

“Physical process version” of the first law and the generalized second law for charged and rotating black holes

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We investigate both the “physical process” version of the first law and the generalized second law of black hole thermodynamics for charged and rotating black holes. We begin by deriving general formulas for the first order variation in the Arnowitt-Deser-Misner mass and angular momentum for linear perturbations off a stationary, electrovac background in terms of the perturbed nonelectromagnetic stress-energy δT_{ab} and the perturbed charge current density δj^a . Using these formulas, we prove the “physical process version” of the first law for charged, stationary black holes. We then investigate the generalized second law of thermodynamics (GSL) for charged, stationary black holes for processes in which a box containing charged matter is lowered toward the black hole and then released (at which point the box and its contents fall into the black hole and/or thermalize with the “thermal atmosphere” surrounding the black hole). Assuming that the thermal atmosphere admits a local, thermodynamic description with respect to observers following orbits of the horizon Killing field, and assuming that the combined black-hole/thermal atmosphere system is in a state of maximum entropy at fixed mass, angular momentum, and charge, we show that the total generalized entropy cannot decrease during the lowering process or in the “release process.” Consequently, the GSL always holds in such processes. No entropy bounds on matter are assumed to hold in any of our arguments.

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I. INTRODUCTION

The close mathematical and physical connection between the laws of black hole physics and the laws of thermodynamics provides the main foundation for ideas and speculations on the nature of quantum gravity in the strong field regime. Many aspects of black hole thermodynamics are on a completely firm foundation, such as the classical laws of black hole mechanics and the fact that black holes radiate via the Hawking process as perfect black bodies (of finite size) at temperature

$$T_H = \frac{\kappa}{2\pi} \quad (1)$$

where κ denotes the surface gravity of the black hole (see, e.g., [1] for a recent review). Nevertheless, there remain some unresolved and/or controversial issues in black hole thermodynamics.

One relatively minor unresolved issue concerns the “physical process version” of the first law of classical black hole mechanics for charged black holes. Consider a linear perturbation of a stationary, electrovac black hole corresponding to taking one to another stationary, electrovac black hole. Then, as originally shown by Bardeen, Carter, and Hawking [2] (see [3] for a generalized version) the first order variations of the area A , mass M , angular momentum J , and charge Q are related by

$$\frac{1}{8\pi} \kappa \delta A = \delta M - \Omega_H \delta J - \Phi_{\text{bh}} \delta Q \quad (2)$$

where Ω_H denotes the angular velocity of the horizon and Φ_{bh} denotes the electrostatic potential of the horizon (i.e., $\Phi_{\text{bh}} = -A_a \xi^a$, where ξ^a is the horizon Killing field and A_a is the vector potential). However, it also is possible to consider a “physical process” wherein some charged matter is thrown

into an initially stationary, electrovac black hole. Assuming that the black hole eventually settles down to a final stationary state, one may calculate the change in black hole area, δA , using the Raychaudhuri equation and compare it with δM , δJ , and δQ . If Eq. (2) were to fail, this would give rise to an inconsistency with the assumption that the black hole settles down to a final stationary state, and would thereby provide strong evidence against cosmic censorship. Conversely, a proof of the “physical process” version of the first law would provide support for cosmic censorship.

A proof of the “physical process” version of the first law for uncharged black holes was given in [4]. However, some difficulties arise in extending this proof to the charged case.¹ One of the purposes of this paper is to remedy these difficulties by showing that Eq. (2) holds for all physical processes.

A crucial issue in black hole thermodynamics is the validity of the generalized second law (GSL), which states that the total generalized entropy $S' \equiv S + S_{\text{bh}}$ never decreases [5], where S is the ordinary entropy of matter outside the black hole and, in general relativity, $S_{\text{bh}} = \frac{1}{4}A$. Early arguments by Bekenstein for the validity of this law in quasi-static lowering processes required the assumption that ordinary matter must obey an entropy bound of the form [6]

$$S \leq 2\pi ER \quad (3)$$

in order to prevent the box from being lowered too close to the black hole. An alternative resolution not requiring any entropy bounds on matter was given by Unruh and Wald [7], taking into account the quantum buoyancy force of the thermal atmosphere surrounding the black hole. This analysis has been criticized by Bekenstein on a variety of grounds

¹We are indebted to A. Ashtekar for pointing out these difficulties to us.

[8–10]; see [11] and [12] for responses to [8] and [9]. Recently, it has been argued that even stronger entropy bounds than (3) are needed for charged and rotating black holes [13–15]. These arguments have been countered for charged black holes by Shimomura and Mukohyama [16].

In view of the above situation, it seems worthwhile to give a new, general analysis of the validity of the GSL in quasi-static lowering processes that is applicable to charged and rotating black holes and invokes no model dependent assumptions concerning the thermal atmosphere and the contents of the box, or assumptions about the size and shape of the box (other than that it is much smaller than the black hole but large enough that a thermodynamic treatment of the thermal atmosphere is adequate). In this paper we shall give such an analysis. Our key assumptions are as follows:

(i) The thermal atmosphere admits a suitable local thermodynamic description with respect to observers following orbits of the horizon Killing field, ξ^a . Furthermore, the thermal atmosphere is in thermal equilibrium with itself. More precisely, we cannot increase the entropy of the thermal atmosphere by any rearrangement of it that keeps fixed its total mass, angular momentum, and charge, as well as other conserved quantities, such as the number of particles of a given species.

(ii) The thermal atmosphere is in thermal equilibrium with the black hole at temperature Eq. (1). More precisely, we cannot increase the total generalized entropy of the black hole/thermal atmosphere system by any rearrangement that keeps fixed the total mass, angular momentum, and charge of the total system.

We consider processes in which a box containing arbitrary matter and charge is quasistatically lowered toward the black hole and then “released,” so that the box or its contents are dropped into the black hole and/or allowed to thermalize with the thermal atmosphere. We will show that if the contents of the box are in thermal equilibrium, no decrease in the total generalized entropy can occur during the “lowering phase” of a quasistatic process. However, in the “release phase,” the total mass, angular momentum, and charge do not change. Consequently, since the black-hole–thermal atmosphere system is assumed to have maximum generalized entropy at fixed mass, angular momentum, and charge, the generalized entropy cannot decrease in the “release phase” either.

A key ingredient in our analysis of both the “physical process” version of the first law and the GSL is a general formula for the variation of the Arnowitt-Deser-Misner (ADM) mass, δM , and angular momentum, δJ , for perturbations of stationary or, respectively, axisymmetric electrovac spacetimes. In Sec. II, we will prove that these quantities are given by²

$$\delta M = - \int_{\Sigma} \epsilon_{dabc} (t^e \delta T^d_e + A_e t^e \delta j^d) + \int_{\partial \Sigma} (\delta Q[t] - t \cdot \Theta) \quad (4)$$

²Note that for perturbations of Minkowski spacetime (with Σ taken to be a slice so that $\partial \Sigma$ is empty), these equations reduce to the frequently used—but seldom, if ever, derived—formulas $\delta M = - \int_{\Sigma} \epsilon_{dabc} t^e \delta T^d_e$ and $\delta J = \int_{\Sigma} \epsilon_{dabc} \varphi^e \delta T^d_e$.

$$\delta J = \int_{\Sigma} \epsilon_{dabc} (\varphi^e \delta T^d_e + A_e \varphi^e \delta j^d) - \int_{\partial \Sigma} (\delta Q[\varphi] - \varphi \cdot \Theta). \quad (5)$$

Here Σ is an arbitrary asymptotically flat hypersurface, possibly possessing an inner boundary $\partial \Sigma$ (which may be—but need not be—the horizon of a black hole), δT_{ab} denotes the perturbed *non-electromagnetic* stress-energy tensor, δj^a denotes the perturbed electromagnetic charge-current vector, and A_a denotes the vector potential of the background in a gauge compatible with the symmetries [i.e., $\mathcal{L}_{\xi} A_a = 0$ in the stationary case, Eq.(4), and $\mathcal{L}_{\varphi} A_a = 0$ in the axisymmetric case, Eq.(5)]. The quantities Q and Θ are given by Eqs. (27) and (21)–(22) of Sec. II below.

In Sec. III, we will give a proof of the physical process version of the first law based on the above formulas. In Sec. IV, we will establish properties of the thermal atmosphere around a black hole that follow from the assumptions stated above. The process of quasi-statically lowering a box filled with matter towards a black hole and then releasing it will be considered in Sec. V, and it will be shown that the GSL holds in such processes. Our analysis of the lowering process is compatible with (i.e., it does not conflict with) the recent analysis of Shimomura and Mukohyama [16] for charged, nonrotating black holes, but some of our arguments are quite different from theirs, and we also clarify and generalize some aspects of their derivation.³ We make some concluding remarks in Sec. VI. In particular, we give an independent argument that if the GSL could be violated in a quasi-static lowering and release process involving a black hole, then there should be a corresponding process involving a self-gravitating system that does not contain a black hole in which the ordinary second law would be violated.⁴ Finally, in the Appendix we give a general derivation of the force needed to hold in place a box containing charged matter in a stationary (but not necessarily static) spacetime.

II. FIRST ORDER VARIATION OF MASS AND ANGULAR MOMENTUM

In this section, we first consider the general issue of calculating the first order variation of conserved quantities in a diffeomorphism covariant theory of gravity in the case where the first order perturbation is not required to satisfy the source free equations (except near infinity). We will then

³In particular, in [16] the formula for the gravitational force on the box is not derived, and it is unclear at certain points whether their energy density, ρ , includes (or should include) the electromagnetic interaction energy of the charged matter with the background electromagnetic field. Also, a proper justification for setting the chemical potential, μ , to zero on the horizon of the black hole was not given.

⁴This thereby provides a response to [10] by showing that if the considerations of that paper could lead to a violation of the GSL, then they also should give rise to a violation of the ordinary second law.

specialize to the Einstein-Maxwell case and derive formulas (4) and (5) above.

Consider a diffeomorphism covariant theory in n -dimensions derived from a Lagrangian L , where the dynamical fields consist of a Lorentz signature metric g_{ab} and other fields ψ . We will follow the notational conventions of [3], and, in particular, we will collectively refer to (g_{ab}, ψ) as ϕ and will use boldface letters to denote differential form. The first order variation of the Lagrangian can always be expressed in the form

$$\delta L = E(\phi) \delta\phi + d\Theta(\phi, \delta\phi) \quad (6)$$

where $E(\phi)$ is locally constructed out of ϕ and its derivatives and Θ is locally constructed out of ϕ , $\delta\phi$ and their derivatives. The equations of motion then can be read off as

$$E(\phi) = 0. \quad (7)$$

The symplectic current $(n-1)$ -form ω is defined by

$$\omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\Theta(\phi, \delta_2\phi) - \delta_2\Theta(\phi, \delta_1\phi). \quad (8)$$

Let ξ^a be any smooth vector field on the spacetime. We associate to ξ^a and ϕ a Noether current $(n-1)$ -form, defined by

$$\mathcal{J} = \Theta(\phi, \mathcal{L}_\xi\phi) - \xi \cdot L \quad (9)$$

where “ \cdot ” denotes contraction of the vector field ξ^a into the first index of the differential form L . A simple calculation yields

$$d\mathcal{J} = -E(\phi) \mathcal{L}_\xi\phi. \quad (10)$$

It was proven in the Appendix of [17] that there exists an $(n-2)$ -form Q (called the *Noether charge*), which is locally constructed from ϕ , ξ^a and their derivatives, such that

$$\mathcal{J}[\xi] = dQ[\xi] + \xi^a C_a \quad (11)$$

where C_a is an $(n-1)$ -form (with an extra dual vector index) which is locally constructed out of the dynamical fields and is such that $C_a = 0$ when the equations of motion are satisfied.

Now suppose that the spacetime satisfies asymptotic conditions at infinity corresponding to “case I” of [18] and that ξ^a is an asymptotic symmetry. (“case I” of [18] is the case where a true Hamiltonian corresponding to every asymptotic symmetry exists, thereby giving rise to a conserved quantity, H_ξ , associated with ξ^a . It includes the case of spacetimes that are asymptotically flat at spatial infinity in general relativity.) Let $\delta\phi$ satisfy the linearized equations of motion, $\delta E(\phi) = 0$, in a neighborhood of infinity, but not necessarily throughout the spacetime. Then the variation of the conserved quantity, δH_ξ associated with ξ^a is given by [18]

$$\delta H_\xi = \int_\infty (\delta Q[\xi] - \xi \cdot \Theta). \quad (12)$$

Here by “ \int_∞ ” we mean the following: Let Σ be a hypersurface in M that extends smoothly to the boundary representing infinity in the unphysical spacetime. We perform the integral of Eq. (12) over an $(n-2)$ -surface in Σ and then take the limit as this $(n-2)$ -surface goes to infinity along Σ ; see [18] for further details.

Using Stokes’ theorem, we may re-write Eq. (12) as

$$\delta H_\xi = \int_\Sigma [\delta dQ[\xi] - d(\xi \cdot \Theta)] + \int_{\partial\Sigma} (\delta Q[\xi] - \xi \cdot \Theta) \quad (13)$$

where $\partial\Sigma$ denotes any “interior boundary” of Σ (which would be empty if Σ is a slice and there are no other asymptotic regions, but we keep this term since we may wish to terminate Σ at, e.g., the event horizon of a black hole). Using the identity [3]

$$\delta \mathcal{J} = \omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) + d(\xi \cdot \Theta) \quad (14)$$

we may eliminate the term $d(\xi \cdot \Theta)$ from Eq. (13) in favor of $\delta \mathcal{J}$ and ω .

We now restrict consideration to the case where ξ^a is a Killing field of the background spacetime and is also a symmetry of any background matter fields ψ . Then $\omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) = 0$, so Eq. (13) becomes

$$\begin{aligned} \delta H_\xi &= \int_\Sigma (\delta dQ[\xi] - \delta \mathcal{J}[\xi]) + \int_{\partial\Sigma} (\delta Q[\xi] - \xi \cdot \Theta) \\ &= - \int_\Sigma \xi^a \delta C_a + \int_{\partial\Sigma} (\delta Q[\xi] - \xi \cdot \Theta) \end{aligned} \quad (15)$$

where Eq. (11) was used in the last step. It is worth noting that for an arbitrary perturbation, $\delta\phi$ —i.e., $\delta\phi$ need not satisfy the linearized field equations or have any symmetries—of a solution, ϕ , of the equations of motion $E(\phi) = 0$ also satisfying $\mathcal{L}_\xi\phi = 0$, we have, from Eq. (11),

$$d(\xi^a \delta C_a) = d\delta \mathcal{J}[\xi] - d^2 Q[\xi] = 0 \quad (16)$$

where the variation of Eq. (10) was used in the last step, together with the fact that ϕ satisfies both $E(\phi) = 0$ and $\mathcal{L}_\xi\phi = 0$. Thus, provided only that ϕ satisfies the equations of motion and $\mathcal{L}_\xi\phi = 0$, the current

$$\alpha^a = -\frac{1}{3!} \epsilon^{abcd} \delta C_{bcde} \xi^e \quad (17)$$

is always conserved, $\nabla_a \alpha^a = 0$, where ϵ_{abcd} is the metric compatible volume element of the background spacetime.

Equation (15) is our desired general formula for the first order variation of conserved quantities. It holds for an arbitrary diffeomorphism covariant theory of gravity derived from a Lagrangian with an asymptotic region satisfying the conditions of “case I” of [18], provided only that ξ^a is a symmetry of the background spacetime (i.e., $\mathcal{L}_\xi\phi = 0$) and that $\delta\phi$ satisfies the source-free linearized equations of motion near infinity. Note that if $\delta\phi$ satisfies the linearized equations of motion throughout the spacetime, then $\delta C_a = 0$ and the integral over Σ in Eq. (15) vanishes. If, in addition, Σ has no interior boundary—i.e., if $\partial\Sigma = \emptyset$ —then Eq. (15)

reduces to simply $\delta H_\xi = 0$ (see [19]). On the other hand, if $\delta\phi$ satisfies the linearized equations of motion throughout the spacetime but $\partial\Sigma$ is the bifurcation surface of the event horizon of a stationary black hole, then Eq. (15) yields the general form of the first law of black hole mechanics when ξ^a is chosen to be the horizon Killing field [19,3]. Our Eq. (15) generalizes these results by allowing $\delta\phi$ to fail to satisfy the linearized equations except near infinity (i.e., by allowing for the presence of sources for Einstein's equation as well as the equations for the matter fields), as well as by allowing $\partial\Sigma$ to be arbitrary.

We now specialize to Einstein-Maxwell theory, in order to obtain explicit formulas for the variation of mass and angular momentum in that case. The Einstein-Maxwell Lagrangian is

$$\mathbf{L} = \frac{1}{16\pi} (\epsilon R - \epsilon g^{ac} g^{bd} F_{ab} F_{cd}) \quad (18)$$

where ϵ is the volume element associated with the metric. Computing the first order variation of \mathbf{L} , we obtain

$$\delta\mathbf{L} = \frac{1}{16\pi} \epsilon (-G^{ab} + 8\pi T_{\text{EM}}^{ab}) \delta g_{ab} + \frac{1}{4\pi} \epsilon (\nabla_a F^{ab}) \delta A_b + d\Theta \quad (19)$$

where T_{EM}^{ab} is the stress energy tensor of the electromagnetic field

$$(T_{\text{EM}})_{ab} = \frac{1}{4\pi} \left\{ F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{de} F^{de} \right\} \quad (20)$$

and

$$\Theta_{abc}(\phi, \delta\phi) = \frac{1}{16\pi} \epsilon_{dabc} v^d \quad (21)$$

with

$$v_d = \nabla^b \delta g_{db} - g^{ce} \nabla_d \delta g_{ce} - 4F_d{}^b \delta A_b. \quad (22)$$

The (source free) Einstein-Maxwell equations can then be read off from Eq. (19):

$$G^{ab} - 8\pi T_{\text{EM}}^{ab} = 0 \quad (23)$$

$$\nabla_a F^{ab} = 0. \quad (24)$$

From Eq. (9), we find that the Noether current 3-form with respect to ξ^a is given by

$$\begin{aligned} \mathcal{J}_{abc} = & dQ_{abc}^{\text{GR}} + \frac{1}{16\pi} \epsilon_{dabc} (2G^d{}_e \xi^e + \xi^d F_{fg} F^{fg}) \\ & - \frac{1}{4\pi} \epsilon_{dabc} F^{df} (\xi^e \nabla_e A_f + A_e \nabla_f \xi^e) \end{aligned} \quad (25)$$

where

$$Q_{ab}^{\text{GR}} = -\frac{1}{16\pi} \epsilon_{abcd} \nabla^c \xi^d.$$

Writing $\nabla_e A_b = F_{eb} + \nabla_b A_e$ in the last term of Eq.(25) and differentiating by parts, we obtain

$$\begin{aligned} \mathcal{J}_{abc} = & dQ_{abc}^{\text{GR}} + \frac{1}{8\pi} \epsilon_{dabc} (G^{de} - 8\pi T_{\text{EM}}^{de}) \xi_e \\ & - \frac{1}{4\pi} \nabla_g (\epsilon_{dabc} F^{dg} A_e \xi^e) + \frac{1}{4\pi} \epsilon_{dabc} A_e \xi^e \nabla_f F^{df} \\ = & (dQ)_{abc} + \frac{1}{8\pi} \epsilon_{dabc} (G^{de} - 8\pi T_{\text{EM}}^{de}) \xi_e \\ & + \frac{1}{4\pi} \epsilon_{dabc} A_e \xi^e \nabla_f F^{df} \end{aligned} \quad (26)$$

where

$$Q_{ab} = -\frac{1}{16\pi} \epsilon_{abcd} \nabla^c \xi^d - \frac{1}{8\pi} \epsilon_{abcd} F^{cd} A_e \xi^e. \quad (27)$$

Equation (26) is precisely of the required form (11), so we may identify Q_{ab} as the Noether charge, and read off \mathbf{C}_a to be given by

$$C_{bcda} = \frac{1}{8\pi} \epsilon_{ebcd} (G^e{}_a - 8\pi T^{\text{EM}e}{}_a) + \frac{1}{4\pi} \epsilon_{ebcd} A_a \nabla_f F^{ef}. \quad (28)$$

Clearly, as is required, we have $\mathbf{C}_a = 0$ whenever the source-free Einstein-Maxwell equations (23) and (24) hold. When the source-free Einstein-Maxwell equations do not hold, we write

$$8\pi T^{de} = G^{de} - 8\pi T_{\text{EM}}^{de} \quad (29)$$

$$4\pi j^d = \nabla_b F^{db}. \quad (30)$$

Then T_{ab} has the interpretation of being the *non-electromagnetic* contribution to the stress energy tensor (i.e., $T^{de} = T_{\text{total}}^{de} - T_{\text{EM}}^{de}$) and j^a is the charge-current of the Maxwell sources. In terms of these sources, we have

$$C_{bcda} = \epsilon_{ebcd} (T^e{}_a + j^e A_a). \quad (31)$$

Now, let (g_{ab}, A_a) be a solution of the source-free Einstein-Maxwell equations (23) and (24), and let $(\delta g_{ab}, \delta A_a)$ be a linearized perturbation which satisfies the linearized Einstein-Maxwell equations with sources δT_{ab} and δj^a . Then, we have

$$\delta C_{bcda} = \epsilon_{ebcd} (\delta T^e{}_a + A_a \delta j^e). \quad (32)$$

Substituting Eq. (32) into Eq. (15), we obtain the explicit formula

$$\delta H_\xi = - \int_\Sigma \epsilon_{ebcd} (\xi^a \delta T^e{}_a + \xi^a A_a \delta j^e) + \int_{\partial\Sigma} (\delta Q[\xi] - \xi \cdot \Theta) \quad (33)$$

where Q_{ab} is given by Eq. (27) and Θ_{abc} is given by Eqs. (21) and (22). Finally, choosing ξ^a to be an asymptotic time

translation, t^a , and writing $M = H_t$, we obtain Eq. (4) above, whereas choosing ξ^a to be an asymptotic rotation, φ^a , and writing $J = -H_\varphi$, we obtain Eq. (5) above.

III. PHYSICAL PROCESS VERSION OF THE FIRST LAW OF BLACK HOLE MECHANICS

Consider a classical, stationary black hole solution to the Einstein-Maxwell equations (23) and (24). Suppose we perturb the black hole by dropping in some (possibly charged) matter. If we assume that the black hole is not destroyed in this process and that it eventually settles down to a stationary final state, we can compute its change in mass and angular momentum using Eqs. (4) and (5) above. We also can compute its change in electric charge from the flux of charge-current through the horizon, and we can compute its change in area using the Raychaudhuri equation. In [4], it was proven that Eq. (2) above holds in the case where the unperturbed black hole has no electromagnetic field, as is necessary for consistency with the first law of black hole mechanics [2,3]. In this section, we shall generalize this “physical process” version of the first law to the case of charged black holes.

Let (g_{ab}, A_a) be a solution to the source free Einstein-Maxwell equations (23) and (24) corresponding to a stationary black hole. Let

$$\xi^a = t^a + \Omega_H \varphi^a \quad (34)$$

denote the horizon Killing field of this black hole [4]. Let Σ_0 be an asymptotically flat hypersurface which terminates on the event horizon \mathcal{H} of the black hole. We wish to consider initial data on Σ_0 for a linearized perturbation $(\delta g_{ab}, \delta A_a)$ with matter sources δT_{ab} and δj^a [see Eqs. (29) and (30) above]. (We emphasize that δT_{ab} denotes the perturbation in the *non-electromagnetic* contribution to the stress-energy tensor.) We require that (i) δT_{ab} and δj^a vanish near infinity and (ii) the initial data for δg_{ab} and δA_a (and, hence, δT_{ab} and δj^a) vanish in a neighborhood of the horizon \mathcal{H} on Σ_0 —so that at the initial “time,” Σ_0 , the black hole is unperturbed. We assume that all of the matter and charge eventually fall into the black hole, and that the black hole eventually settles down to another stationary black hole solution of the source free Einstein-Maxwell equations. Our goal is to compute δM , δJ , δQ , and δA for the final state black hole and verify that Eq. (2) holds.

Since the perturbation vanishes near the internal boundary $\partial\Sigma_0$ of the initial hypersurface, it follows immediately from Eq. (33) [or equivalently, from Eqs. (4) and (5)] that the perturbed spacetime satisfies

$$\delta M - \Omega_H \delta J = - \int_{\Sigma_0} \epsilon_{dabc} (\xi^e \delta T^d_e + A_e \xi^e \delta j^d). \quad (35)$$

Thus, the perturbed mass and angular momentum of the final black hole will satisfy this relation. In terms of the current, α ,

$$\alpha^a = \xi^b \delta T^a_b + A_b \xi^b \delta j^a \quad (36)$$

[see Eq. (17) and Eq. (32)], we have

$$\delta M - \Omega_H \delta J = \int_{\Sigma_0} \alpha^d n_d \tilde{\epsilon}_{abc} \quad (37)$$

where n^a denotes the future-directed unit normal to Σ_0 and $\tilde{\epsilon}_{abc} = n^d \epsilon_{dabc}$. Using the conservation of α^a and our assumption that all of the matter eventually falls into the black hole, we can rewrite Eq. (37) as⁵

$$\delta M - \Omega_H \delta J = \int_{\mathcal{H}} \alpha^d k_d \tilde{\epsilon}_{abc} \quad (38)$$

where k^a is tangent to the affinely parametrized null geodesic generators of the event horizon, \mathcal{H} , of the unperturbed black hole and $\tilde{\epsilon}_{abc}$ satisfies

$$\frac{1}{4} \epsilon_{abcd} = -k_{[a} \tilde{\epsilon}_{bcd]}. \quad (39)$$

The second term in α^a in Eq. (36) yields a contribution to the integral in Eq. (38) of the form

$$I = - \int_{\mathcal{H}} \Phi_{\text{bh}} \delta j^d k_d \tilde{\epsilon}_{abc} \quad (40)$$

where we have written $\Phi_{\text{bh}} = -(\xi^a A_a)|_{\mathcal{H}}$. However, Φ_{bh} is constant over the horizon of the black hole [20], as can be seen as follows. We have

$$\nabla_a (A_b \xi^b) = \mathcal{L}_\xi A_a + \xi^b (dA)_{ab} = \mathcal{L}_\xi A_a + \xi^b F_{ab}. \quad (41)$$

But $\mathcal{L}_\xi A_a$ vanishes since ξ^b is a symmetry of the background solution. Furthermore, by the Raychaudhuri equation (see e.g. [21])

$$\frac{d\theta}{dV} = -\frac{1}{2} \theta^2 - \sigma_{ab} \sigma^{ab} - R_{ab} k^a k^b \quad (42)$$

(where V denotes the affine parameter corresponding to k^a) together with the fact that the expansion, θ , and shear, σ_{ab} , vanish in the stationary background, we have

$$0 = [T_{\text{EM}}]_{ab} k^a k^b |_{\mathcal{H}} = F_{ac} F_b^c k^a k^b. \quad (43)$$

Consequently, $F_{ab} k^a$ is null, and since $F_{ab} k^a k^b = 0$ by the antisymmetry of F_{ab} it follows that $F_{ab} k^a$ is proportional to k_b . Hence, the pullback of $F_{ab} k^a$ to \mathcal{H} vanishes. Thus, the pullback of $\nabla_a \Phi_{\text{bh}}$ to \mathcal{H} also vanishes, i.e., Φ_{bh} is constant on \mathcal{H} , as we desired to show. Consequently, we have

⁵In fact, Eq. (38) should hold for the black hole final state even if all of the matter and charge do not eventually fall into the black hole, with δM and δJ being the perturbed mass and angular momentum of the black hole—which no longer equal the perturbed mass and angular momentum of the spacetime because of the presence of matter outside of the black hole.

$$I = -\Phi_{\text{bh}} \int_{\mathcal{H}} \delta j^d k_d \tilde{\epsilon}_{abc} = \Phi_{\text{bh}} \delta Q \quad (44)$$

where δQ denotes the net flux of charge into the black hole. From Eqs. (38), (36) and (44), we obtain

$$\delta M - \Omega_H \delta J - \Phi_{\text{bh}} \delta Q = \int_{\mathcal{H}} \delta T^d_e \xi^e k_d. \quad (45)$$

We now compute the change in area of the black hole. To simplify the calculation, we use our diffeomorphism freedom in identifying the perturbed spacetime with the background spacetime to make the null geodesic generators of the event horizon of the perturbed black hole coincide (as unparametrized curves) with the null geodesic generators of the unperturbed stationary black hole. As a result of this gauge choice, the perturbation in the location of the horizon vanishes, and we have $\delta k^a \propto k^a$. The perturbed Raychaudhuri equation (42) yields

$$\begin{aligned} \frac{d(\delta\theta)}{dV} &= -8\pi \delta([T_{\text{total}}]_{ab} k^a k^b)|_{\mathcal{H}} \\ &= -8\pi \delta([T_{\text{EM}}]_{ab}) k^a k^b|_{\mathcal{H}} - 8\pi (\delta T_{ab}) k^a k^b|_{\mathcal{H}} \end{aligned} \quad (46)$$

where we have used the fact that $[T_{\text{EM}}]_{ab} k^a k^b|_{\mathcal{H}} = 0$ [see Eq. (43)] together with $\delta k^a \propto k^a$ to eliminate such terms as $[T_{\text{EM}}]_{ab} k^a \delta k^b$. However, we have

$$\begin{aligned} \delta([T_{\text{EM}}]_{ab}) k^a k^b|_{\mathcal{H}} &= \left(2F_{ac} \delta F_b^c - \frac{1}{4} \delta g_{ab} F_{de} F^{de} \right. \\ &\quad \left. - \frac{1}{2} g_{ab} F_{de} \delta F^{de} \right) k^a k^b. \end{aligned} \quad (47)$$

The last two terms vanish since k^a is null in both the perturbed and unperturbed spacetimes. On the other hand, we showed above that $F_{ac} k^c \propto k_a$, so $F_{ac} \delta F_b^c k^a k^b \propto \delta F_{bc} k^b k^c = 0$ by antisymmetry of δF_{bc} . Thus, we obtain

$$\frac{d(\delta\theta)}{dV} = -8\pi \delta T_{ab} k^a k^b|_{\mathcal{H}}. \quad (48)$$

A calculation identical to that given in [4] then yields

$$\kappa \delta A = 8\pi \int_{\mathcal{H}} \delta T^d_e \xi^e k_d. \quad (49)$$

Substitution of this result into Eq. (45) then yields Eq. (2), as we desired to show.

IV. THE THERMAL ATMOSPHERE AROUND A BLACK HOLE

The remainder of this paper will be devoted to analyzing the validity of the generalized second law (GSL) for processes in which some (possibly charged) matter is quasi-statically lowered toward a (possibly charged and rotating) black hole and then released. In this section, we will state our

assumptions about the thermal atmosphere surrounding the black hole and derive certain properties of it.

An isolated black hole would continuously emit Hawking radiation to infinity, and quantum fields around the black hole cannot come to thermal equilibrium. However, thermal equilibrium should be possible if the black hole is enclosed in a box. Nevertheless, even for an uncharged and nonrotating black hole, this equilibrium will be unstable unless the box enclosing the black hole is sufficiently small [22]. For a rotating black hole, a more serious problem occurs: There cannot exist a thermal equilibrium state of quantum fields outside the black hole unless the black hole is enclosed in a box that is sufficiently small that the horizon Killing field ξ^a , Eq. (34), is timelike everywhere within the box [23]. Similarly, for a charged black hole, thermal equilibrium of a charged field outside the black hole would not be possible unless the box is sufficiently small that the electrostatic potential differences inside the box are insufficient to permit pair creation. In the following, we will restrict consideration to the case where we have a (possibly charged and rotating) black hole enclosed in a sufficiently small box⁶ that the quantum fields outside of the black hole are in a thermal equilibrium state with respect to the notion of ‘‘time translations’’ provided by the horizon Killing field ξ^a .

We shall refer to observers following orbits of ξ^a , i.e., observers with 4-velocity

$$u^a = \xi^a / (-\xi^c \xi_c)^{1/2} \quad (50)$$

as *stationary observers*. We shall assume that the thermal atmosphere admits a local thermodynamic description with respect to stationary observers. There is a natural ground state of the quantum field associated with ξ^a [24,4], which we shall refer to as the *Boulware vacuum state*. It would be natural for stationary observers to consider the non-electromagnetic stress-energy tensor T_{ab} and charge-current j^a relative to the Boulware vacuum, so we define

$$\tilde{T}_{ab} = T_{ab} - (T_0)_{ab} \quad (51)$$

$$\tilde{j}^a = j^a - j_0^a \quad (52)$$

where $(T_0)_{ab}$ and j_0^a denote, respectively, the (true, renormalized) non-electromagnetic stress energy and charge-current of the Boulware vacuum state.

It will be assumed that in thermal equilibrium, the non-electromagnetic energy current $\tilde{T}_{ab} \xi^b$ and the charge current \tilde{j}^a relative to the Boulware vacuum state are proportional to ξ^a . We also shall allow for the possibility that other locally defined conserved currents $(\sigma_i)^a$ may exist—such as, e.g., the number currents of various species of particles that represent additional conservation laws beyond conservation of

⁶Note that if the black hole were enclosed in a box that was large enough for ξ^a to become spacelike and/or allow a large electrostatic potential difference, the black hole would presumably simply lose angular momentum and/or discharge until the quantum fields outside the black hole could reach a thermal equilibrium state.

charge. We also shall assume that in thermal equilibrium the corresponding currents $(\tilde{\sigma}_i)^a$ measured relative to the Boulware vacuum state are proportional to ξ^a . Consequently, in thermal equilibrium, the above currents can be characterized, respectively, by the energy density $\tilde{\rho} = \tilde{T}_{ab} u^a u^b$, the charge density $\tilde{q} = -\tilde{j}_a u^a$, and the quantities $\tilde{\lambda}_i = -(\tilde{\sigma}_i)_a u^a$, all measured relative to the Boulware vacuum state.

It also will be assumed that in local thermal equilibrium the stationary observers would assign an entropy current \tilde{s}^a to the thermal atmosphere, which also will be assumed to be proportional to ξ^a , so that it also can be described by the entropy density $\tilde{s} = -\tilde{s}_a u^a$, relative to the Boulware vacuum state. We explicitly allow for the possibility that some “renormalization” of entropy may occur [25], so that the “true” entropy density, s , of the thermal atmosphere is given by

$$s = \tilde{s} + s_0 \quad (53)$$

where s_0 is the (true) entropy density of the Boulware vacuum. It would appear that such a renormalization of entropy must occur to avoid a divergence in the contribution of the thermal atmosphere to the total entropy due to \tilde{s} becoming arbitrarily large near the horizon.

We will assume that $\tilde{\rho}$, \tilde{q} , and $\tilde{\lambda}_i$ serve as “state variables” that characterize the local thermodynamic state, so that the entropy density, \tilde{s} , can be expressed as a function of these state variables

$$\tilde{s} = \tilde{s}(\tilde{\rho}, \tilde{q}, \tilde{\lambda}_i). \quad (54)$$

However, it should be emphasized that we make no assumptions about the explicit functional form of \tilde{s} .

The renormalized non-electromagnetic stress energy, T_{ab} , and charge-current, j^a , of the thermal atmosphere will, of course, perturb the spacetime metric, g_{ab} , and electromagnetic field, A_a , around the black hole. These perturbations of g_{ab} and A_a will, in turn, affect the distribution and properties of the thermal atmosphere. However, for a macroscopic black hole (i.e., a black hole of mass much greater than the Planck mass), T_{ab} and j^a will be small compared with scales set by the background curvature, and we shall assume that they can be treated as linear perturbations of the classical electrovac black hole spacetime. In particular, we will assume that the formulas of Sec. II apply for the contribution of the thermal atmosphere to the total mass and angular momentum of the spacetime. The effects of the perturbations of g_{ab} and A_a on the thermal atmosphere would then be of second and higher order, and therefore will be neglected.

In the following, we shall consider the effects of perturbing the state of the thermal atmosphere to a nearby state that is locally in thermal equilibrium. In accordance with the remarks in the previous paragraph, we shall neglect the effects of the resulting perturbations of g_{ab} and A_a when calculating the changes in the renormalized T_{ab} and j^a of the thermal atmosphere caused by the perturbation of its state. We shall similarly neglect the effects of these perturbations of g_{ab} and

A_a when calculating the changes in \tilde{T}_{ab} and \tilde{j}^a . In view of Eqs. (51) and (52), this additional assumption amounts to assuming that the perturbations of $(T_0)_{ab}$ and j_0^a are small compared with the perturbations of T_{ab} and j^a . Similar remarks apply to $(\sigma_i)^a$ and $(\tilde{\sigma}_i)^a$. Finally, we also shall neglect the effects of the perturbations of g_{ab} and A_a when calculating the changes in s^a and \tilde{s}^a .

We now consider perturbing the state of the thermal atmosphere to a nearby state that is locally in thermal equilibrium, characterized by the state variables $\tilde{\rho} + \delta\tilde{\rho}$, $\tilde{q} + \delta\tilde{q}$, $\tilde{\lambda}_i + \delta\tilde{\lambda}_i$. Variation of Eq. (54) yields the local form of the ordinary first law of thermodynamics

$$\delta\tilde{s} = \frac{1}{T} \delta\tilde{\rho} + \Psi \delta\tilde{q} + \sum_i \gamma_i \delta\tilde{\lambda}_i \quad (55)$$

where the temperature, T , and the potentials Ψ and γ_i are defined by appropriate partial derivatives of \tilde{s} with respect to the state variables. In a small volume, V , the locally measured energy relative to the Boulware vacuum is $\tilde{U} = \tilde{\rho}V$, and we similarly have $\tilde{Q} = \tilde{q}V$, and $\tilde{\Lambda}_i = \tilde{\lambda}_i V$. Hence, we obtain

$$\begin{aligned} \delta\tilde{S} &= \delta(\tilde{s}V) = V \left(\frac{1}{T} \delta\tilde{\rho} + \Psi \delta\tilde{q} + \sum_i \gamma_i \delta\tilde{\lambda}_i \right) + \tilde{s} \delta V \\ &= \frac{\delta\tilde{U}}{T} + \Psi \delta\tilde{Q} + \sum_i \gamma_i \delta\tilde{\Lambda}_i \\ &\quad + \left(\tilde{s} - \frac{\tilde{\rho}}{T} - \Psi \tilde{q} - \sum_i \gamma_i \tilde{\lambda}_i \right) \delta V. \end{aligned} \quad (56)$$

We interpret the coefficient of δV in this formula as \tilde{P}/T , where \tilde{P} denotes the pressure of the thermal atmosphere relative to the Boulware vacuum. We thereby obtain the integrated form of the Gibbs-Duhem relationship for the thermal atmosphere:

$$\tilde{P} = T\tilde{s} - \tilde{\rho} - T\Psi\tilde{q} - T\sum_i \gamma_i \tilde{\lambda}_i. \quad (57)$$

Since only differences are taken in Eq. (55), the ground state contributions cancel out because, as discussed in the previous paragraph, we neglect changes in the ground state quantities. Therefore, Eq. (55) also holds for the true, renormalized quantities, i.e.,

$$\delta s = \frac{1}{T} \delta\rho + \Psi \delta q + \sum_i \gamma_i \delta\lambda_i. \quad (58)$$

In the following, we will work with the true, renormalized quantities.

As discussed above, the contribution,

$$E \equiv M - \Omega_H J \quad (59)$$

of the thermal atmosphere to the total “energy” conjugate to ξ^a of the spacetime is given by [see Eq. (33)]

$$E = - \int_{\Sigma} \epsilon_{abcd} (\xi^a T^e{}_a + \xi^a A_a j^e). \quad (60)$$

Similarly, the contribution of the thermal atmosphere to the total electric charge is

$$Q = \int_{\Sigma} \epsilon_{abcd} j^e \quad (61)$$

and its contribution to the globally conserved quantities, Λ_i , associated with $(\sigma_i)^a$ is

$$\Lambda_i = \int_{\Sigma} \epsilon_{abcd} (\sigma_i)^e. \quad (62)$$

If the state of the thermal atmosphere is perturbed, variation of the above formulas yields

$$\delta E = \delta M - \Omega_H \delta J = - \int_{\Sigma} \epsilon_{abcd} (\xi^a \delta T^e{}_a + \xi^a A_a \delta j^e) \quad (63)$$

$$\delta Q = \int_{\Sigma} \epsilon_{abcd} \delta j^e \quad (64)$$

$$\delta \Lambda_i = \int_{\Sigma} \epsilon_{abcd} \delta (\sigma_i)^e. \quad (65)$$

However, since as discussed above, we have $\delta T_{ab} = \delta \tilde{T}_{ab}$, $\delta j^a = \delta \tilde{j}^a$, and $\delta (\sigma_i)^a = \delta (\tilde{\sigma}_i)^a$ and since the ‘‘tilded’’ currents have been assumed to be proportional to ξ^a , we may rewrite Eqs. (63)–(65) as

$$\delta E = \delta M - \Omega_H \delta J = \int_{\Xi} (\chi \delta \rho - \xi^a A_a \delta q) \quad (66)$$

$$\delta Q = \int_{\Xi} \delta q \quad (67)$$

$$\delta \Lambda_i = \int_{\Xi} \delta \lambda_i \quad (68)$$

where Ξ denotes the manifold of orbits [26] of ξ^a with the natural volume element $\bar{\epsilon}_{abc} = \epsilon_{dabc} u^d$ on Ξ understood. In Eq. (66), the ‘‘redshift factor’’ χ is defined by

$$\chi \equiv (-\xi^a \xi_a)^{1/2}. \quad (69)$$

We now impose the assumption that the thermal atmosphere is in thermal equilibrium with itself. More precisely, we assume that at fixed total ‘‘energy,’’ E , fixed total charge, Q , and fixed Λ_i , the total entropy $S = \int_{\Xi} s$ is maximum. In order for this to be the case, it is necessary for S to be an extremum with respect to all first order variations that preserve the above constraints. We have

$$\delta S = \int_{\Xi} \delta s = \int_{\Xi} \left(\frac{1}{T} \delta \rho + \Psi \delta q + \sum_i \gamma_i \delta \lambda_i \right). \quad (70)$$

The necessary and sufficient conditions for δS to vanish for all variations $\delta \rho$, δq and $\delta \lambda_i$ satisfying $\delta E = \delta Q = \delta \Lambda_i = 0$ can be determined as follows. If we set $\delta q = \delta \lambda_i = 0$ (so that we automatically satisfy $\delta Q = \delta \Lambda_i = 0$), we see that by a suitable choice of $\delta \rho$ we can make $\delta S \neq 0$ while preserving the remaining constraint $\delta E = 0$ unless the temperature, T , obeys the Tolman law

$$T = T_0 / \chi \quad (71)$$

where T_0 is a constant. The first term in Eq. (70) can then be written as

$$\int_{\Xi} \frac{1}{T} \delta \rho = \frac{1}{T_0} \int_{\Xi} \chi \delta \rho = \frac{\delta E}{T_0} + \frac{1}{T_0} \int_{\Xi} \xi^a A_a \delta q. \quad (72)$$

Substituting this into Eq. (70), we find that by a suitable choice of δq we can make $\delta S \neq 0$ unless Ψ is of the form

$$\Psi = - \frac{1}{T_0} (\xi^a A_a + \Phi_0) \quad (73)$$

where Φ_0 is a constant. Finally, it is easily seen that extremization of S under the constraints requires each γ_i to be constant,⁷

$$\gamma_i = \gamma_{0i}. \quad (74)$$

It is easily seen that Eqs. (71), (73), and (74) are also sufficient for S to be an extremum.

Finally, taking Eqs. (71), (73), and (74) into account in Eq. (70), we see that for a perturbation that does not necessarily preserve E , Q , or Λ_i , the change in the total entropy, S , of the thermal atmosphere is given by

$$\begin{aligned} \delta S &= \int_{\Xi} \left(\frac{\chi}{T_0} \delta \rho - \frac{1}{T_0} (\xi^a A_a + \Phi_0) \delta q + \sum_i \gamma_{0i} \delta \lambda_i \right) \\ &= \frac{1}{T_0} (\delta E - \Phi_0 \delta Q) + \sum_i \gamma_{0i} \delta \Lambda_i \\ &= \frac{1}{T_0} (\delta M - \Omega_H \delta J - \Phi_0 \delta Q) + \sum_i \gamma_{0i} \delta \Lambda_i. \end{aligned} \quad (75)$$

Equation (75) is the ‘‘global form’’ of the first law of thermodynamics for the thermal atmosphere. Note that our above analysis and results are applicable to any thermodynamic

⁷If the conserved current $(\sigma_i)_a$ corresponds to a particle number current of a neutral particle, then Λ_i can be varied independently of Q and the other particle species. The chemical potential, μ_i , of this species would then be given in terms of γ_i by $\mu_i = -T \gamma_i$. Taking Eq. (71) into account, we see that the behavior of the chemical potential corresponding to Eq. (74) is $\mu_i = -T_0 \gamma_{0i} / \chi$. If $(\sigma_i)_a$ is the particle number current of a charged particle, then a variation of Λ_i holding the number of other particle species fixed requires a corresponding variation of Q . The chemical potential for a species of charged particles would be given by $\mu_i = (-T_0 \gamma_{0i} + e_i [\Phi_0 + \xi^a A_a]) / \chi$, where e_i denotes the charge per particle of this species.

system that is locally “at rest” with respect to a timelike Killing field ξ^a , i.e., we did not use any special properties of the thermal atmosphere to derive Eqs. (71), (73), (74), or (75), although in our formula for E , we did assume that the non-electromagnetic stress-energy, T_{ab} , and charge-current, j^a , could be treated as a linear perturbation of an electrovac spacetime.

We now impose our assumption that the thermal atmosphere is in thermal equilibrium with the black hole, i.e., that the total generalized entropy, $S_C \equiv S + S_{\text{bh}}$, of the combined black hole/thermal atmosphere system is at its maximum possible value for the given values of the combined mass, $M_C = M + M_{\text{bh}}$, angular momentum, $J_C = J + J_{\text{bh}}$, and charge, $Q_C = Q + Q_{\text{bh}}$, of the total system. By the first law of black hole mechanics, we have

$$\delta S_{\text{bh}} = \frac{1}{T_H} (\delta M_{\text{bh}} - \Omega_H \delta J_{\text{bh}} - \Phi_{\text{bh}} \delta Q_{\text{bh}}). \quad (76)$$

Comparing Eqs. (75) and (76), we see that S_C will be an extremum under interchange of mass, angular momentum, and charge between the thermal atmosphere and the black hole if and only if we have

$$T_0 = T_H \quad (77)$$

$$\Phi_0 = \Phi_{\text{bh}} \quad (78)$$

and

$$\gamma_{0i} = 0. \quad (79)$$

Note that the vanishing of γ_{0i} is essentially a consequence of the “no hair theorems”: Although for matter (including the thermal atmosphere) there may be locally conserved currents $(\sigma_i)^a$ aside from mass, angular momentum, and charge, there is no global conservation law for these quantities when a black hole is present, since the “charges” Λ_i can fall into the black hole, which retains no “memory” of them. If the charge Λ_i corresponds to the number of particles of a particular species, then the chemical potential for that species (see footnote 7) is given by

$$\mu_i = e_i (\Phi_{\text{bh}} - \Phi) / \chi. \quad (80)$$

It follows from Eq. (80) that μ_i vanishes on the horizon of the black hole, as had previously been claimed in [16].

Taking into account the above relations, we find that the integrated Gibbs-Duhem relation (57) for the thermal atmosphere now takes the form

$$\chi \tilde{P} = T_H \tilde{s} - \chi \tilde{\rho} - (\Phi - \Phi_{\text{bh}}) \tilde{q}. \quad (81)$$

We also note that the combined black hole/thermal atmosphere system satisfies

$$\delta S_C = \frac{1}{T_H} (\delta M_C - \Omega_H \delta J_C - \Phi_{\text{bh}} \delta Q_C) = \frac{1}{T_H} (\delta E_C - \Phi_{\text{bh}} \delta Q_C). \quad (82)$$

V. LOWERING PROCESS

In this section, we will consider a process in which a box containing charged matter is quasi-statically lowered toward the black hole and then dropped into the black hole or otherwise allowed to thermalize with the black hole/thermal atmosphere system. The box will be assumed to be “perfectly insulating,” i.e., the walls will be assumed to be perfectly reflecting with respect to the fields inside and outside of the box. However, as will be discussed further below, the walls of the box will necessarily radiate energy, charge, etc. into or out of the box as the box is lowered [7]. We make no assumptions concerning the size, shape, or contents of the box other than that its size is small compared with that of the black hole and that its contents satisfy the ordinary thermodynamic laws (see below). In particular, we do not make the “thin box” approximation [7], nor do we even assume that the walls of the box are rectangular in shape.

The process under consideration consists of two distinct stages: (i) A quasi-static process in which the box is slowly lowered toward the black hole. In this process, work may be done by an external agent, so that the total energy contained in the box/black hole/thermal atmosphere system may change. (ii) A non-quasi-static process in which the box is dropped into the black hole or otherwise destroyed, and the system is allowed to thermalize. No change in the total energy or charge of the complete system occurs in this stage. We will argue that the total generalized entropy, S' , cannot decrease in either of these stages.

Consider, first, the quasi-static lowering process. At any given time during this process, the total generalized entropy, S' of the system can be written as

$$S' = S_B + S_C - S_D. \quad (83)$$

Here, S_B denotes the total entropy contained in the box, and S_C denotes the total entropy that the combined black hole/thermal atmosphere system would have at the given values of T_H and Φ_{bh} if the box were not present. Finally, S_D denotes the entropy of the displaced thermal atmosphere, i.e., the entropy that would have been contained in the thermal atmosphere (at the given values of T_H and Φ_{bh}) within the region occupied by the box.

We now focus attention on S_B . It is instructive to consider, first, the case where the box initially is “empty,” i.e., the initial state of the fields inside the box is the natural vacuum/ground state relative to the notion of time translations defined by ξ^a (see [24]). Then, we claim that if the lowering process is sufficiently slow, no “particle creation” will occur as the box is lowered, and the box will remain in its ground state. In other words, an empty box will remain empty as it is quasi-statically lowered. By definition, the Casimir energy of the box is the difference between the energy contained in the empty box and the energy that would be contained in the Boulware vacuum in the region of space occupied by the box. (Recall here that by “energy,” E , we mean the conserved quantity conjugate to the horizon Killing field ξ^a rather than the asymptotic time translation t^a , i.e., $E = M - \Omega_H J$ [see Eq. (59) above].) If we neglect possible

changes in the Casimir energy of the box as it is lowered, it follows that in the lowering process for an empty box, we have

$$\Delta E_B = \Delta E_0 \quad (84)$$

where ΔE_B denotes the difference in the energies contained in the box at two stages of the lowering process, and ΔE_0 denotes the difference in the Boulware vacuum energies in the corresponding volumes. Here we have written “ Δ ” rather than “ δ ” to emphasize that we are taking differences of quantities associated with different regions of space rather than differences of quantities associated with the same region of space, as in the previous section.

Similarly, the total charge Q_B within the empty box satisfies

$$\Delta Q_B = \Delta Q_0. \quad (85)$$

It should be noted that changes in the charge contained within the box can occur only as a result of radiation by the walls of the box. However, this radiation by the walls of the box is independent of the contents of the box. Therefore, in all cases (i.e., whether or not the box is empty), the change in the charge of the box as it is lowered is given by Eq. (85). However, Eq. (84) holds only for the empty box, since the total energy contained within the box can vary due to redshifting of the energy of the contents of the box as well as by radiation by the walls of the box.

Now suppose that the box is initially filled with (possibly charged) matter that is in thermal equilibrium. As the box is lowered, the matter will, in general, redistribute itself within the box due to the changing electromagnetic and gravitational fields. However, if the box is lowered sufficiently slowly, the matter will remain in thermal equilibrium as it is lowered. Observers inside of the box will view the process as being isentropic for the same reason as slow variations of parameters in the Hamiltonian result in isentropic processes in flat spacetime physics. Thus, the entropy above the ground state must remain constant⁸ as the box is lowered. Taking into account the possibility that the ground state entropy is nonzero due to “renormalization” as discussed in the previous section, we see that in a slow lowering process where the matter is initially in thermal equilibrium, we have $\Delta S_B = \Delta S_0$, where S_0 denotes the entropy of the Boulware vacuum in the region occupied by the box. Equivalently, we have $\Delta \tilde{S}_B = 0$, where in accord with the notation of the previous section, $\tilde{S}_B \equiv S_B - S_0$.

Finally, suppose now that the box is initially filled with arbitrary matter, not necessarily in thermal equilibrium. Then, as the box is lowered, the matter may (partially or fully) thermalize. Observers inside the box will see an

increase—or, at least, a non-decrease—of entropy relative to the ground state during the lowering process for the same reason that the ordinary second law of thermodynamics holds in flat spacetime physics. Consequently, we conclude that whatever is initially placed inside the box, we have

$$\Delta \tilde{S}_B \equiv \Delta S_B - \Delta S_0 \geq 0 \quad (86)$$

during the lowering process.

We now turn our attention to the calculation of the change, ΔS_C , in S_C during the lowering process. By Eq. (82), we have

$$\Delta S_C = \delta S_C = \frac{1}{T_H} (\delta E_C - \Phi_{\text{bh}} \delta Q_C) \quad (87)$$

so we need to calculate the energy, δE_C , and charge, δQ_C delivered to the black hole/thermal atmosphere system during the lowering process. The total charge of the entire system is

$$Q' = Q_B + Q_C - Q_D \quad (88)$$

so by conservation of charge, we have

$$\delta Q_C = -\Delta Q_B + \Delta Q_D. \quad (89)$$

We already found above that $\Delta Q_B = \Delta Q_0$ [see Eq. (85)], so we have

$$\delta Q_C = -\Delta Q_0 + \Delta Q_D = \Delta \tilde{Q}_D. \quad (90)$$

To calculate δE_C , we note that this quantity cannot depend upon what is placed inside the box during the lowering process, so it suffices to restrict attention to the case where the box is empty. In that case, the change in total energy inside the box is given by Eq. (84), so by conservation of energy, we have

$$\delta E_C = W - \Delta E_0 + \Delta E_D = W + \Delta \tilde{E}_D \quad (91)$$

where W denotes the work done by the external agent during the lowering process. We calculate W as follows. As shown in the Appendix, the net force F^a exerted by an external agent at redshift χ_0 on a stationary box whose size is much smaller than the scales set by curvature is given by⁹

$$\chi_0 F^a = e^a \int_{\partial B} \chi \tilde{P} e_b n^b dA. \quad (92)$$

Here e^a denotes the unit “upward pointing” tangent to the string (defined over the volume of the box via parallel transport—see the Appendix for details) and n^a is the unit outward pointing normal to the surface, ∂B , of the box in the

⁸This result also can be derived from energy balance considerations, as outlined in footnote 2 of [12]. However, this argument assumes that no “extra energy” (i.e., “heat”) is fed into or taken out of the box as it is lowered, and thus, in essence, *assumes* that the lowering process is isentropic.

⁹Here we neglect any contributions of the Casimir energy and/or the walls of the box to $\tilde{\rho}$. These will not, in any case, affect the energy delivered to the black hole/thermal atmosphere system, provided that their locally measured stress-energy remains constant during the lowering process, so that Eq. (84) holds.

manifold of orbits of ξ^a . If the box is lowered, the work done by the external agent during the lowering process is

$$W = -\chi_0 \int (F^a e_a)|_l dl = - \int \chi \tilde{P} e_a n^a dA dl \quad (93)$$

where l denotes proper length along the path of lowering in the manifold of orbits. Equation (93) is easiest to analyze in the case of a rectangular box with “top” and “bottom” faces perpendicular to e^a . In that case, Eq. (93) corresponds to performing a volume integral of $\chi \tilde{P}$ over the spatial region swept out by the top face of the box, and subtracting from it the similar integral over the spatial region swept out by the bottom face. The result is

$$W = \Delta \int_B \chi \tilde{P} dV \quad (94)$$

where the integral is taken over the volume of the box, and Δ denotes the difference between the final and initial values of this integral in the lowering process. By a similar argument, it is not difficult to see that Eq. (94) remains valid for a box of arbitrary shape.

We now apply the integrated Gibbs-Duhem relation (81) for the thermal atmosphere. We thereby obtain

$$\begin{aligned} W &= \Delta \int [T_H \tilde{s} - \chi \tilde{\rho} - (\Phi - \Phi_{\text{bh}}) \tilde{q}] dV \\ &= T_H \Delta \tilde{S}_D - \Delta \tilde{E}_D + \Phi_{\text{bh}} \Delta \tilde{Q}_D \end{aligned} \quad (95)$$

where the subscript D denotes quantities associated with the displaced thermal atmosphere and Eq. (66) was used. Combining Eqs. (87), (90), (91), and (95), we obtain

$$\begin{aligned} \Delta S_C &= \frac{1}{T_H} [\delta E_C - \Phi_{\text{bh}} \delta Q_C] = \frac{1}{T_H} [W + \Delta \tilde{E}_D - \Phi_{\text{bh}} \delta Q_C] \\ &= \Delta \tilde{S}_D \end{aligned} \quad (96)$$

combining this equation with Eqs. (83) and (86), we obtain

$$\Delta S' = \Delta \tilde{S}_B \geq 0 \quad (97)$$

which shows that the generalized second law holds during the lowering process.

Now, consider the “dropping process,” i.e., we suppose that—after completion of the above lowering process—the box is released and allowed to fall into the black hole (or that the box is destroyed and its contents are allowed to thermalize with the black hole/thermal atmosphere system). We assume that at the end of this process, the final state of the system is that of a black hole in equilibrium with its thermal atmosphere. Although the “dropping process” is a highly nonequilibrium process and we cannot analyze the time evolution of the total entropy during this process, it is clear that if the second assumption of Sec. I holds, the total generalized entropy, S' , cannot decrease in this process. Namely, during the “dropping process” the total mass, angular momentum, and charge of the system remain constant. However, assump-

tion (ii) asserts that the final state maximizes the total generalized entropy at the given values of total mass, angular momentum, and charge. Therefore, the initial state could not have had more generalized entropy than the final state.

Consequently, under the assumptions stated in Sec. I as well as our assumptions about the thermodynamic properties of the thermal atmosphere stated in Sec. IV, the GSL cannot be violated in any quasi-static lowering/dropping process.

VI. CONCLUDING REMARKS ON THE VALIDITY OF THE GSL

In this paper, we have established the validity of the GSL in arbitrary quasi-static lowering/dropping processes for charged and rotating black holes, without the need to assume any entropy bounds for matter. Our analysis depends, primarily, only on the two very general assumptions stated in Sec. I. It thereby generalizes and simplifies previous analyses of such processes.

However, it should be noted that during the course of our analysis, we made several simplifying assumptions concerning the thermodynamic properties of the thermal atmosphere. In particular, it was assumed that stationary observers would (i) assign a locally homogeneous entropy density \tilde{s} to the thermal atmosphere that is valid on all relevant scales and (ii) that \tilde{s} is a function only of $\tilde{\rho}$, \tilde{q} , and $\tilde{\lambda}_i$. The first assumption need not be valid if one considers boxes whose size is small compared with the wavelength of the ambient thermal atmosphere (see [10]). The second assumption could fail because the formula for \tilde{s} might also depend nontrivially upon “location in the gravitational field” (e.g., depend upon the local value of the curvature and/or derivatives of ξ^a).

In this concluding section, we wish to argue that if the breakdown of any such simplifying assumptions about the thermal atmosphere were to allow one to violate the GSL, they also would allow one to violate the ordinary second law. Specifically, suppose that—on account of, say, the breakdown of assumptions (i) or (ii) of the previous paragraph—it were possible to violate the GSL in a quasi-static lowering/dropping process. Since the validity of the GSL during the “dropping” phase does not depend upon any simplifying assumptions about the thermal atmosphere, it follows that a violation of the GSL must occur in the lowering phase. Now, in the lowering phase, the box will stay some finite distance, ϵ , outside of the horizon of the black hole. Then, it should be possible, in principle, to construct a (charged and rotating) shell with a perfectly reflecting surface whose exterior gravitational and electromagnetic fields coincide with those of the black hole at distances greater than ϵ from the horizon. We can enclose this shell in a cavity of the same size as used for the black hole and then fill this cavity with “real” thermal radiation in such a way that its temperature is T_H/χ , its potential Φ_0 is equal to Φ_{bh} and the potentials γ_{0i} are equal to zero. (This can be accomplished by supplying the atmosphere with the appropriate amounts of energy, charge, and other conserved quantities.) There may be slight differences between the properties of the thermal atmosphere of the black hole and those of the “real” atmosphere around the

shell that we have put in “by hand” on account of slight differences between the ground states and modes in the two cases. However, these differences can be made arbitrarily small by choosing the radius of the shell to be arbitrarily close to the horizon radius of the black hole.

Now, the analysis of the lowering process given in the previous section applies without change to a lowering process where the black hole/thermal atmosphere system is replaced by the shell surrounded by a “real” atmosphere, provided that the subscript “C” is now interpreted as referring to the “real” atmosphere around the shell. The values of S_C , E_C , etc. may be very different for the “real” atmosphere around the shell as compared with the black hole/thermal atmosphere system, but the variations of these quantities during the lowering process will be the same (provided that the radius of the shell is sufficiently close to the horizon radius of the black hole). It follows that if a lowering process that decreases the total generalized entropy can be done in the black hole case, a corresponding lowering process in the case of the shell will decrease the total ordinary entropy. Thus, if the GSL can be violated, then a corresponding process will violate the ordinary second law.

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APPENDIX: FORCE EXERTED ON A STATIONARY BOX

In this appendix, we consider a stationary (but not necessarily static) spacetime, with timelike Killing vector field ξ^a . We consider a stationary box in this spacetime which is held in place by an agent who holds a massless string that is connected to the box (see Fig. 1). The box may be of arbitrary shape and may contain charged matter. An external electromagnetic field may be present and there also may be additional matter outside of the box (which may exert a “buoyancy force” on the box). We wish to calculate the force that the agent must exert on the “far end” of the string in order to hold the box in place under the following assumptions:

We assume that the world sheet of the string is invariant under ξ^a , and has stress-energy of the form

$$T_S^{ab} = P e^a e^b \quad (\text{A1})$$

where e^a is a unit vector that is tangent to the world sheet of the string and is orthogonal to ξ^a . (We choose the direction of e^a to point “towards the agent,” i.e., “away from the box.”) This stress tensor corresponds to a massless string, which we consider for simplicity; it is straightforward to allow the string to have mass, but then the weight of the string would contribute to the force exerted by the external agent. The “ P ” in Eq. (A1) is understood to be proportional to a delta-function on the world sheet of the string. We also assume that the string does not contain any electromagnetic charge or current.

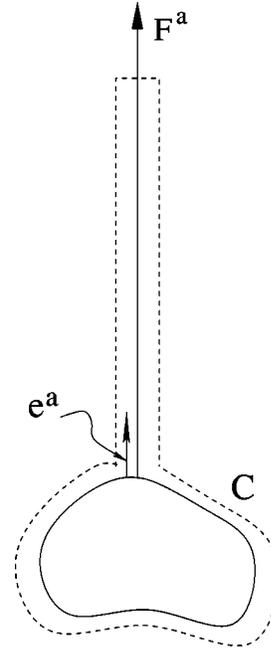


FIG. 1. In a stationary spacetime, a box is held in place by an agent who holds a massless string connected to the box. The surface, C , enclosing the box and string is represented by the dotted line.

We decompose the stress-energy tensor of everything—including the contents of the box, the walls of the box, the string, and the matter outside of the box—into its electromagnetic and non-electromagnetic parts

$$T_{\text{total}}^{ab} = T^{ab} + T_{\text{EM}}^{ab} \quad (\text{A2})$$

where T_{EM}^{ab} denotes the stress tensor of the electromagnetic field [see Eq. (20)]. We assume that the electromagnetic field is stationary

$$\mathcal{L}_\xi A_a = 0. \quad (\text{A3})$$

This implies, of course, that T_{EM}^{ab} and the electromagnetic charge-current vector, j^a , also are Lie derived by ξ^a . We further assume that j^a takes the form

$$j^a = q u^a = \frac{q}{\chi} \xi^a \quad (\text{A4})$$

where $u^a = \xi^a / \chi$ with $\chi = (-\xi^a \xi_a)^{1/2}$, i.e., we assume that the charges are “at rest” (no current flow) with respect to the stationary observers. Similarly, we assume that the total non-electromagnetic stress-energy tensor, T_{ab} , is stationary

$$\mathcal{L}_\xi T_{ab} = 0 \quad (\text{A5})$$

and we further assume that T_{ab} takes the form

$$T^{ab} = \rho u^a u^b + t^{ab} \quad (\text{A6})$$

with $t_{ab} u^a = 0$, i.e., we assume that the non-electromagnetic stress-energy tensor has no “time-space” components (i.e., no momentum density) relative to the stationary observers.

Note that our assumptions concerning T^{ab} are compatible with our assumed form of the stress-energy tensor of the string, Eq. (A1), which is included in T^{ab} .

It is convenient to work on the manifold, Ξ , of orbits of ξ^a (see [26]). All tensor fields on M that are Lie derived by ξ^a and have all indices perpendicular to ξ^a have a natural projection to Ξ , and we will not distinguish in our notation such spacetime tensors from their projections to Ξ . In particular, Ξ naturally acquires a Riemannian metric h_{ab} given by

$$h_{ab} = g_{ab} + u_a u_b. \quad (\text{A7})$$

We denote by D_a the derivative operator on Ξ associated with h_{ab} . Our final assumption is that the size of the box is small compared with the scales of curvature in the manifold of orbits.

The string stress-energy T_S^{ab} , Eq. (A1), must be conserved everywhere except at the end points of the string. This implies that e^a must be a geodesic in spacetime, $e^b \nabla_b e^a = 0$. It follows immediately that $e^b D_b e^a = 0$, i.e., the projection of the string to the manifold of orbits, Ξ , is a geodesic in the manifold of orbits. We now choose a surface C in Ξ which encloses the box and string in the manner shown in Fig. 1. We extend the definition of e^a to the interior of C by parallel transport (with respect to D_a) along geodesics (with respect to D_a) starting from the point at which the string is attached to the box. (Note that since the size of the box has been assumed to be small compared with scales set by curvature, parallel transport over the box will be essentially path independent in any case.)

Conservation of the total stress energy, Eq. (A2), yields

$$\begin{aligned} 0 &= \nabla_b T_{\text{total}}^{ab} = \nabla_b (\rho u^a u^b) + \nabla_b t^{ab} + \nabla_b T_{\text{EM}}^{ab} \\ &= \rho u^b \nabla_b u^a + \nabla_b t^{ab} - F^{ab} j_b \end{aligned} \quad (\text{A8})$$

where we have used $u^b \nabla_b \rho = 0$ and $\nabla_b u^b = 0$ in the last line. Since $u^a = \xi^a / \chi$, we obtain

$$u^b \nabla_b u^a = \frac{1}{\chi} D^a \chi. \quad (\text{A9})$$

On the other hand,

$$\begin{aligned} F_{ab} j^b &= (\nabla_a A_b - \nabla_b A_a) \frac{q}{\chi} \xi^b = \frac{q}{\chi} [-\mathcal{L}_{\xi} A_a + \nabla_a (A_b \xi^b)] \\ &= -\frac{q}{\chi} D_a \Phi \end{aligned} \quad (\text{A10})$$

where $\Phi \equiv -A_a \xi^a$. Thus, we obtain

$$0 = \frac{\rho}{\chi} D^a \chi + \nabla_b t^{ab} + \frac{q}{\chi} D^a \Phi. \quad (\text{A11})$$

We now contract this equation with e_a , using

$$e_a \nabla_b t^{ab} = \nabla_b (e_a t^{ab}) - t^{ab} \nabla_b e_a = \frac{1}{\chi} D_b (\chi e_a t^{ab}) - t^{ab} D_b e_a. \quad (\text{A12})$$

Here we have used the identity

$$\nabla_b v^b = \frac{1}{\chi} D_b (\chi v^b) \quad (\text{A13})$$

that holds for any vector field v^a in the class that projects to Ξ , and we also changed ∇_b to D_b in the second term since t^{ab} has both indices perpendicular to ξ^a . We thus obtain

$$0 = \rho e^a D_a \chi + q e^a D_a \Phi + D_b [\chi t^{ab} e_a] - \chi t^{ab} D_b e_a. \quad (\text{A14})$$

By construction, $D_b e_a$ vanishes at the point where the string is attached to the box. If the geometry of Ξ were flat, then $D_b e_a$ would vanish identically throughout the box. Since the geometry of Ξ is not flat, $D_b e_a$ is, in general, nonvanishing. However, its magnitude is bounded by the size of the box times the curvature of Ξ . Therefore, for a box whose size is small compared with the scales set by the curvature of Ξ , the last term in Eq. (A14) will be negligible compared with the other terms in that equation. Therefore, we shall drop this term.

Integrating the remaining three terms in Eq. (A14) over the volume, V , enclosed by C and using Gauss’s law, we obtain

$$0 = \int_V [\rho e^a D_a \chi + q e^a D_a \Phi] + \int_C \chi t^{ab} e_a n_b dS \quad (\text{A15})$$

where the natural volume elements (with respect to h_{ab}) on V and C are understood, and n^a is the unit, outward pointing normal to C . We now “shrink C down” so that it just barely encloses the box and string. In this limit, the volume integral receives no contribution from the string (since we assume that $\rho = q = 0$ on the string), and the surface term also receives no contribution from the portion surrounding the string (since the area of this portion goes to zero), except for the contribution $\int \chi P = \chi_0 \int P$ arising from the end point of the string held by the external agent, where χ_0 denotes the value of χ at this end point. Since $F^a = -e^a \int P$ is just the force that the external agent must exert to counterbalance the tension/pressure of the string and thereby hold the box in place, we obtain the desired general expression for the force needed to hold the box in place,

$$\chi_0 F^a = e^a \left(\int_B [\rho e^b D_b \chi + q e^b D_b \Phi] + \int_{\partial B} \chi t^{cb} e_c n_b dS \right) \quad (\text{A16})$$

where the volume integral is now taken over the box, B , and the surface integral is taken over the boundary of the box, ∂B . The first term in the volume integral can be interpreted as the “weight” of the contents of the box. Note that ρ includes only the non-electromagnetic energy density, i.e., neither the electromagnetic self-energy of charges within the box nor their interaction energy with external electromag-

netic fields contributes to the first term. The second term is just the Lorentz force on the charge distribution in the box (including “self-force” effects). The final term corresponds to the buoyancy force exerted on the box by the matter surrounding the box.

We now specialize this result to the case of a box held near a black hole, which is surrounded by the thermal atmosphere of the black hole. In this case, we take ξ^a to be the horizon Killing field, Eq. (34). However, there is no reason to expect the true, renormalized charge-current, j^a , will be of the form (A4) nor do we expect the renormalized nonelectromagnetic stress-energy tensor, T^{ab} , to be of the form (A6), since the charge-current and stress-energy of the Boulware vacuum would not be expected to have this form. However, it seems reasonable to expect that the differences, \tilde{j}^a and \tilde{T}^{ab} , between the true charge-current and stress-energy and those of the Boulware vacuum [see Eqs. (51) and (52) above] will have this form. Now, the total stress-energy, $(T_0)^{ab}_{\text{total}}$, of the Boulware vacuum must be conserved, so $\tilde{T}_{\text{total}}^{ab}$ also is conserved. If treat both T_{EM}^{ab} and $(T_0)^{ab}_{\text{EM}}$ as small perturbations of the electromagnetic stress-energy tensor of the black hole (so that only linear terms in the deviation from the back-

ground black hole electromagnetic stress-energy are kept), then a repetition of the steps in the above derivation shows that in this case, Eq. (A16) continues to hold, provided only that ρ , q , and t^{ab} are replaced by their “tilded” counterparts, i.e., we have

$$\chi_0 F = \int_B [\tilde{\rho} e^a D_a \chi + \tilde{q} e^a D_a \Phi] + \int_{\partial B} \chi \tilde{t}^{ab} e_a n_b dS \quad (\text{A17})$$

where $F = F^a e_a$ and $\Phi = -A_a \xi^a$ with A_a is the vector potential of the background black hole.

Finally, for the case where the thermal atmosphere surrounds the box, \tilde{T}^{ab} outside of the box will have a perfect fluid form, so Eq. (A17) further simplifies to

$$\chi_0 F = \int_B [\tilde{\rho} e^a D_a \chi + \tilde{q} e^a D_a \Phi] + \int_{\partial B} \chi \tilde{P} e_a n^a dS. \quad (\text{A18})$$

For the case of an empty box ($\tilde{\rho} = \tilde{q} = 0$), we obtain Eq. (92) used in our analysis in Sec. V.

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