}$ drives the internal dynamics of the environment. The interaction between $\mathcal{S}$ and $\mathcal{E}$ is described by H', with g a small, dimensionless coupling constant.

In many instances, the initial state of the total system $\mathcal{S}+\mathcal{E}$ can be taken to be in factorized form: $\rho_{\rm tot}=\rho\otimes\rho_{\mathcal{E}}$. This is surely justified in the case of the neutrino system: Since the mechanism of neutrino production is different from the one responsible for the dissipative effects, system and environment are surely uncorrelated at the moment of the emission.[2] Then, the time evolution of the state ρ of the system $\mathcal{S}$ can be obtained by tracing over the environment degrees of freedom:

$$\rho\equiv\rho(0)\mapsto\rho(t)=\mathrm{Tr}_{\mathcal{E}}[e^{-iH_{\rm tot}t}(\rho\otimes\rho_{\mathcal{E}})e^{iH_{\rm tot}t}], \tag{2.5}$$

In general, the resulting map $\rho(0)\to\rho(t)$ turns out to be rather involved, developing nonlinearity and memory effects. Nevertheless, when the interaction between $\mathcal{S}$ and $\mathcal{E}$ is weak, an evolution equation for $\rho(t)$ local in time naturally emerges. The technical details are presented in the Appendix. As discussed there, the environment can be modeled as a gas of quanta, obeying infinite statistics; this description is in line with the idea that the dissipative effects originate from the low energy string dynamics at a fundamental scale M_F (e.g., Planck's mass). Then, in the weak coupling limit, i.e., when the coupling constant g becomes very small, the resulting dynamical equation for the subsystem state $\rho(t)$ turns out to be precisely of the form (2.2) [1–3,21].

This result allows a rough estimate of the magnitude of the effects produced by the nonstandard piece $L[\rho]$: They should be proportional to powers of the typical energy of the system $\mathcal{S}$, while suppressed by inverse powers of the characteristic energy scale of $\mathcal{E}$. In the case of the neutrino system, these effects should be very small, since the typical energy scale of the environment can be assimilated to the fundamental scale M_F. For any fixed neutrino source and observational conditions, an upper bound on the magnitude of the effects induced by $L[\rho]$ can be evaluated to be of order E^2/M_F, where E is the average neutrino energy.

As a further outcome of the weak coupling limit procedure, the Hamiltonian part of the evolution equation for $\rho(t)$ gets modified by the presence of the environment. Indeed, the effective Hamiltonian $H_{\rm eff}$ in Eq. (2.2) does not coincide in general with the starting system Hamiltonian H in Eq. (2.4): Suitable dissipative contributions to H, generated by the interaction H', need to be taken into account [1–3,21]. As we shall see in the following, this fact has interesting consequences in neutrino physics: One can have oscillations among different flavors induced by dissipative effects even for massless (or mass-degenerate) neutrinos. In other words, originally massless neutrinos can get an effective nonzero mass via the interaction with the environment.

III. QUANTUM DYNAMICAL SEMIGROUPS AND NEUTRINO OSCILLATIONS

In the case of the neutrino system, much of the considerations and discussions of the previous section about the evolution equation (2.2) can be made more transparent and explicit. In particular, both for the effective Hamiltonian $H_{\rm eff}$ and for the extra piece $L[\rho]$, simple expressions can be given.

We shall be as general as possible and include in our

[2]Even in presence of an initially correlated total system $\mathcal{S}+\mathcal{E}$, the factorized approximation becomes a very good approximation when the short-time correlations have died out [4].

discussion effects due to the propagation of neutrinos in a medium made of ordinary matter. Because of the interactions of the neutrinos with the particles in the medium, an effective potential can be generated, that has different effects for different flavors. In the case of ordinary matter, the electron neutrinos interact with the electrons in the medium, so that their average energy effectively receive an extra contribution $A=\sqrt{2}G_F n_e$ with respect to the energy of the muon neutrinos (G_F is the Fermi constant, while n_e represents the electron number density in the medium) [33–37,42]. In the ordinary case, this contribution can significally change the oscillation pattern between ν_e and ν_μ states [the so-called Mikheyev-Smirnov-Wolfenstein (MSW) effect] [43,44]. As we shall see, this phenomenon can be substantially modified by the presence of nonstandard, dissipative effects.

On the basis introduced in the previous section, the 2×2 matrix representing the effective Hamiltonian can be taken to be of the form

$$H_{\rm eff}=\begin{pmatrix} E-\omega_0-\omega_3 & \omega_1-i\omega_2 \\ \omega_1+i\omega_2 & E+\omega_0+\omega_3\end{pmatrix} + \frac{A}{2}\begin{pmatrix} 1+\cos 2\theta & e^{-i\varphi}\sin 2\theta \\ e^{i\varphi}\sin 2\theta & 1-\cos 2\theta\end{pmatrix}. \tag{3.1}$$

In the first piece, E represents the average neutrino energy, while $\omega_0=\Delta m^2/4E$ takes into account the square mass difference Δm^2 of the two mass eigenstates; these are the usual contributions that give rise to the standard oscillation pattern in vacuum. The extra real parameters ω_1, ω_2, and ω_3 are the consequence of the interaction with the environment; as explained in the previous section (and discussed in detail in the Appendix), they represent the contribution of the dissipative phenomena to the system Hamiltonian.

Both ω_0 and ω_1, ω_2, ω_3 contribute to the level splitting $\omega=[(\omega_0+\omega_3)^2+\omega_1^2+\omega_2^2]^{1/2}$ between the two mass eigenstates, so that they all contribute to the oscillation phenomena in vacuum. Therefore, even for initially degenerate mass eigenstates, $\Delta m^2=0$, vacuum oscillations can occur between the two flavors due to the dissipative effects induced by the fundamental dynamics at the large scale M_F. Although in general all three parameters ω_1, ω_2, and ω_3 are nonvanishing, in the following, in order to simplify the treatment, we shall assume $\omega_1=\omega_2=0$; this working assumption allows for more manageable formulas, while keeping unaffected their physical meaning and implications.[3]

The final contribution to $H_{\rm eff}$ in Eq. (3.1) takes into account the interaction of the propagating neutrinos with ordinary matter; it would be diagonal in the flavor basis (only electron neutrinos are affected), but assumes a more complicated expression involving the mixing angle θ and the phase φ in the chosen basis. Since the coefficient A is proportional to the density of electrons in the mean, for propagation in nonhomogeneous matter $H_{\rm eff}$ will in general be a function of the position of the neutrinos. Nevertheless, one can always approximate a nonhomogeneous medium by a collection of media, each with a constant density, while having different thickness; in view of this, in the following we shall assume the parameter A be a constant (see also the discussion in Sec. IV).

As mentioned before, although the effective Hamiltonian $H_{\rm eff}$ gets also dissipative contributions, only when the additional piece $L[\rho]$ in the evolution equation (2.2) is nonvanishing, irreversibility and mixing enhancing effects are possible. In the present case, its explicit expression in terms of the six phenomenological constants a, b, c, α, β, and γ in Eq. (2.3) can be most simply given by expanding the 2×2 matrix ρ in terms of the Pauli matrices σ_i, $i=1,2,3$, and the identity σ_0:

$$\rho=\tfrac{1}{2}\sum_{\mu=0}^{3}\rho_\mu\sigma_\mu. \tag{3.2}$$

In this way, the linear map L acting on ρ can be represented by the following, symmetric 4×4 matrix $[L_{\mu\nu}]$, acting on the four-vector of components $(\rho_0,\rho_1,\rho_2,\rho_3)$:

$$[L_{\mu\nu}]=-2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & b & \alpha & \beta \\ 0 & c & \beta & \gamma\end{pmatrix}. \tag{3.3}$$

The form of the evolution equation (2.2) can be further simplified by recalling that it is trace preserving. From the initial normalization condition $\mathrm{Tr}[\rho(0)]=1$, one immediately obtains that the component of $\rho(t)$ along the identity is equal to one for all times. Then, the evolution equation for the remaining three components of $\rho(t)$ can be rewritten in a Schrödinger-like form:

$$\frac{\partial}{\partial t}|\rho(t)\rangle=-2\mathcal{H}|\rho(t)\rangle, \tag{3.4}$$

where the three-vector $|\rho\rangle$ has components (ρ_1,ρ_2,ρ_3), while

$$\mathcal{H}=\begin{pmatrix} a & b+\mu & c-\nu\sin\varphi \\ b-\mu & \alpha & \beta+\nu\cos\varphi \\ c+\nu\sin\varphi & \beta-\nu\cos\varphi & \gamma\end{pmatrix}, \tag{3.5}$$

with

$$\mu=\frac{A}{2}\cos 2\theta-\omega, \quad \nu=\frac{A}{2}\sin 2\theta. \tag{3.6}$$

The solution of Eq. (3.4) involves the formal exponentiation of the matrix $\mathcal{H}$:

$$|\rho(t)\rangle=\mathcal{M}(t)|\rho(0)\rangle, \quad \mathcal{M}(t)=e^{-2\mathcal{H}t}. \tag{3.7}$$

As discussed in Ref. [18], expressions for the entries of $\mathcal{M}(t)$ can always be obtained by solving the eigenvalue problem for the 3×3 matrix in Eq. (3.5):

[3]When $\Delta m^2=0$, this is no longer an assumption: In this case, one can always choose to work in a basis for which ω_1 and ω_2 vanish.

$$\mathcal{H}|v^{(k)}\rangle=\lambda^{(k)}|v^{(k)}\rangle,\quad k=1,2,3. \tag{3.8}$$

The three eigenvalues $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$ satisfy the cubic equation:

$$\lambda^3+r\lambda^2+s\lambda+w=0, \tag{3.9}$$

with real coefficients:

$$r\equiv-(\lambda^{(1)}+\lambda^{(2)}+\lambda^{(3)})=-(a+\alpha+\gamma), \tag{3.10a}$$

$$\begin{aligned}s&\equiv\lambda^{(1)}\lambda^{(2)}+\lambda^{(1)}\lambda^{(3)}+\lambda^{(2)}\lambda^{(3)}\\&=a\alpha+a\gamma+\alpha\gamma-b^2-c^2-\beta^2+\mu^2+\nu^2,\end{aligned} \tag{3.10b}$$

$$\begin{aligned}w&\equiv-\lambda^{(1)}\lambda^{(2)}\lambda^{(3)}\\&=a(\beta^2-\nu^2\cos^2\varphi)+\alpha(c^2-\nu^2\sin^2\varphi)\\&\quad+\gamma(b^2-\mu^2)-a\alpha\gamma-2bc\beta-b\nu^2\sin 2\varphi\\&\quad-2\mu\nu(\beta\sin\varphi+c\cos\varphi).\end{aligned} \tag{3.10c}$$

The solutions are either real, or one is real and the remaining two are complex conjugate, according to the sign of the associated discriminant: $D=p^3+q^2$, $p=s/3-(r/3)^2$, $q=(r/3)^3-rs/6+w/2$ (degenerate, real solutions occur when $D=0$) [45]. Then, recalling that the matrix $\mathcal{H}$ itself satisfies Eq. (3.9), one can derive the following expression for the entries of $\mathcal{M}(t)$:

$$\mathcal{M}_{ij}(t)=\sum_{k=1}^{3}e^{-2\lambda^{(k)}t}\times\left[\frac{([\lambda^{(k)}]^2+r\lambda^{(k)}+s)\delta_{ij}+(\lambda^{(k)}+r)\mathcal{H}_{ij}+\mathcal{H}^2_{ij}}{3[\lambda^{(k)}]^2+2r\lambda^{(k)}+s}\right],\quad i,j=1,2,3. \tag{3.11}$$

Although rather formal, this formula allows a general discussion on the behavior of $\mathcal{M}(t)$. For $\mu=\nu=0$, due to the inequalities in Eq. (2.3), the matrix $\mathcal{H}$ results are real, symmetric, and non-negative; its eigenvalues are all real and non-negative. Only when $|\mu|$ and $|\nu|$ are sufficiently large can complex eigenvalues appear, although with a non-negative real part, since in general the evolution generated by Eq. (2.2) is bounded for any t [46]. In this case an oscillatory behavior is possible, while for small μ, ν, the damping terms prevail and dissipation is the dominant phenomena.

In particular, since generically $\det\mathcal{H}\equiv-w\neq0$, in presence of dissipation the real part of $\lambda^{(1)},\lambda^{(2)},\lambda^{(3)}$ are all strictly positive; therefore $\mathcal{M}(t)$ asymptotically vanishes for large enough times.[4] This has clearly dramatic consequences in the study of neutrino flavor transitions.

Let us assume that at $t=0$ the neutrinos are generated to be of type ν_e. In the formalism of density matrices, the probability of having a transition into neutrinos of type ν_μ at time t is given by

$$\mathcal{P}_{\nu_e\to\nu_\mu}(t)\equiv\mathrm{Tr}[\rho_{\nu_e}(t)\rho_{\nu_\mu}]=\frac{1}{2}\left[1+\sum_{i,j=1}^{3}\rho^i_{\nu_\mu}\rho^j_{\nu_e}\mathcal{M}_{ij}(t)\right], \tag{3.12}$$

where $\rho_{\nu_e}(t)$ is the solution of Eq. (2.2) with the initial condition given by the matrix $\rho_{\nu_e}(0)\equiv\rho_{\nu_e}$, while $\rho^i_{\nu_e},\rho^j_{\nu_\mu},i,j=1,2,3$ are the components of the three-vectors $|\rho_{\nu_e}\rangle$, $|\rho_{\nu_\mu}\rangle$ corresponding to the density matrices in Eq. (2.1). Using the explicit expressions for these components, one finds

$$\begin{aligned}\mathcal{P}_{\nu_e\to\nu_\mu}(t)=\frac{1}{2}\big(1&-\cos^2 2\theta\,\mathcal{M}_{33}(t)-\sin^2 2\theta\{\mathcal{M}_{11}(t)\cos^2\varphi\\&+\mathcal{M}_{22}(t)\sin^2\varphi+[\mathcal{M}_{12}(t)+\mathcal{M}_{21}(t)]\\&\times\sin\varphi\cos\varphi\}-\cos 2\theta\sin 2\theta\\&\times\{[\mathcal{M}_{13}(t)+\mathcal{M}_{31}(t)]\cos\varphi\\&+[\mathcal{M}_{23}(t)+\mathcal{M}_{32}(t)]\sin\varphi\}\big).\end{aligned} \tag{3.13}$$

One of the interesting features of this formula is its explicit dependence on the phase φ; in the presence of dissipative effects, it is therefore possible, at least in principle, to distinguish between Dirac and Majorana neutrinos by studying the oscillation pattern in Eq. (3.13). This peculiarity disappears when the nonstandard, dissipative pieces in Eq. (2.2) are absent; indeed, in that case, one has

$$\mathcal{M}_{ij}(t)=\delta_{ij}-\frac{\sin 2\omega_M t}{\omega_M}\mathcal{H}_{ij}+\frac{2\sin^2\omega_M t}{\omega_M^2}\mathcal{H}^2_{ij}\quad i,j=1,2,3, \tag{3.14}$$

where $\mathcal{H}$ is now as in Eq. (3.5) with a, b, c, α, β, and γ all equal to zero, while $\omega_M=\sqrt{\mu^2+\nu^2}$, and Eq. (3.13) reduces to the well-known standard expression for the oscillation probability in an homogeneous medium [33–37,42]:

$$\begin{aligned}\mathcal{P}^{(0)}_{\nu_e\to\nu_\mu}(t)&=\sin^2 2\theta_M\sin^2\omega_M t,\\ \sin^2 2\theta_M&=\frac{\sin^2 2\theta}{(A/2\omega-\cos 2\theta)^2+\sin^2 2\theta}.\end{aligned} \tag{3.15}$$

Another distinctive characteristic of the transition probability in the presence of dissipation given in Eq. (3.13) is its asymptotic behavior for large times, which turns out to be independent from the mixing angle θ, the phase φ and the matter coefficient A:

$$\mathcal{P}_{\nu_e\to\nu_\mu}(t)\underset{t\to\infty}{\sim}\tfrac{1}{2}. \tag{3.16}$$

This result is a direct consequence of the vanishing of the matrix $\mathcal{M}(t)$ in Eq. (3.7). Nevertheless, as discussed below,

[4] In the presence of vanishing eigenvalues, this decoherence effect is only partial [18]; however, note that having $\det\mathcal{H}=0$ requires a unnatural fine-tuning among the parameters in Eq. (3.10c).

the regime of validity of this asymptotic limit can seldom be reached in practical experimental conditions.

IV. TRANSITION PROBABILITY IN MATTER

The general expression (3.13) for the transition probability is rather involved and is not particularly useful for studying in more detail its physical properties. Therefore, in the present and following sections we shall discuss various approximations in which $\mathcal{P}_{\nu_e\to\nu_\mu}(t)$ assumes a more manageable form. These simplified expressions, besides being appropriate for theoretical analysis, could also be used to fit actual experimental data [32,47].

As already mentioned in Sec. II, the values of the constants a, b, c, α, β, and γ parametrizing the nonstandard effects are expected to be very small, with an upper bound of order $E^2/M_F \simeq 10^{-19}$ GeV for $E\simeq 1$ GeV and for M_F, the Planck mass. Nevertheless, this estimate is not far from the values that the standard oscillation parameter $\omega_0=\Delta m^2/4E$ assumes for typical neutrino sources. Indeed, the ratio of a, b, c, α, β, and γ with ω_0 can be evaluated to be at most of order $10^{-10}E^3/\Delta m^2$, with E expressed in MeV and the neutrino mass difference Δm^2 in eV^2; this ratio turns out to be about 10^2 for atmospheric neutrinos, of order one for solar neutrinos, while for accelerator neutrinos it can be as small as 10^{-2}. Therefore, the effects induced by dissipation can interfere with those producing oscillations via a nonvanishing ω_0, resulting in observable modifications of the oscillation pattern. Present and, most likely, future dedicated neutrino experiments should be able to detect these modifications, or at least put stringent limits on the magnitude of the nonstandard phenomena.

Let us first consider the case in which the dissipative parameters a, b, c, α, β, and γ are of the same order or larger than the remaining constants in Eq. (3.1). In this case a very useful approximation is to assume $a=\alpha=\gamma$ and $c=0$, conditions perfectly compatible with the inequalities (2.3), provided $\alpha^2 \geqslant b^2+\beta^2$. For simplicity, we further assume the extra phase φ to be vanishingly small. A manageable expression for the transition probability can then be derived:

$$\mathcal{P}_{\nu_e\to\nu_\mu}(t)=\frac{1}{2}(1-e^{-2\alpha t})+\left[\frac{\tilde{\nu}^2-\tilde{\beta}^2}{\Omega_M^2}\right]e^{-2\alpha t}\sin^2(\Omega_M t), \tag{4.1}$$

where

$$\Omega_M=[\mu^2+\nu^2-b^2-\beta^2]^{1/2}, \tag{4.2}$$

$$\tilde{\nu}=\omega\sin 2\theta, \quad \tilde{\beta}=\beta\cos 2\theta+b\sin 2\theta. \tag{4.3}$$

The oscillating behavior in Eq. (4.1) depends on the magnitude of the combination $\mu^2+\nu^2=(A/2-\omega\cos 2\theta)^2+\omega^2\sin^2 2\theta$ with respect to $b^2+\beta^2$; in regions for which $b^2+\beta^2 \geqslant \mu^2+\nu^2$, the frequency Ω_M becomes purely imaginary and $\mathcal{P}_{\nu_e\to\nu_\mu}(t)$ contains only exponential terms. Anyway, the α-dependent damping terms in Eq. (4.1) dominate for large times, and the asymptotic limit (3.16) is thus recovered.

In absence of dissipation, $\alpha=b=\beta=0$, the factor in front of the sine term in Eq. (4.1) becomes parametrized as in Eq. (3.15), with a modified mixing angle θ_M. When the matter parameter A is close to $A_R\equiv 2\omega\cos 2\theta$, the transition probability gets enhanced, and oscillations between the two neutrino species is possible even when the original mixing angle θ is small. This phenomenon is at the root of the so-called MSW effect [42–44].

In the presence of dissipation, however, the physical consequences of this effect are in general much more modest. In this case, one cannot parametrize $\mathcal{P}_{\nu_e\to\nu_\mu}(t)$ in terms of a modified mixing angle; in spite of this, the expression in Eq. (4.1) as a function of A, at fixed time, has a critical point for $A=A_R$. This point is a maximum for $\tilde{\nu}^2 \geqslant \tilde{\beta}^2$, and indeed as A approaches A_R an enhancement in $\mathcal{P}_{\nu_e\to\nu_\mu}(t)$ occurs: $(\tilde{\nu}^2-\tilde{\beta}^2)/\Omega_M^2>1$; however, the exponentially damping factors in Eq. (4.1) greatly reduce in practice its effectiveness. Furthermore, when $\tilde{\nu}^2<\tilde{\beta}^2$, the probability $\mathcal{P}_{\nu_e\to\nu_\mu}(t)$ in Eq. (4.1) is maximally suppressed at the critical point: It is dominated by the damping factors.

This discussion might appear spoiled by the initial assumption of a constant matter parameter A: The occurrence of the MSW effect requires a medium with a (slowly) varying density. As already pointed out, the assumption of a constant A is not really a limitation: One can always approximate, with arbitrary accuracy, the traveling of neutrinos through varying density matter as the propagation in a series of media with different constant densities and different thickness. The total time evolution will be given by the composition of the evolutions in the various matter slices, so that the matrix $\mathcal{M}$ in Eq. (3.7) becomes

$$\mathcal{M}(t)=\mathcal{M}_n(t_n)\cdots\mathcal{M}_2(t_2)\mathcal{M}_1(t_1), \quad t=t_1+t_2+\cdots+t_n, \tag{4.4}$$

where, $t_1,t_2,\ldots,t_n$ are the total times spent by the neutrinos in the various media, while $\mathcal{M}_i$, $i=1,\ldots,n$ are the corresponding propagation matrices.

As an example, let us consider the case of an initial electron neutrino traveling for a time t_1 into a medium with matter parameter A, which is then detected in vacuum at a later time $t=t_1+t_2$; this situation can roughly represent a solar neutrino model. Using Eq. (4.4), the probability of detecting the original ν_e as a muon neutrino is given by

$$\begin{aligned}\mathcal{P}_{\nu_e\to\nu_\mu}(t)=\frac{1}{2}\Bigg\{&1-e^{-\alpha(t_1+t_2)}-e^{-\alpha(t_1+t_2)}\left(\frac{\tilde{\nu}^2-\tilde{\beta}^2}{\Omega_0^2}\right)\\ &\times\Bigg[\frac{4\Omega_M^2-A(A+2b+2\omega\cos 2\theta)}{\Omega_M^2}\\ &\times\sin^2(\Omega_M t_1)\sin^2(\Omega_0 t_2)\\ &-\frac{\Omega_0}{\Omega_M}\sin(2\Omega_M t_1)\sin(2\Omega_0 t_2)-2\sin^2(\Omega_0 t_2)\\ &-2\frac{\Omega_0^2}{\Omega_M^2}\sin^2(\Omega_M t_1)\Bigg]\Bigg\},\end{aligned} \tag{4.5}$$

where $\Omega_0 = \sqrt{\omega^2 - b^2 - \beta^2}$, while Ω_M is as in Eq. (4.2). One can check that, with the appropriate choice of parameters (see the previous discussion), the probability $\mathcal{P}_{\nu_{e\to\nu_\mu}}(t)$ can indeed get an enhancement for A close to the critical point; the effect is, however, modest and further suppressed by the damping factors. Nevertheless, with the appropriate values of A, t_1, and t_2, the expression (4.5) can be used to fit solar neutrino data; the total flight time t is large, so that a good sensitivity at least on the dissipative parameter α is surely attainable.

A different approximation of the full expression (3.13) for the transition probability $\mathcal{P}_{\nu_{e\to\nu_\mu}}$ can be obtained when the dissipative parameters a, b, c, α, β, and γ can be considered small with respect to the level-splitting term ω; as mentioned before, this typically occur for neutrino beams generated at accelerators. In this case the additional term $L[\rho]$ in the evolution equation (2.2) can be treated as a perturbation. To first order in the small parameters, explicit expressions for the entries of the evolution matrix $\mathcal{M}(t)$ in Eq. (3.7) can be easily obtained; then, using Eq. (3.12), one finds

$$\mathcal{P}_{\nu_{e\to\nu_\mu}}(t) = \left(\frac{\tilde{\mu}^2}{\omega_M^2}\right) e^{-2\lambda_1 t} + e^{-\lambda_2 t}\left[\left(\frac{\tilde{\nu}^2}{\omega_M^2}\right)\cos(2\omega_M t) + \left(\frac{N}{\omega_M^2}\right)\sin(2\omega_M t)\right], \tag{4.6}$$

where $\omega_M = \sqrt{\mu^2 + \nu^2}$ as in the previous section, while

$$\tilde{\mu} = \mu\cos 2\theta + \nu \sin 2\theta = \frac{a}{2} - \omega\cos 2\theta, \tag{4.7a}$$

$$\tilde{\nu} = \nu\cos 2\theta - \mu\sin 2\theta = \omega \sin 2\theta; \tag{4.7b}$$

the parameters λ_1, λ_2, and N contain the dependence on the dissipative constants:

$$\lambda_1 = (a\nu^2 + 2c\mu\nu + \gamma\mu^2)/\omega_M^2, \tag{4.8a}$$

$$\lambda_2 = \alpha + (a\mu^2 - 2c\mu\nu + \gamma\nu^2)/\omega_M^2, \tag{4.8b}$$

$$N = \frac{\tilde{\nu}^2}{2}[\alpha - a - \nu(3a\nu + 2c\mu)/\omega_M^2] + 3\nu\tilde{\nu}(a-\gamma)\cos 2\theta + c(2\mu\tilde{\mu}^2\nu - \omega_M^4 \sin 4\theta)/\omega_M^2. \tag{4.8c}$$

In expression (4.6), we have reconstructed the exponential factors by consistently putting together the terms linear in t. Notice that the result in Eqs. (4.8) is in agreement with the discussion in Sec. III concerning the eigenvalues of the matrix $\mathcal{H}$. In this case the algebraic equation (3.9) has one real, $\lambda^{(1)}$, and two complex conjugate solutions, $\lambda^{(2,3)} = \lambda_R \pm i\lambda_I$; within our approximation, $\lambda^{(1)} = \lambda_1$, $2\lambda_R = \lambda_2$, so that the first condition in Eq. (3.10) is satisfied, while the remaining two fix the imaginary part λ_I.

The expression (4.6) for the transition probability can be used to fit experimental data. With respect to the standard case, it contains three additional parameters, λ_1, λ_2, and N, that signal the presence of nonstandard phenomena, through the constants a, b, c, α, β, and γ. If at least one of these three parameters is found to be nonzero, it would clearly signal the presence of dissipative effects in neutrino physics; this is surely the most simple experimental check on the generalized evolution equation (2.2) that can be performed with accelerator neutrino beams.

V. ADIABATIC APPROXIMATION

For time evolutions with a semigroup composition law, the appropriate way to follow neutrino propagation in a medium is through the successive applications of the finite evolution matrices $\mathcal{M}(t)$ to the initial state $|\rho(0)\rangle$, as shown in Eq. (4.4). In more traditional approaches, one usually adopts a different approximation, based on the assumption of (adiabatic) slowly varying matter density. This approximation can be easily discussed also in the framework of density matrices and quantum dynamical semigroups.

Let us consider the case of a neutrino, created at $t=0$ in matter of high density (i.e., in the core of the sun), propagating towards regions of smaller density. In this case, the effective Hamiltonian (3.1) is no longer constant, and the propagating matrix $\mathcal{M}(t)$ involves a time-ordered exponentiation of $\mathcal{H}$. Nevertheless, at any instant of time, the 3×3 matrix $\mathcal{H}$ can be diagonalized by a similarity transformation:

$$\mathcal{H} = T\mathcal{D}T^{-1}. \tag{5.1}$$

Using this decomposition in Eq. (3.4), one can derive the evolution equation for the transformed three-vector, $|\tilde{\rho}\rangle = T|\rho\rangle$; explicitly, one finds

$$\frac{\partial}{\partial t}|\tilde{\rho}(t)\rangle = -2\left(\mathcal{D} + \frac{\partial T}{\partial t}T^{-1}\right)|\tilde{\rho}(t)\rangle. \tag{5.2}$$

The adiabatic approximation amounts to neglecting the last term in this equation; this is justified when the matter density parameter A is slowly varying. In this approximation, the neutrino state essentially evolves in time as an eigenstate of $\mathcal{H}$.

In order to make the discussion more explicit, as in the previous section, we shall take $a=\alpha=\gamma$ and $c=0$, while neglecting the extra phase φ and the dissipative contributions ω_1, ω_2 to $\mathcal{H}$. With these choices, one has

$$\mathcal{D} = -2\begin{pmatrix} \alpha & & \\ & \alpha + i\Omega_M & \\ & & \alpha - i\Omega_M \end{pmatrix}, \tag{5.3}$$

with Ω_M as in Eq. (4.2), while the transformation matrix T takes the form

$$T = \frac{1}{\sqrt{2}\Omega_M}\begin{pmatrix} \sqrt{2}(\beta+\nu) & b+\mu & b+\mu \\ 0 & i\Omega_M & -i\Omega_M \\ -\sqrt{2}(b-\mu) & \beta-\nu & \beta-\nu \end{pmatrix}. \tag{5.4}$$

The entries of T can be parametrized in terms of two real variables ξ and ζ and an angle ϕ, that could be complex:

$$\frac{\beta \pm \nu}{\Omega_M} = \pm (e^{\pm \zeta} \cos 2\theta \sin 2\phi + e^{\pm \xi} \sin 2\theta \cos 2\phi), \tag{5.5a}$$

$$\frac{b \pm \mu}{\Omega_M} = \pm (e^{\mp \zeta} \sin 2\theta \sin 2\phi - e^{\mp \xi} \cos 2\theta \cos 2\phi); \tag{5.5b}$$

for later convenience, an explicit dependence on the mixing angle θ has been extracted from the entries of T.

Using a compact vector notation, the transition probability $\mathcal{P}_{\nu_e \to \nu_\mu}$ in Eq. (3.12) can now be written as

$$\mathcal{P}_{\nu_e \to \nu_\mu}(t) = \tfrac{1}{2}\{1 + \langle \rho_{\nu_\mu} | T_f \cdot \mathcal{M}_{\mathcal{D}}(t_f, t_i) \cdot T_i^{-1} | \rho_{\nu_e} \rangle\}, \tag{5.6}$$

where

$$\mathcal{M}_D(t_f, t_i) = e^{-2\alpha t} \begin{pmatrix} 1 & & \\ & \exp\left(-2i\int_0^t d\tau \Omega_M(\tau)\right) & \\ & & \exp\left(2i\int_0^t d\tau \Omega_M(\tau)\right) \end{pmatrix} \tag{5.7}$$

while T_i and T_f are the matrices that diagonalize $\mathcal{H}$ at the initial time $t_i = 0$ and final time $t_f = t$; they can be written as in Eq. (5.4), with parameters ξ_i, ζ_i, ϕ_i and ξ_f, ζ_f, ϕ_f, respectively. Explicitly, one finds

$$\mathcal{P}_{\nu_e \to \nu_\mu}(t) = \frac{1}{2}\Bigg\{1 - e^{-2\alpha t}\Bigg[e^{\xi_f - \xi_i} \cos 2\phi_f \cos 2\phi_i + e^{-(\zeta_f - \zeta_i)} \sin 2\phi_f \sin 2\phi_i \times \cos\left(2\int_0^t d\tau\, \Omega_M(\tau)\right)\Bigg]\Bigg\}. \tag{5.8}$$

In absence of dissipation, $\xi_i = \zeta_i = \xi_f = \zeta_f = 0$, $\alpha = b = \beta = 0$, one recovers the familiar expression for the adiabatic transition probability.

When the adiabatic approximation ceases to be valid, the previous treatment needs to be generalized. Indeed, in this case, the neutrino state no longer remains in a specific eigenstate of $\mathcal{H}$ for the whole time evolution; rather, it can mix with the remaining eigenstates. In order to take into account this possibility, the expression for the transition probability (5.6) needs to be modified:

$$\mathcal{P}_{\nu_e \to \nu_\mu}(t) = \tfrac{1}{2}\{1 + \langle \rho_{\nu_\mu} | T_f \cdot \mathcal{M}_{\mathcal{D}}(t_f, t_c) \cdot \Delta \cdot \mathcal{M}_{\mathcal{D}}(t_c, t_i) \cdot T_i^{-1} | \rho_{\nu_e} \rangle\}, \tag{5.9}$$

where t_c is the time at which the neutrino crosses the critical region, while Δ is the mixing matrix that encodes the possible hopping between the instantaneous eigenstates of the effective Hamiltonian. For simplicity, we are not taking into account the possibility of hoppings induced by dissipative effects: They can be considered to be negligible with respect to the matter induced ones. As a consequence, Δ can be taken to be the most general 3×3 unitary matrix, which preserves appropriate consistent conditions: They assure the reality of the transition probability. Taking into account these conditions, Δ can be parametrized in terms of two complex numbers u and v, such that $|u|^2 + |v|^2 = 1$:

$$\Delta = \begin{pmatrix} |u|^2 - |v|^2 & \sqrt{2}\bar{u}v & \sqrt{2}u\bar{v} \\ -\sqrt{2}\bar{u}\bar{v} & \bar{u}^2 & -\bar{v}^2 \\ -\sqrt{2}uv & -v^2 & u^2 \end{pmatrix}. \tag{5.10}$$

The explicit expression for the probability in Eq. (5.9) is now rather involved; however, it simplifies when neglecting the fast oscillating terms:

$$\langle \mathcal{P}_{\nu_e \to \nu_\mu}(t) \rangle = \tfrac{1}{2}\{1 - e^{-2\alpha t} e^{\xi_f - \xi_i}(1 - 2|v|^2) \cos 2\phi_f \cos 2\phi_i\}. \tag{5.11}$$

In practical applications, the interesting case occurs when the neutrinos are generated in a medium with very large matter density, $e^{-\xi_i} \cos 2\phi_i \simeq -1$, while detected at a later time t in vacuum, $\xi_f \simeq 0$, $\phi_f = \theta$. In this case, one finds

$$\langle \mathcal{P}_{\nu_e \to \nu_\mu}(t) \rangle = \tfrac{1}{2}(1 - e^{-2\alpha t}) + e^{-2\alpha t}[\cos^2 \theta - |v|^2 \cos 2\theta]. \tag{5.12}$$

This is the most simple form that the transition probability formula takes in the presence of dissipative and matter effects: With respect to the familiar expression, it contains exponential damping factors. Taking into account that neutrinos are relativistic, the flight time between emission and detection is with very good approximation the same as the distance l between source and detector. One can then use Eq. (5.12) to derive a rough order of magnitude limits on the nonstandard parameter α. The best bounds are expected from solar neutrinos, where $1/l$ can be as low as 10^{-27} GeV.

VI. TRANSITION PROBABILITY IN VACUUM

One of the most interesting properties of the quantum dynamical semigroup approach to neutrino propagation is the possibility of probing the nature of the neutrinos by studying their oscillation pattern. Indeed, the transition probability $\mathcal{P}_{\nu_e \to \nu_\mu}$ in Eq. (3.13) explicitly depends on the extra phase φ, which can be nonvanishing for Majorana neutrinos. In this section we shall discuss to what extent the phase φ can be extracted from $\mathcal{P}_{\nu_e \to \nu_\mu}$. With in mind possible applications to atmospheric neutrinos, we shall limit our considerations to oscillations in vacuum, and present various explicit formulas for the transition probability in different approximations.

As a working assumption, let us first take the dissipative parameters c and β in Eq. (3.5) to be much smaller than the remaining constants. To lowest order, a closed form for the entries of the evolution matrix $\mathcal{M}(t)$ in Eq. (3.8) can then be obtained. Using the general formula (3.13), one finds

$$\mathcal{P}_{\nu_e \to \nu_\mu}(t) = \tfrac{1}{2}\Bigg\{ 1 - e^{-2\gamma t}\cos^2 2\theta - e^{-(a+\alpha)t}\sin^2 2\theta \times\left[\cos(2\Omega t) + \frac{|B|}{2\Omega}\sin(2\Omega t)\cos(\phi_B + 2\varphi)\right]\Bigg\}, \tag{6.1}$$

where $B = \alpha - a + 2ib \equiv |B| e^{i\phi_B}$ and $\Omega = \sqrt{\omega^2 - |B|^2/4}$. In this case the dependence on the phase φ is very mild, and cannot be extracted by studying $\mathcal{P}_{\nu_e \to \nu_\mu}$ alone: An independent determination of the combination B is necessary.

The situation is even worse when $\gamma = 0$; in this case, the inequalities (2.3) automatically guarantee $c = \beta = 0$ and further impose $b = 0$, $a = \alpha$. In this case, φ completely disappears from the expression of the transition probability, which reduces to [18]

$$\mathcal{P}_{\nu_e \to \nu_\mu}(t) = \frac{1}{2}\sin^2 2\theta [1 - e^{-2\alpha t}\cos(2\omega t)]. \tag{6.2}$$

In view of this, analysis of the experimental data based on Eq. (6.2) along the lines of Ref. [32] is totally insensitive to φ; a fit with more than one nonvanishing dissipative parameters is in general needed, although this condition is certainly not enough, as shown by Eq. (6.1).

In this respect, a more interesting situation occurs when $c = 0$ and $a = \alpha = \gamma$, as considered in the previous sections. All the entries of the evolution matrix $\mathcal{M}(t)$ are now nonvanishing, and the transition probability can be written as

$$\begin{aligned}\mathcal{P}_{\nu_e \to \nu_\mu}(t) = \frac{1}{2}\Bigg\{ 1 - e^{-2\alpha t}\Bigg[\left(1 + \frac{2\beta^2}{\Omega_0^2}\sin^2(\Omega_0 t)\right)\cos^2 2\theta \\ + \sin^2 2\theta\left(\cos(2\Omega_0 t) - \frac{2\beta^2}{\Omega_0^2}\sin^2(\Omega_0 t)\cos^2\varphi \right.\\ \left. - \frac{b}{\Omega_0}\sin(2\Omega_0 t)\sin 2\varphi\right) + \frac{2\beta}{\Omega_0}\cos 4\theta \sin(\Omega_0 t) \\ \times\left(\frac{b}{\Omega_0}\sin(\Omega_0 t)\cos\varphi - \cos(\Omega_0 t)\sin\varphi\right)\Bigg]\Bigg\},\end{aligned} \tag{6.3}$$

where $\Omega_0 = \sqrt{\omega^2 - b^2 - \beta^2}$ as before. In this case, the dependence of $\mathcal{P}_{\nu_e \to \nu_\mu}(t)$ on time is significally altered by the presence of a nonvanishing φ.

Atmospheric neutrino data are the most suitable for an experimental study of Eq. (6.3), since in this case the time dependence can actually be probed. Nevertheless, it should be stressed that the expression in Eq. (6.3) contains four additional parameters besides the standard ones, ω and θ, so that the fitting procedure might turn out to be difficult in practice. In order to simplify the analysis, one can further assume one of the two nonstandard parameters, b or β, to be zero, but not both: Here again, when α is the only nonvanishing dissipative parameter, the dependence on φ in $\mathcal{P}_{\nu_e \to \nu_\mu}(t)$ disappears. Despite these difficulties, the amount of data on atmospheric neutrinos is constantly growing, so that at least some information on the presence of a nonvanishing φ, together with some of the dissipative parameters, will surely be attainable in the near future.

VII. DISCUSSION

The study of open systems by means of quantum dynamical semigroups offers a physically consistent, general approach to the discussion of phenomena leading to irreversibility and dissipation. When this formalism is applied to the analysis of the propagation of neutrinos, both in vacuum and in matter, it gives precise predictions on the pattern of oscillation phenomena: The new, nonstandard effects manifest themselves through a set of phenomenological parameters, ω_1, ω_2, and ω_3 (the Hamiltonian ones), a, b, c, α, β, and γ (the purely dissipative ones). Their presence allows oscillating phenomena even for mass-degenerate neutrinos, accompanied by possible *CP*-violating effects. Further, these predictions can be experimentally probed. Indeed, fits of experimental data along the lines discussed in Ref. [32] can be repeated for the more complete expressions of the transition probability $\mathcal{P}_{\nu_e \to \nu_\mu}(t)$ presented in the previous sections.

For sure, the fitting procedure will be more difficult and uncertain than in the standard case, due to the presence of more unknown parameters. Furthermore, one has to take into account that the effects induced by the presence of the new parameters are expected to be small. By viewing the neutrinos as an open system in weak interaction with an environment generated by a fundamental ''stringy'' dynamics, the effects of irreversibility and dissipation can be roughly estimated to be proportional to the square of the average neutrino energy, divided by the characteristic energy scale of the environment. Assimilating this scale to the Planck mass produces estimates of order 10^{-27} GeV for solar neutrinos, while larger values are expected for more energetic neutrinos. Despite these difficulties, a new generation of dedicated neutrino experiments are currently collecting data or will shortly start construction, so that stringent bounds on the dissipative effects can surely be expected in the future.

The above estimate on the magnitude of the nonstandard, dissipative effects is based on very general and physically motivated considerations about open systems; therefore, it is rather robust and quite independent from the details of the microscopic, fundamental dynamics responsible for the interaction between the neutrino subsystem and the environment. Nevertheless, it has been questioned on the basis of a formal similarity of a particular, simplified version of the evolution equation (2.2) with those describing the phenomenon of the so-called dynamical reduction of the wave packet [48]. The analogy is rather superficial: The physical process leading to dissipation and the reduction process are quite distinct and act at different energy scales. Furthermore, as a more complete analysis would reveal, quantum dynamical semigroups generated by equations of the form (2.2) are unable to properly describe dynamical reduction processes [9].

A different criticism on the use of quantum dynamical semigroups for the description of dissipative effects advocates the use of nonlinear evolution equations [49]. Once more the general theory of open systems offers a clarifying discussion on this point (for further technical details, see the Appendix).

As pointed out in Sec. II, the dynamics of a small system $\mathcal{S}$ in interaction with a large environment $\mathcal{E}$ is in general very complex and cannot be described by means of evolution equations that are linear in time: Possible initial correlations and the continuous exchange of energy as well as entropy between $\mathcal{S}$ and $\mathcal{E}$ produces memory effects and nonlinear phenomena. Nevertheless, when the typical time scale in the evolution of the subsystem $\mathcal{S}$ is much larger than the characteristic time correlations in the environment, the subdynamics simplifies and a mathematically precise description in terms of quantum dynamical semigroups naturally emerges [1–3,21].

This limiting procedure is general and can be applied to all physical situations for which the interaction between $\mathcal{S}$ and $\mathcal{E}$ is weak and for not-too-short times, so that the nonlinear disturbances due to possible initial correlations have died out [4]. These are precisely the conditions that are expected to be fulfilled in the case of neutrino systems: The characteristic time correlations in the environment, induced by the fundamental (gravitational or stringy) dynamics, is certainly much smaller than the neutrino propagation time, while the interaction between neutrinos and environment is surely weak.

The description of neutrino propagation in terms of quantum dynamical semigroups automatically guarantees the fulfillment of basic physical requirements, as forward in time composition, entropy increase (irreversibility), and complete positivity. This is a clear advantage over alternative formulations. Based on ideas originally presented in Ref. [23], generalized dynamics for the neutrino system incorporating some of these properties have been discussed before (see Refs. [50–52]). However, those dynamics do not satisfy the condition of complete positivity; as already mentioned, this could lead to serious inconsistencies that can be avoided in all situations only by adopting evolutions of the form (2.2) [41].

Nonlinear dynamics in the description of the neutrino system naturally emerge when the requirement of weak coupling between neutrinos and environment is not satisfied. This typically happens in extreme conditions, as those found in the core of a supernova or the early universe; more conventional dissipative phenomena then arise due to the scattering and absorption processes in the medium [53–55]. In order to properly deal with these situations, a second-quantized, field-theoretical formalism has been constructed, using specific effective interaction Hamiltonians as starting point. Although derived using techniques similar to the ones described before, the resulting kinetic evolution equations are quite distinct from Eq. (2.2); they give rise to decoherence effects that modify the pattern of neutrino oscillations in a very different way with respect to the expressions discussed in the previous sections. Nevertheless, also in this case the condition of complete positivity needs to be satisfied for consistency, and this requirement might produce further constraints on the modified dynamics.

Decoherence effects in neutrino physics have been further discussed in connection with the uncertainties in the emission and detection processes [56]. By smearing the familiar expression for the transition probability over energy and time (or position) with an appropriate Gaussian distribution, an exponential damping factor is generated, so that the resulting expression for the averaged probability looks similar to the one presented in Eq. (6.2). The analogy is once more only superficial, since the transition probability (6.2) [and more generally the expression in Eq. (3.13)] has an explicit time (position) dependence that cannot be reproduced via a Gaussian average. Further, the physical mechanisms leading to the modified probability expressions are clearly different: the detector ''noise'' in one case, a fundamental dynamics in the other. In turn, this leads to a different dependence of the damping factors on the average neutrino energy.

As mentioned before, quantum dynamical semigroups can be employed to model a large variety of physical situations. It is not a surprise that they have been used to study the effects of density waves in the propagation of neutrinos in fluctuating media, in particular, in the interior of the sun; these phenomena are also described by equations of the form (2.2) and induce modifications on the neutrino oscillation probabilities [57]. However, it should be stressed that these density fluctuations have their origin in the dynamics of the sun and operate at energy scales quite different from the Planck mass; therefore, they can be easily isolated from the dissipative effects discussed before, which are not expected to be influenced by long-range phenomena.

As a final remark, let us point out that several unconventional phenomena affecting neutrino propagation have been discussed in the literature. They include neutrino decay, flavor-changing neutral currents, violation of Lorentz invariance, or the equivalent principle (e.g., see Refs. [58–64]). All these phenomena lead to modifications of the standard oscillation pattern; however, the resulting transition probability $\mathcal{P}_{\nu_e \to \nu_\mu}(t)$ has a dependence on time (or path length) and neutrino energy that differ from the one discussed in the previous sections.

Indeed, the dependence of the observable $\mathcal{P}_{\nu_e \to \nu_\mu}(t)$ on the phenomenological parameters a, b, c, α, β, and γ is very distinctive of the presence of dissipative phenomena and cannot be mimicked by other unconventional mechanisms. This further strengthens the possibility of identifying the dissipative contributions from the analysis of experimental data, quite independently from other effects.

APPENDIX: THE WEAK COUPLING LIMIT

For a neutrino system in interaction with an environment $\mathcal{E}$, the total Hamiltonian can be decomposed as in Eq. (2.4):

$$H_{\rm tot} = H \otimes \mathbf{1} + \mathbf{1} \otimes H_{\mathcal{E}} + gH', \tag{A1}$$

where, neglecting for simplicity matter effects, the system Hamiltonian H can be taken to be of the form

$$H = \begin{pmatrix} E - \omega_0 & 0 \\ 0 & E + \omega_0 \end{pmatrix}. \tag{A2}$$

Assuming no initial correlations between the two systems, the evolution in time of the neutrino state $\rho(t)$ follows the general rule (2.5):

$$\rho \mapsto \rho(t) = {\rm Tr}_{\mathcal{E}}[e^{-iH_{\rm tot}t}(\rho \otimes \rho_{\mathcal{E}}) e^{iH_{\rm tot}t}], \tag{A3}$$

This evolution map is in general very complicated, developing irreversibility and memory effects. However, it simplifies when the interaction between the neutrino subsystem and the environment is weak.

There are essentially two different ways of implementing in practice this condition [1–3]: They correspond to the two ways of making the ratio $\tau/\tau_{\mathcal{E}}$ large. Here τ is the typical variation time of $\rho(t)$, while $\tau_{\mathcal{E}}$ represents the typical decay time of the correlations in the environment. Only when $\tau \gg \tau_{\mathcal{E}}$, one expects the memory effects in Eq. (A2) to be negligible, and a local in time evolution for the state $\rho(t)$ to be valid.

When $\tau_{\mathcal{E}}$ becomes small, while τ remains finite, one speaks of "singular coupling limit," since the typical time correlations of the environment approach a δ function. In the other case, it is τ that becomes large, while $\tau_{\mathcal{E}}$ remains finite: One then works in the framework of the "weak coupling limit." In practice, this is obtained by suitably rescaling the time variable, $t \to t/g^2$, and by sending the coupling constant g to zero (van Hove limit) [1–3].

The choice between these two limiting procedures clearly depends on specific physical considerations about the system under study. In the case of the neutrino system, both limits appear physically acceptable. Since the singular coupling limit has been presented elsewhere [21], in the following we shall concentrate on the discussion of the weak coupling limit. In this case, following the steps presented in Ref. [21], from the finite time evolution (A3) one can derive a differential equation for $\rho(t)$ local in time; it is of the form (2.2a),

$$\frac{\partial \rho(t)}{\partial t} = -iH\rho(t) + i\rho(t)H + \mathcal{L}[\rho(t)], \tag{A4}$$

with

$$\mathcal{L}[\rho] = -\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} ds \int_0^\infty dt \times {\rm Tr}_{\mathcal{E}}\{e^{iH_{\rm tot}s}[e^{iH_0 t} H' e^{-iH_0 t}, [H', \rho \otimes \rho_{\mathcal{E}}]] e^{-iH_{\rm tot}s}\}, \tag{A5}$$

where H_0 represents the limit of $H_{\rm tot}$ for a vanishing coupling constant g.

The general form of the additional term $\mathcal{L}[\rho]$ in Eq. (A5) does not actually depend very much on the details of the environment dynamics; an effective description that takes into account its most fundamental characteristic properties is enough to allow an explicit evaluation of the integrals in Eq. (A5). Following the idea that the dissipative effects are low-energy phenomena that originate from the fundamental gravitational or stringy dynamics at some large scale M_F, we shall model the environment as a gas of quanta, obeying infinite statistics, in thermodynamic equilibrium at inverse temperature $\beta_F = 1/M_F$.[5]

Further, taking into account that the interaction between the neutrino system and the environment is weak, we shall assume the interaction Hamiltonian H' to be linear both in the neutrino and the environment dynamical variables:

$$H' = \sum_{\mu=0}^{3} \sigma_\mu \otimes B_\mu; \tag{A6}$$

an explicit expression for the environmental operators B_μ will be discussed below.

In order to proceed further, it is convenient to introduce a spectral decomposition, and use the auxiliary matrices $\sigma_\mu^{(\lambda)}$, $\lambda = -1,0,1$ [1],

$$\sigma_\mu^{(0)} = P_1 \sigma_\mu P_1 + P_2 \sigma_\mu P_2, \quad \sigma_\mu^{(+)} = P_1 \sigma_\mu P_2, \quad \sigma_\mu^{(-)} = P_2 \sigma_\mu P_1, \tag{A7}$$

with

[5] For a motivation of this choice in terms of the dynamics of extended objects ($D0$ branes), see Refs. [21] and [65–67].

$$P_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad P_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{A8}$$

Then, the limit in Eq. (A5) can be explicitly performed and the result expressed in terms of the following 4×4 Hermitian matrices:

$$a_{\mu\nu}^{(\lambda)}=g^2\int_{-\infty}^{\infty}dt\, e^{-2i\lambda\omega_0 t}\langle B_\mu(t)B_\nu\rangle, \tag{A9a}$$

$$b_{\mu\nu}^{(\lambda)}=ig^2\left\{\int_0^{\infty}dt\, e^{-2i\lambda\omega_0 t}\langle B_\mu(t)B_\nu\rangle - \int_0^{\infty}dt\, e^{2i\lambda\omega_0 t}\langle B_\mu B_\nu(t)\rangle\right\}, \tag{A9b}$$

involving thermal correlations of the environment operators,

$$\langle B_\mu(t)B_\nu\rangle=\mathrm{Tr}_{\mathcal{E}}[B_\mu(t)B_\nu\rho_{\mathcal{E}}], \quad \rho_{\mathcal{E}}=\frac{e^{-\beta_F H_{\mathcal{E}}}}{\mathrm{Tr}_{\mathcal{E}}(e^{-\beta_F H_{\mathcal{E}}})}. \tag{A10}$$

Explicitly, one finds

$$\mathcal{L}[\rho]=i[\rho,\tilde{H}]+L[\rho], \tag{A11}$$

where

$$L[\rho]=\tfrac{1}{2}\sum_{\lambda\in\{0,\pm1\}}\left\{\sum_{i,j=1}^{3}a_{ij}^{(\lambda)}[2\sigma_j^{(\lambda)}\rho\sigma_i^{(\lambda)}-\sigma_i^{(\lambda)}\sigma_j^{(\lambda)}\rho - \rho\sigma_i^{(\lambda)}\sigma_j^{(\lambda)}]\right\}, \tag{A12}$$

and

$$\tilde{H}=\tfrac{1}{2}\sum_{\lambda\in\{0,\pm1\}}\left\{\sum_{\mu,\nu=0}^{3}b_{\mu\nu}^{(\lambda)}\sigma_\mu^{(\lambda)}\sigma_\nu^{(\lambda)}\right\}+i\sum_{i=1}^{3}(a_{0i}^{(0)}-a_{i0}^{(0)})\sigma_i. \tag{A13}$$

The first term on the RHS of (A11) is of Hamiltonian form. This is a general feature of the reduced dynamics: Even in the absence of an initial system dynamics, a nontrivial Hamiltonian contribution is always generated by the dissipative piece, Eq. (A5). This mechanism has deep consequences in neutrino physics: As remarked in the text, it allows oscillation phenomena even in case of initially massless neutrinos.

Further information on the coefficient matrices $a_{\mu\nu}^{(\lambda)}$ and $b_{\mu\nu}^{(\lambda)}$ in Eq. (A9) can be obtained by studying the behavior of the environment correlations in Eq. (A10). The operators B_μ can be taken to be a general linear expression in the environment variables; these are the creation, $A_a^\dagger(k)$, and annihilation, $A_a(k)$, operators for the quanta representing the environment modes, living in an abstract n-dimensional space:

$$B_\mu(t)=\frac{1}{M_F^{(n-4)/2}}\sum_{a,b}\int\frac{d^{n-1}k}{[2(2\pi)^{n-1}\varepsilon(k)]^{1/2}}\times f^a(k)\chi_\mu^{ab}[A_b(k)e^{-i\varepsilon(k)t}+A_b^\dagger(k)e^{i\varepsilon(k)t}]. \tag{A14}$$

The coefficients χ_μ^{ab} "embed" the n-dimensional environment modes into the effective two-dimensional neutrino Hilbert space, while $f^a(k)$ are appropriate test functions necessary to make the operator B_μ and its correlations well-defined; it can be taken to be of the form $(|k|/M_F)^{m/2}g^a(k)$, for some positive integer m, with $g^a(k)$ of Gaussian form. For sake of definiteness, in the following we shall use $g^a(k)=g^a(\Omega)e^{-\eta^2\varepsilon^2/2}$, with $g^a(\Omega)$ depending only on the angle variables and η a real constant. The function $\varepsilon(k)$ gives the dispersion relation obeyed by the environment modes; for simplicity we shall adopt an ultrarelativistic law: $\varepsilon(k)=|k|\equiv\varepsilon$. The powers of M_F, characterizing the energy scale of the environment, are necessary to give B_μ the right dimension of energy.

We assume an indefinite statistics for these modes, so that the creation and annihilation operators obey generalized commutation relations:

$$A_a(k)A_b^\dagger(k')-qA_b^\dagger(k')A_a(k)=\delta_{ab}\delta^{(n-1)}(k-k'); \tag{A15}$$

the real parameter q determines the mode statistics: The case $q=1$ corresponds to standard bosons, while for $q=0$ one obtains the degenerate algebra discussed in Refs. [65–67], in connection with $D0$ branes and black holes. Without loss of generality, we shall assume $q<1$. Furthermore, the single-mode Hamiltonian can be taken to be proportional to the corresponding number operator, so that the total environment Hamiltonian $H_{\mathcal{E}}$ satisfies the relation

$$[H_{\mathcal{E}},A_a^\dagger(k)]=\varepsilon(k)A_a^\dagger(k), \quad [H_{\mathcal{E}},A_a(k)]=-\varepsilon(k)A_a(k), \tag{A16}$$

implicit in the time dependence of Eq. (A14).

The thermal correlations involved in the definitions in Eq. (A9) can now be readily computed. For instance, one explicitly gets

$$a_{ij}^{(\lambda)}=\frac{1}{2M_F^{m+n-4}}\int_{-\infty}^{\infty}dt\, e^{-2i\lambda\omega_0 t}\int_0^{\infty}d\varepsilon\, \varepsilon^{m+n-3}[X_{ij}(\varepsilon)e^{i\varepsilon t}+X_{ji}(\varepsilon)e^{-i\varepsilon(t+i\beta_F)}]\frac{1}{e^{\beta_F\varepsilon}-q}, \tag{A17}$$

where

$$X_{ij}(\varepsilon)=\sum_{a,b,c}\left[\int\frac{d\Omega_{n-1}}{(2\pi)^{n-1}}g^a(k)\chi_i^{ab}\chi_j^{cb}g^c(k)\right]\equiv e^{-\eta^2\varepsilon^2}X_{ij}(0) \tag{A18}$$

involves the integration over the angle variables; notice that $X_{ij}(\varepsilon)$ is a real, symmetric matrix.

This matrix is not generic: It turns out that in order to satisfy the condition of entropy increase for finite β_F, $X_{ij}(\varepsilon)$ must vanish for $i,j=1,2$. With this choice, one finds that the nonvanishing contributions to $L[\rho]$ in Eq. (A12) can come only from the coefficient $a_{33}^{(0)}$, provided $m=3-n$. Explicitly, one obtains

$$a_{33}^{(0)}=\frac{\pi}{1-q}g^2M_FX_{33}(0). \tag{A19}$$

The dimensionless coupling constant g should be expressible in terms of the relevant energy scales, i.e., the average neutrino energy E and the mass M_F characteristic of the environment. Since g is small, it must be at most of order E/M_F. As a consequence, it turns out that the dissipative parameter $a_{33}^{(0)}$ must scale as E^2/M_F. As mentioned in the text, this is a general prediction of the open system approach to dissipation.

Using the expansion $\rho=\Sigma_\mu\rho_\mu\sigma_\mu/2$ as in the text, one immediately finds that the dissipative contribution $L[\rho]$ in Eq. (A12) is of the form (3.3), with $a=\alpha=a_{33}^{(0)}$ and $b=c=\beta=\gamma=0$. In the weak coupling limit a special form of the matrix (3.3) is then selected: It is expressible in terms of only one nonstandard parameter. This situation does not hold any more in the case of the singular coupling limit. In that case, all six parameters a, b, c, α, β, and γ are in general nonvanishing (for details, see Ref. [21]).

In a similar way, the Hamiltonian contribution in Eq. (A13) can be explicitly computed. Taking into account the results obtained in the evaluation of the coefficients $a_{ij}^{(\lambda)}$, one sees that the 2×2 matrix $\tilde{H}$ becomes diagonal:

$$\tilde{H}=b_{03}^{(0)}\sigma_3, \tag{A20}$$

where

$$b_{03}^{(0)}=g^2M_FX_{03}(0)G(\beta_F/\eta), \tag{A21}$$

and

$$G(\beta_F/\eta)=\int_0^\infty dt\int_0^\infty d\varepsilon\,\sin\varepsilon t\,e^{-\varepsilon^2\eta^2}\left(\frac{1-e^{-\beta_F\varepsilon}}{1-qe^{-\beta_F\varepsilon}}\right). \tag{A22}$$

In the case of infinite statistics, $q=0$, the function G can be explicitly evaluated in terms of generalized hypergeometric functions:

$$G(x)=\frac{\sqrt{\pi}}{2}x\;{}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{x^2}{4}\right)-\frac{x^2}{4}\;{}_2F_2\left(1,1;\frac{3}{2},2;\frac{x^2}{4}\right). \tag{A23}$$

The operator in Eq. (A20) contributes via the parameter $\omega_3=b_{03}^{(0)}$ to the effective Hamiltonian $H_{\rm eff}$ in Eq. (3.1): Even in the absence of the standard piece ω_0, the quantity ω_3 would still generate a level splitting between the two neutrino mass eigenstates, making possible oscillation phenomena.

As a further remark, notice that although both generated via the interaction with the environment, the magnitude of the Hamiltonian contribution ω_3 could differ from the dissipative one in Eq. (A19), since their ratio involves the function G. Although in a different context, this phenomenon has also been observed in Ref. [57].

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