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Curved dilatonic brane worlds

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We construct a broad family of exact solutions to the five-dimensional Einstein equations coupled to a scalar field with an exponential potential. Embedding a three-brane in these bulk space-times in a particular way, we obtain a class of self-tuned curved brane worlds in which the vacuum energy on the brane is *gravitationally idle*, the four-dimensional geometry being insensitive to the value of the brane tension. This self-tuning arises from cancellations, enforced by the junction conditions, between the scalar field potential, the brane vacuum energy, and the matter on the brane. Finally, we study some physically relevant examples and their dynamics.

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I. INTRODUCTION

Brane worlds, the models in which our Universe appears as some kind of domain wall or brane embedded in a higher-dimensional space-time, have gained considerable attention recently. Although the idea goes back to the 1980s [1], its current appeal comes from several directions. As shown by Hořava and Witten [2], the $E_8 \times E_8$ heterotic string theory at strong coupling is described in terms of M theory in an eleven-dimensional space-time with boundaries, where the ten-dimensional gauge degrees of freedom exist on the "branes at the end of the world." This confinement of the gauge fields to a lower-dimensional submanifold, in contrast with the gravitational field that can propagate into the bulk, might help to explain the hierarchy between the electroweak and the Planck scales in four dimensions [3]. With the aim of solving the hierarchy problem, Randall and Sundrum [4] put forward a proposal, inspired by the AdS/CFT correspondence, where our four-dimensional universe is embedded in a nonfactorizable way into five-dimensional anti–de Sitter (AdS) space-time. Gravity in this scenario is "trapped" on the four-dimensional brane due to the geometry of the bulk space-time [4,5].

The phenomenological viability of brane world models is now being extensively discussed in the literature [6]. In addition to the possible imprints of these models detectable in high-energy experiments, cosmology naturally emerges as a very promising arena to study the possible consequences of living inside a brane (for an incomplete list of references see [7–9]). One of the important issues in cosmology is to explain how our homogeneous and isotropic universe could have emerged from "generic" initial conditions. A popular mechanism to address this question is to have a certain period of inflation in the early universe, during which the possible initial anisotropies and inhomogeneities would be smoothed out. In the context of the brane world the issue of inflation [7,8,10] as well as the perturbative deviations from isotropy [11,12] have been investigated recently.

One of the most interesting consequences of considering our universe as a brane inside a higher-dimensional space-time is that the Einstein equations in four dimensions do not form a closed system [11]. As a consequence, for a four-dimensional observer it is not sufficient to know the distribution of energy-matter in her/his universe to determine its geometry, the missing element coming from the geometrical features of the space-time outside the four-dimensional universe. This "out of this world" ingredient to the right-hand side of the Einstein equations is crucial in analyzing the cosmological dynamics of the universe in four dimensions. However, in many instances in the literature the five-dimensional solution in the bulk associated with a four-dimensional brane cosmology is not known and its effects on the brane world have to be modeled using some simplifying assumptions, or neglected altogether.

Thus, in order to gauge to what extent the five-dimensional geometry influences the cosmological dynamics on the brane, it is important to consider exact bulk solutions with various deviations from homogeneity and isotropy (see, for example, [13–15] for some studies in this direction). In this paper we will propose a systematic way of constructing five-dimensional homogeneous and inhomogeneous cosmologies coupled to a scalar field with an exponential potential. We will construct brane cosmologies using these bulk metrics and study the effect that the bulk dynamics has on the cosmological evolution of the brane world.

The cosmological constant problem remains a central issue to be solved in theoretical physics [16]. Brane cosmology provides new approaches that might help in the solution of this long-standing problem [17]. One of the proposals recently put forward is a self-tuning mechanism that tunes the four-dimensional cosmological constant to zero, indepen-

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dently of the value of the cosmological constant in the bulk [18–21]. In this paper we propose rather a new realization of this self-tuning property in which vacuum energy on the brane cancels despite the ''bare'' four-dimensional cosmological constant being nonzero. This is due to a nontrivial counterbalance between the nonvanishing ''bare'' cosmological term and the matter induced on the brane.

The paper is outlined as follows. In the next section we will briefly describe the dynamics of dilaton brane cosmologies. After this, in Sec. III, we present a solution generating technique to build five-dimensional scalar cosmologies with exponential potential starting with a vacuum solution in five dimensions. Section IV will be devoted to the study of brane cosmologies embedded in the class of five-dimensional metrics obtained here, and how for these particular embeddings there is a self-tuning mechanism at work, the four-dimensional geometry being independent of the vacuum energy on the brane and the value of the scalar field potential. In Sec. V we illustrate our discussion with some physically interesting examples, and in Sec. VI we summarize our results.

II. BRANE DYNAMICS

In the spirit of the brane world picture, we assume that the four-dimensional universe is described by a domain wall (M,g) located at some hypersurface $Y(x^A)=0$ in the five-dimensional bulk space-time $(\mathcal{M},\mathcal{G})$. The only matter in the bulk will be a massless scalar field with an exponential potential. Therefore, the action governing the dynamics is[1]

$$S_{5\mathrm{D}}=\int d^5x\sqrt{-\mathcal{G}}\left[\frac{1}{2\kappa_5^2}\mathcal{R}-\frac{1}{2}\partial_A\phi\partial^A\phi-\Lambda e^{-(2/3)k\phi}\right]+\int_{Y=0}d^4x\sqrt{-g}L_{\mathrm{brane}}, \tag{1}$$

where k and Λ are constants and

$$L_{\mathrm{brane}}=-\lambda(\phi)+\frac{1}{\kappa_5^2}K^{\pm}+e^{4b\phi}L(e^{2b\phi}g_{\mu\nu},\ldots)_{\mathrm{matter}}.$$

Here $\lambda(\phi)$ is a ϕ-dependent vacuum energy (i.e., tension) on the brane, $K^{\pm}$ is the extrinsic curvature on either side of the brane, and L_{matter} is the Lagrangian of the matter degrees of freedom confined to the brane world minimally coupled to the metric $e^{2b\phi}g_{\mu\nu}$, $b\in\mathbf{R}$ (cf. [22]). In the following we use units in which $\kappa_5=1$. The induced metric on the brane is the projection of the five-dimensional metric onto the brane world. If we denote by n^A the unit spacelike vector normal to the brane, the four-dimensional metric will be given by

$$g_{AB}=\mathcal{G}_{AB}-n_An_B.$$

Because of the embedding there will be two sources of curvature for the four-dimensional universe. One will be the intrinsic curvature induced by the ambient space-time and it will be given by the projection of the five-dimensional Riemann tensor onto the brane. The second one is due to the embedding itself and it is governed by the extrinsic curvature $K_{\mu\nu}=g_\mu^C g_\nu^D\nabla_C n_D$. Using the Gauss-Codazzi equations [23] we can write the Einstein equations in four dimensions as [22,24,25] (for a review see [26])

$$\begin{aligned}R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=&\tfrac{2}{3}[\mathcal{T}_{AB}g_\mu^A g_\nu^B+(\mathcal{T}_{AB}n^An^B-\tfrac{1}{4}g^{AB}\mathcal{T}_{AB})g_{\mu\nu}]\\&+KK_{\mu\nu}-K_\mu{}^\sigma K_{\nu\sigma}-\tfrac{1}{2}g_{\mu\nu}(K^2-K^{\alpha\beta}K_{\alpha\beta})\\&-E_{\mu\nu},\end{aligned} \tag{2}$$

where we have denoted by $K\equiv K_\mu^\mu$ the trace of the second fundamental form, $E_{\mu\nu}$ is written in terms of the five-dimensional Weyl tensor $\mathcal{C}^A_{BCD}$ as

$$E_{\mu\nu}=\mathcal{C}^E{}_{AFB}n_En^Fg_\mu^Ag_\nu^B,$$

and the five-dimensional energy-momentum tensor in the bulk derived from Eq. (1) is

$$\mathcal{T}_{AB}=\partial_A\phi\partial_B\phi-\mathcal{G}_{AB}(\tfrac{1}{2}\partial_C\phi\partial^C\phi+\Lambda e^{-(2/3)k\phi}). \tag{3}$$

It is important to point out that the right-hand side of Eq. (3) can be evaluated on any side of the brane, the Einstein tensor on the brane being uniquely defined. In the $\mathbf{Z}_2$-symmetric case to be studied below, both the extrinsic curvature and the derivatives of the dilaton field on the two sides of the brane differ just by a sign, so every term in the right-hand side of Eq. (3) is well defined on the brane. If the reflection symmetry is relaxed, on the other hand, the uniqueness of the Einstein tensor is ensured in a nontrivial way [27].

In addition to this, the scalar field will satisfy the wave equation

$$\nabla^2\phi+\frac{2k}{3}\Lambda e^{-(2k/3)\phi}=\frac{\sqrt{-g}}{\sqrt{-\mathcal{G}}}[\lambda'(\phi)-bg^{\mu\nu}\tau_{\mu\nu}]\delta(Y), \tag{4}$$

where $\tau_{\mu\nu}$ is the energy-momentum tensor of the matter action, as derived from $e^{4b\phi}L_{\mathrm{matter}}$, and the prime denotes differentiation with respect to ϕ. Upon projection we obtain the four-dimensional equation for the dilaton

$$D_\mu D^\mu\phi-a_C\partial^C\phi+K\mathcal{L}_n\phi+\mathcal{L}_n^2\phi+\frac{2k}{3}\Lambda e^{-(2/3)k\phi}=\frac{\sqrt{-g}}{\sqrt{-\mathcal{G}}}[\lambda'(\phi)-bg^{\mu\nu}\tau_{\mu\nu}]\delta(Y) \tag{5}$$

with D_μ the covariant derivative with respect to the induced metric, $a^C=n^B\nabla_Bn^C$, and $\mathcal{L}_n\phi=n^A\nabla_A\phi$ the Lie derivative in the direction n^A.

[1]In the following we will use capital Latin indices for the five-dimensional coordinates, whereas the coordinates on the brane world will be denoted by Greek indices. To avoid the use of super-indices to indicate the dimension, tensors in five dimensions will be indicated by capital script letters and their four-dimensional counterparts will be denoted by the corresponding italic types.

As usual, we will take coordinates (χ,x^{μ}) in such a way that the brane world lies on the hypersurface defined by $\chi=0$. For later convenience we will consider that the five-dimensional metric takes the form

$$ds^2_{5\mathrm{D}}=N(x,\chi)^2d\chi^2+g_{\mu\nu}(x,\chi)dx^{\mu}dx^{\nu},$$

where the "shift" function $N(x,\chi)$ depends on all five-dimensional coordinates. Therefore $n\equiv N(x,\chi)^{-1}\partial_{\chi}$ and, as a consequence,

$$K_{\mu\nu}=\frac{1}{2N(x,\chi)}\,\partial_{x}g_{\mu\nu}(x,\chi),$$

$$a_{\mu}=-\frac{1}{N(x,\chi)}\,\partial_{\mu}N(x,\chi),\quad a_{\chi}=0,$$

and $E_{\mu\nu}=\mathcal{C}_{\mu\chi\nu}{}^{\chi}$.

The discontinuity of the derivatives of the metric across the brane due to the energy-momentum localized on the hypersurface $Y(x^A)=0$ is given by the Israel junction conditions [24,28]. They relate the jump in the first derivative of the metric at $\chi=0$ to the total energy-momentum tensor on the brane, namely,

$$[K_{\mu\nu}]=-(S_{\mu\nu}-\tfrac{1}{3}g_{\mu\nu}S),\tag{6}$$

where we have used the usual notation $[A]\equiv A^{+}-A^{-}$, and $S_{\mu\nu}$ is the total brane energy-momentum tensor:

$$S_{\mu\nu}=-\frac{2}{\sqrt{-g}}\,\frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}L_{\mathrm{brane}})\equiv-\lambda(\phi)g_{\mu\nu}+\tau_{\mu\nu}.\tag{7}$$

In a similar fashion we can find the jump in the derivative of the scalar field by integrating Eq. (4) across $\chi=0$: namely,

$$[\partial_{\chi}\phi]=N(x,0)[\lambda'(\phi)-bg^{\mu\nu}\tau_{\mu\nu}].\tag{8}$$

The Einstein equations in four dimensions can be now written from Eq. (2) as

$$\begin{aligned}R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=&\tfrac{2}{3}[\partial_{\mu}\phi\partial_{\nu}\phi+\tfrac{5}{8}(\partial\phi)^2g_{\mu\nu}]+\frac{(\partial_{\chi}\phi)^2}{4N(x,0)^2}g_{\mu\nu}\\&-\tfrac{1}{2}\Lambda e^{-(2/3)k\phi}g_{\mu\nu}+KK_{\mu\nu}-K_{\mu}{}^{\sigma}K_{\nu\sigma}\\&-\tfrac{1}{2}g_{\mu\nu}(K^2-K^{\alpha\beta}K_{\alpha\beta})-E_{\mu\nu}.\end{aligned}\tag{9}$$

As we discussed above, the only discontinuities on the right-hand side of Eq. (9) are contained in the extrinsic curvature $K_{\mu\nu}$, the derivative of the dilaton field "normal" to the brane $\partial_{\chi}\phi$, and, eventually, the potential. These terms have to be evaluated at any side of the brane, the sum of them being independent of the side chosen. All other terms involving the scalar field and their "tangent" derivatives are continuous and can be evaluated individually without ambiguity on the hypersurface $\chi=0$.

If we assume that our brane world is at the fixed point of a $\mathbf{Z}_2$ orbifold, as is the case in the Hořava-Witten scenario, $K^{+}_{\mu\nu}=-K^{-}_{\mu\nu}$ and the Israel junction condition (6) completely determines the extrinsic curvature in terms of the energy-momentum tensor on the brane. In the same way $(\partial_{\chi}\phi)^2$ can be read from Eq. (8). Substitution into Eq. (9) leads then to (cf. [22])

$$\begin{aligned}R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=&\tfrac{2}{3}[\partial_{\mu}\phi\partial_{\nu}\phi-\tfrac{5}{8}g_{\mu\nu}(\partial\phi)^2]+\tfrac{1}{6}\lambda(\phi)\tau_{\mu\nu}\\&-\Lambda_4g_{\mu\nu}-\tfrac{1}{16}[2\lambda'(\phi)-b\tau]b\tau g_{\mu\nu}+\pi_{\mu\nu}\\&-E_{\mu\nu},\end{aligned}\tag{10}$$

where $\tau\equiv g^{\alpha\beta}\tau_{\alpha\beta}$ and the four-dimensional cosmological constant and the tensor $\pi_{\mu\nu}$ are given, respectively, by

$$\Lambda_4=\tfrac{1}{2}[\Lambda e^{-(2k/3)\phi}+\tfrac{1}{6}\lambda(\phi)^2-\tfrac{1}{8}\lambda'(\phi)^2],$$

$$\pi_{\mu\nu}=\tfrac{1}{12}\tau\tau_{\mu\nu}+\tfrac{1}{8}g_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta}-\tfrac{1}{4}\tau_{\mu\alpha}\tau^{\alpha}_{\nu}-\tfrac{1}{24}\tau^2g_{\mu\nu}.\tag{11}$$

Looking at the Einstein equations on the brane, Eq. (10), we find that in general the dynamics of the four-dimensional universe is not uniquely determined by the distribution of energy inside the universe as encoded in $\tau_{\mu\nu}$ [11,22], as it is the case with "ordinary" Einstein equations. Indeed, the only ingredient on the right-hand side of Eq. (10) that cannot be related to the matter content of the four-dimensional universe (i.e., either four-dimensional matter or the scalar field ϕ) is the tensor $E_{\mu\nu}$, which is determined by the five-dimensional Weyl tensor. In order to study the influence of the five-dimensional geometry on the cosmological evolution of the brane world via $E_{\mu\nu}$, it is necessary to consider not only the four-dimensional metric on the brane but also the higher-dimensional ambient geometry. Thus, to address this problem, we will proceed to construct explicitly five-dimensional solutions to the Einstein equations in which our brane worlds will be embedded.

III. SCALAR FIELD COSMOLOGIES IN FIVE DIMENSIONS WITH AN EXPONENTIAL POTENTIAL

Since we will be interested in studying the effects of the bulk on the four-dimensional brane world, our starting point will be the five-dimensional geometry in which the brane is embedded. In particular, we want to consider solutions to the equations of motion derived from the bulk terms in Eq. (1), i.e., geometries coupled to a massless scalar field with a Liouville potential.

Such cosmologies can be constructed using a higher-dimensional generalization of the theorem presented in Ref. [29]. Related models were studied by Lidsey [30] in the context of heterotic M theory; some particular examples of constant curvature dilatonic branes were also discussed in [31]. Let us consider a *vacuum* solution to the Einstein equations in five dimensions of the form

$$ds^2_{(\mathrm{vac,5D})}=\epsilon e^{4Q(x)}d\chi^2+e^{-2Q(x)}h_{\mu\nu}(x)dx^{\mu}dx^{\nu},\tag{12}$$

where $\epsilon=\pm1$ depending on whether χ is a spatial direction or the time coordinate, and the metric functions $Q(x)$ and $h_{\mu\nu}(x)$ are independent of the χ coordinate. The new metric

$$ds^2_{5D}=\epsilon e^{(4k/\sqrt{k^2+6})Q(x)+a_1\xi\chi}d\chi^2 + e^{-(2k/\sqrt{k^2+6})Q(x)+a_2\xi\chi}h_{\mu\nu}(x)dx^\mu dx^\nu \quad (13)$$

and the scalar field

$$\phi(x,\chi)=\frac{6}{\sqrt{k^2+6}}Q(x)+\begin{cases}\dfrac{3k\xi}{k^2-3}\chi, & k^2\neq 3\\ \pm\sqrt{3}\xi\chi, & k^3=3,\end{cases} \quad (14)$$

solve Einstein equations coupled to a massless scalar field with potential $V(\phi)=\Lambda e^{-(2/3)k\phi}$ where

$$\Lambda=\begin{cases}-\dfrac{9}{2}\dfrac{12-k^2}{(k^2-3)^2}\xi^2, & k^2\neq 3,\\ -\dfrac{9}{2}\xi^2, & k^2=3,\end{cases}$$

and

$$a_1=\frac{k^2}{3}a_2\equiv\frac{2k^2}{k^2-3} \quad \text{if } k^2\neq 3,$$
$$a_1=a_2\equiv 2 \quad \text{if } k^2=3. \quad (15)$$

Finally, ξ fixes the scale of the cosmological constant Λ.

Although some algebra is involved, the result can easily be proved following the same steps as in the four-dimensional case [29]; we therefore leave this to the reader. Using the theorem we can construct five-dimensional dilaton gravity solutions with a cosmological constant (negative when $k^2<12$). At a glance, there are several interesting values of k for which the metric (13) admits different physical interpretations. The first one is $k^2=12$ when the potential for the scalar field vanishes. The second and more interesting one is $k=0$, at which the coupling between the dilaton field and the cosmological constant is zero and we are left with a negative cosmological constant $\Lambda=-6\xi^2$. In this case the geometries are characterized by the line element

$$ds^2_{k=0}=\epsilon\, d\chi^2+e^{-2\xi\chi}h_{\mu\nu}(x)dx^\mu dx^\nu \quad (16)$$

which solves the Einstein equations with a negative cosmological constant and a massless scalar field $\phi(x)=\sqrt{6}Q(x)$. Taking $\epsilon=1$ and χ as the bulk coordinate, we are provided with generalizations of the Randall-Sundrum model with a generic four-dimensional metric and a massless scalar field. If $k^2=18$ and $\epsilon=1$ we obtain solutions to the low-energy field equations of the $E_8\times E_8$ heterotic string at strong coupling compactified on a threefold Calabi-Yau space, with the scalar field ϕ representing the breathing mode of the internal manifold [30,32].

Whenever $\epsilon=1$ it can be easily realized that the vacuum five-dimensional metric (12) can be thought of as the "oxidation" of a four-dimensional massless scalar field cosmology with a scalar field $\psi(x)_{4D}=\sqrt{6}Q(x)$ (cf., for example, [33]). Therefore in order to get Eq. (12) one can start with *any* four-dimensional solution to the Einstein scalar equations. If $k=0$ the previous results tell us that *any* four-dimensional metric coupled to a massless scalar field can be embedded into a five-dimensional bulk space-time with negative cosmological constant by the ansatz (16) with $\epsilon=1$. Finally, in the case $Q(x)=0$ the Einstein vacuum equations for Eq. (12) imply that $h_{\mu\nu}$ has to be a Ricci flat four-dimensional metric. For the particular choice $h_{\mu\nu}=\eta_{\mu\nu}$ we recover the family of metrics considered in [18] after the obvious change of coordinates $dx_5=\exp(\frac{1}{2}a_1\xi\chi)d\chi$ and a rescaling of the dilaton field. On the other hand, taking $h_{\mu\nu}$ to be the Schwarzschild metric in four dimensions we can construct embeddings of four-dimensional black holes in five dimensions of the form

$$ds^2=e^{a_1\xi\chi}d\chi^2+e^{a_2\xi\chi}\left[-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-2M/r}+r^2d\Omega_2^2\right],$$

which, for $k=0$, corresponds to the AdS black string considered by Chamblin, Hawking, and Reall [34]. When $k\neq 0$ the four-dimensional black hole is embedded into a five-dimensional bulk space-time with a nontrivial profile for the dilaton field, which, on the other hand, is constant on the brane.

IV. BRANE COSMOLOGIES WITH IDLE VACUUM ENERGY

In the previous section we constructed a generic procedure to obtain five-dimensional cosmologies coupled to a scalar field with an exponential potential (or a negative cosmological constant). The final aim is to use these solutions as bulk geometries of four-dimensional brane worlds. As we reviewed in Sec. II, given the solution in the bulk, the matter/energy content of the brane world is strongly constrained by the junction conditions for both the metric and the scalar field.

Among the different possibilities for embedding a four-dimensional brane world in the five-dimensional solution (13) the simplest ones correspond to taking the codimension of the brane along one of the four spacelike coordinates.

Let us consider the solutions (13) and (14) with $\epsilon=1$ so that the brane world lies on the hypersurface defined by the equation $\chi=0$. This choice leads naturally to warped geometries that generalize the Randall-Sundrum construction to include a scalar field with a Liouville potential.

We begin by assuming $\mathbf{Z}_2$ symmetry around the location of the brane at $\chi=0$. In this case we are led to the following nonfactorizable geometry in five dimensions:

$$ds^2_{5D}=e^{(4k/\sqrt{k^2+6})Q(x)+a_1\xi|\chi|}d\chi^2 + e^{-(2k/\sqrt{k^2+6})Q(x)+a_2\xi|\chi|}h_{\mu\nu}(x)dx^\mu dx^\nu \quad (17)$$

with a_1 and a_2 given by Eq. (15). The dilaton, on the other hand, is given by

$$\phi(x,\chi)=\varphi(x)+\begin{cases}\dfrac{3k\xi}{k^2-3}|\chi|, & k^2\neq 3,\\ \pm\sqrt{3}\xi|\chi|, & k^2=3,\end{cases} \tag{18}$$

where $\varphi(x)=(6/\sqrt{k^2+6})Q(x)$.

In principle, for generic $k\neq 0$, the metric (17) can have a curvature singularity located at $\chi=\pm\infty$ (cf. [18,35]), in addition to the possible singularities of the metric $h_{\mu\nu}$. Depending on the value of k and on the particular solution considered, this singularity can be harmless provided it is located at an infinite proper distance from the brane location. It can easily be seen that this is the case whenever $k^2\geqslant 3$. However, in this case the four-dimensional Planck scale obtained by integrating out the coordinate χ in the five-dimensional action (1) diverges. On the other hand, when $k^2<3$ the singularity is located at finite proper distance from the brane but the four-dimensional Planck scale is finite as well. Whenever needed, a second brane can be located at $\chi_0>0$, imposing reflection symmetry around this point, so the bulk coordinate is restricted to the interval $[0,\chi_0]$ and the singularity is screened.

Since we have completely determined the bulk geometry and assumed reflection symmetry around $\chi=0$, the junction conditions (6) will determine the energy-momentum tensor of the matter fields on the brane. Imposing the matching condition on the scalar field (8) at the brane location for the dilaton (18) we find the following relation:

$$\lambda'(\varphi)-bg^{\mu\nu}\tau_{\mu\nu}=\begin{cases}\dfrac{6k\xi}{k^2-3}e^{-(k/3)\varphi}, & k^2\neq 3,\\ \pm 2\sqrt{3}\xi e^{\mp(1/\sqrt{3})\varphi}, & k^2=3,\end{cases} \tag{19}$$

where we have used the fact that in our case the shift function $N(x,\chi)$ can be written in terms of the scalar field as

$$N(x,\chi)=e^{(k/3)\phi(x,\chi)}. \tag{20}$$

Moreover, using Eq. (19) and the expression of the extrinsic curvature of the hypersurface $\chi=0$ embedded in the metric (17), $K_{\mu\nu}=\frac{1}{2}a_2\xi e^{-(k/3)\varphi}g_{\mu\nu}$, we can write a differential equation for $\lambda(\varphi)$:

$$\lambda'(\varphi)-4b\lambda(\varphi)=\begin{cases}6\xi\dfrac{k+12b}{k^2-3}e-(k/3)\varphi, & k^2\neq 3,\\ 2\xi(12b\pm\sqrt{3})e^{\mp(1/\sqrt{3})\varphi}, & k^2=3.\end{cases} \tag{21}$$

This equation can be solved to get the functional dependence of the vacuum energy on the brane with the dilaton field

$$\lambda(\varphi)=-\rho_0e^{4b\varphi}-\begin{cases}\dfrac{18\xi}{k^2-3}e^{-(k/3)\varphi}, & k^2\neq 3,\\ 6\xi e^{\mp(1/\sqrt{3})\varphi}, & k^2=3,\end{cases} \tag{22}$$

where ρ_0 is an integration constant. We can now substitute Eq. (22) into Eqs. (7) and (6) to obtain the form of the energy-momentum tensor $\tau_{\mu\nu}$. Proceeding in this way we find

$$\tau^{\mu}_{\nu}=-\rho_0e^{4b\varphi}\delta^{\mu}_{\nu}. \tag{23}$$

Therefore the matter content of the brane described by L_{matter} has as its effective equation of state $p=-\rho_0$. Notice that the form of the energy-momentum tensor is independent of the value of k and therefore valid for *all* brane models constructed using the solution generating technique of Sec. III, and that the constant energy density and the equation of state do not depend on the value of the dilaton coupling b. We now evaluate the tensor $\pi_{\mu\nu}$ that appears on the right-hand side of Eq. (10). The result is

$$\pi_{\mu\nu}=-\tfrac{1}{12}\rho_0^2e^{8b\varphi}g_{\mu\nu}.$$

With these expressions for the tensors appearing on the right-hand side of Eq. (10) we can study the source of the brane gravitational field. Remarkably, we find that

$$\tfrac{1}{6}\lambda(\varphi)\tau_{\mu\nu}-\Lambda_4g_{\mu\nu}-\tfrac{1}{16}[2\lambda'(\varphi)-b\tau]b\tau g_{\mu\nu}+\pi_{\mu\nu}=0, \tag{24}$$

so the "effective" energy-momentum tensor in Eq. (10) receives contributions only from the dilaton field and the tidal bulk effects

$$T^{\text{eff}}_{\mu\nu}=\tfrac{2}{3}[\partial_\mu\varphi\partial_\nu\varphi-\tfrac{5}{8}g_{\mu\nu}(\partial\varphi)^2]-E_{\mu\nu}.$$

An interesting thing to notice about the cancellation (24) is that the four-dimensional cosmological constant, as defined by Eq. (11), is different from zero. Only when $\rho_0=0$ and $Q(x)=0$ does the four-dimensional cosmological constant Λ_4 vanish exactly independently of the value of the cosmological constant in the bulk. A particular case is the Randall-Sundrum model [4], where $h_{\mu\nu}=\eta_{\mu\nu}$, $k=0$, and $\lambda^2=-6\Lambda$. In the generic case, however, Λ_4 is canceled by the matter contributions encoded in the energy-momentum tensor and the dilaton-induced terms on the right-hand side of Eq. (10). This counterbalance of the four-dimensional vacuum energy by matter cosmological terms is imposed by the junction conditions, representing a different realization of the self-tuning mechanism of [18,19].

Note, however, that in the case at hand we are not making any *a priori* assumption about the form of the energy-momentum tensor on the brane $\tau_{\mu\nu}$. The Israel junction condition, together with the jump condition for the dilaton across $\chi=0$, force it to be of vacuum type, the net cosmological term (24) being zero independently of the value of ρ_0, b, and ξ.

For five-dimensional metrics of the form (17) we can evaluate the tensor $E_{\mu\nu}$ in terms of the curvature and the scalar field. The result is

$$E_{\mu\nu}=-\frac{1}{3}\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)-\frac{2k^2}{27}\left[\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{4}(\partial\varphi)^2g_{\mu\nu}\right]-\frac{2k}{9}\left(D_\mu D_\nu\varphi-\frac{1}{4}D^2\varphi g_{\mu\nu}\right). \tag{25}$$

Using this expression the effective energy-momentum tensor driving the four-dimensional geometry can be written solely in terms of the four-dimensional projection of the scalar field $\varphi(x)$. The four-dimensional equations are given by

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{2}g_{\mu\nu}(\partial\varphi)^2+\frac{k}{3}\left(D_\mu D_\nu\varphi-\frac{1}{4}g_{\mu\nu}D^2\varphi\right)+\frac{k^2}{9}\left[\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{4}g_{\mu\nu}(\partial\varphi)^2\right]. \tag{26}$$

The first important thing to notice is that when $k=0$ we recover an ordinary four-dimensional scalar cosmology with $\varphi(x)=\sqrt{6}Q(x)$. Actually, by looking at the original ansatz for the vacuum five-dimensional metric in Eq. (12) we realize that this is exactly the four-dimensional geometry whose "oxidation" produces Eq. (12). This is yet another way to see our previous conclusion, that any four-dimensional cosmology coupled to a massless scalar field can be trivially embedded into a five-dimensional cosmology with negative cosmological constant by the ansatz (16). In this case $E_{\mu\nu}$ has a purely "geometrical" origin, being determined just by the four-dimensional curvature, as can be seen from Eq. (25).

If $k\neq 0$ things get more involved, however, since now the effective energy-momentum tensor in Eq. (26) cannot be interpreted as that of a minimally coupled massless scalar field. The nontrivial cosmological dynamics in the χ coordinate translates into Brans-Dicke-like terms in the energy-momentum tensor (cf. [36]). It is interesting to notice that in the large-$|k|$ limit, even if the dilaton $\varphi(x)$ is switched off, there is a residual contribution coming from the last two terms in Eq. (26). Since $\varphi(x)\sim 1/|k|$ for large $|k|$ the weakness of the scalar field is compensated by the extra powers of k in Eq. (26) so a source for the gravitational field is left in that limit, given by the terms in the "effective" energy-momentum tensor with explicit k dependence.

As it turns out, the four-dimensional cosmologies can be conformally related to the equations of low-energy string cosmology. In order to see this note that the vacuum Einstein equations for Eq. (12) imply that $D_h^2\varphi=0$, where the covariant derivative is defined with respect to the metric $h_{\mu\nu}$; when written in terms of the four-dimensional metric $g_{\mu\nu}=e^{-(k/3)\varphi}h_{\mu\nu}$ the condition translates into $D^2\varphi=-(k/3)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$. In this way, $E_{\mu\nu}$ is given by

$$E_{\mu\nu}=-\left(\frac{1}{3}+\frac{k^2}{9}\right)\partial_\mu\varphi\partial_\nu\varphi+\frac{1}{12}g_{\mu\nu}(\partial\varphi)^2-\frac{k}{3}D_\mu D_\nu\varphi,$$

whereas the four-dimensional Einstein equations can be written as

$$R_{\mu\nu}=\partial_\mu\varphi\partial_\nu\varphi+\frac{k}{3}D_\mu D_\nu\varphi+\frac{k^2}{9}\partial_\mu\varphi\partial_\nu\varphi.$$

Dilaton gravity is then recovered through the field redefinitions (cf. [30])

$$\overline{g}_{\mu\nu}=\exp\left[\left(\frac{k}{3}\pm\sqrt{(k^2+6)/3}\right)\varphi\right]g_{\mu\nu},$$
$$\Phi=\pm\sqrt{(k^2+6)/3}\varphi, \tag{27}$$

so we arrive at the usual equations for dilaton gravity in the string frame (with the normalization $\mathcal{L}_{\rm grav}\sim e^{-\Phi}\overline{R}$)

$$\overline{R}_{\mu\nu}=-\overline{D}_\mu\overline{D}_\nu\Phi,$$
$$\overline{D}^2\Phi=\overline{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi. \tag{28}$$

An important stage in the evolution of the early Universe is inflation. As shown above the solutions on the brane can be conformally related to low-energy string cosmology in the string frame. With one further conformal transformation they can, of course, be related to the usual Einstein frame in which the metric is just $h_{\mu\nu}$ and the physics is described by Einstein relativity coupled to a massless scalar field $\psi(x)_{\rm 4D}=\sqrt{6}Q(x)$. In this frame standard inflation does not take place. However, in the string frame there is the possibility of implementing inflation in the framework of the pre-big-bang scenario [37]. In particular, if the four-dimensional metric has a spacelike Killing vector, scale factor duality along that direction will be a symmetry of Eqs. (28).

It is interesting to note that from the point of view of a five-dimensional observer there are inflating solutions on the brane. Using the form of the energy-momentum tensor in five dimensions the strong energy condition reads, for any timelike vector $t^A(t_At^A=-1)$,

$$0<\mathcal{R}_{AB}t^At^B=(t^A\partial_A\phi)^2-\tfrac{2}{3}\Lambda e^{-(2/3)k\phi}.$$

This condition can in principle be violated for $\Lambda>0$, i.e., if $k^2>12$.

V. EXAMPLES

A. Bianchi type-I brane cosmologies

As an illustrating example we can construct Bianchi type-I brane cosmologies generalizing the analysis of [14] to nonzero profiles for a scalar field with a nontrivial potential. We start with a vacuum Kasner-like line element in five dimensions and apply the algorithm described to find, after a suitable redefinition of the constants, the following solution (we will consider that $k^2\neq 3$; the case $k^2=3$ can easily be obtained):

$$ds^2=t^{(2k/3)\beta}e^{[2k^2\xi/(k^2-3)]|\chi|}d\chi^2+e^{6\xi/(k^2-3)|\chi|}\times(-dt^2+t^{2\alpha_1}dx^2+t^{2\alpha_2}dy^2+t^{2\alpha_3}dz^2),$$

$$\phi(t,\chi)=\beta\log t+\frac{3k\xi}{k^2-3}|\chi|,$$

where the constants α_i and β satisfy the following relations:

$$\sum_{i=1}^{3}\alpha_i=1-\frac{k\beta}{3},\quad \sum_{i=1}^{3}\alpha_i^2=1-\beta^2-\left(\frac{k\beta}{3}\right)^2. \tag{29}$$

We notice that for $k=0$ the usual conditions for a Kasner metric coupled to a massless scalar field are retrieved. As we saw in the general analysis of Sec. IV, for vanishing k the four-dimensional effective energy-momentum tensor is that of a massless scalar field. When $k\neq 0$ the usual Kasner conditions are modified. Looking at the scaling of powers of $k\beta$ in Eq. (29) we see that the first condition is modified by the linear terms in the dilaton field in Eq. (26), whereas the modification of the second condition comes from the quadratic ones. In the Hořava-Witten case the conditions on the Kasner exponents reduce to the ones found in [38].

Looking at the constraints (29) it can easily be seen that there is no volume inflation for any value of the parameter k and β, since the average scale factor scales as t^n with $0<n<\frac{2}{3}$.

The case when $|k|\to\infty$ is of some interest. In this limit, regularity of the five-dimensional metric requires that $\beta k\sim$const so the homogeneous part of the scalar field vanishes. The result is a Bianchi type-I metric in four dimensions where the Kasner exponents α_i satisfy the constraints

$$\sum_{i=1}^{3}\alpha_i=1-C, \quad \sum_{i=1}^{3}\alpha_i^2=1-C^2$$

with $|C|\leqslant 1$ a numerical constant. In this case the (traceless) effective energy-momentum tensor is that of an anisotropic fluid with energy density and pressure given by

$$\rho=\frac{C(1-C)}{t^2}, \quad p_i=\frac{C\alpha_i}{t^2}.$$

B. Friedmann-Robertson-Walker models

Another physically interesting example is the Friedmann-Robertson-Walker (FRW) cosmologies. These can be easily constructed by considering the five-dimensional vacuum line element [33]

$$ds^2_{\mathrm{vac}}=\frac{t^2}{1-\kappa t^2}d\chi^2-dt^2+(1-\kappa t^2)\times\left[\frac{dr^2}{1-\kappa r^2}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right]$$

where $\kappa=0,\pm 1$. Taking χ as the bulk coordinate and applying the algorithm of Sec. III we get, after changing into conformal time, the following four-dimensional metrics:

$$ds^2_{4\mathrm{D}}=a^2(\eta)\left[-d\eta^2+\frac{dr^2}{1-\kappa r^2}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right],$$

where

$$a(\eta)=\begin{cases}(\cosh\eta)^{(\sqrt{k^2+6}+k)/(2\sqrt{k^2+6})}(\sinh\eta)^{(\sqrt{k^2+6}-k)/(2\sqrt{k^2+6})}, & \kappa=-1,\\ \eta^{(\sqrt{k^2+6}-k)/(2\sqrt{k^2+6})}, & \kappa=0,\\ (\cos\eta)^{(\sqrt{k^2+6}+k)/(2\sqrt{k^2+6})}(\sin\eta)^{(\sqrt{k^2+6}-k)/(2\sqrt{k^2+6})}, & \kappa=1,\end{cases} \tag{30}$$

while the dilaton field is

$$\varphi(\eta)=\begin{cases}\frac{3}{\sqrt{k^2+6}}\log\tanh\eta, & \kappa=-1,\\ \frac{3}{\sqrt{k^2+6}}\log\eta, & \kappa=0,\\ \frac{3}{\sqrt{k^2+6}}\log\tan\eta, & \kappa=1.\end{cases}$$

In the flat ($\kappa=0$) case we recover the solution discussed in [35]. There is no inflation for any value of k, since the exponent of the scale factor in Eq. (30) is always positive. It is interesting to notice, however, that when $k\to-\infty$ we recover a radiation-dominated Universe. Looking at the effective energy-momentum tensor on the brane, Eq. (26), we see that this corresponds to the dynamics of the universe being dominated by $E_{\mu\upsilon}$. The opposite limit $k\to\infty$ gives Minkowski space-time as a result.

For the nonflat FRW models we find again that inflation does not occur for any value of k. In the case of the negatively curved model ($\kappa=-1$) the Universe approaches a radiation-dominated regime at late times ($\eta\to\infty$) in which the dilaton field is frozen, independently of k. As in the flat case, the limit $k\to-\infty$ retrieves a radiation-dominated open FRW model. When $k\to\infty$, on the other hand, we get a *regular* vacuum FRW model with scale factor $a(\eta)=\cosh\eta$. The conclusions are similar in the case of the models with $\kappa=1$: no value of k renders an inflationary universe.

We will not present further explicit examples, but just mention again that the ansatz (12) is quite generic. One may start directly with any vacuum five-dimensional solution and put it into the (12) form, or rather start with a four-dimensional dilaton solution lifting it to a five-dimensional vacuum geometry. Thus, for example, the solutions of [13] and their generalizations may easily be obtained by using the dilatonic plane wave solutions given in [39].

VI. CONCLUSIONS AND OUTLOOK

Brane cosmology is special, as compared to standard Einstein gravity, in that the four-dimensional world is not dynamically self-contained, in the sense that the matter/energy content of the Universe encoded in the energy-momentum tensor does not determine the gravitational field. The ''missing'' part on the right-hand side of the Einstein equations comes from gravitational effects in the bulk that are not sourced by four-dimensional matter. It is important to notice, however, that the term containing the tidal bulk effects is the only one not suppressed by powers of the five-dimensional Newton's constant.

Here we have studied the physical and cosmological relevance of bulk effects by looking at five-dimensional cosmologies coupled to a Liouville scalar field and embedding the brane world into them. One of the remarkable properties of the class of brane cosmologies under study is that there is a natural self-tuning of the vacuum energy on the brane. This insensitivity of the brane solutions to the value of the brane tension happens because the ''bare'' cosmological constant on the brane is dynamically counterbalanced by the brane matter, which the junction conditions force to be of vacuum energy type. So the dynamics of the four-dimensional universe is driven just by the dilaton field and the nonlocal bulk effects contained in the tensor $E_{\mu\nu}$.

The five-dimensional bulk cosmologies we constructed using the theorem stated in Sec. III generalize and include those studied previously in the literature (see, for example, [13,18,30,31,35,40]). In addition, the brane cosmologies obtained by warped embeddings extend and complement the self-tuning mechanism of [18,20] to generic non-Ricci-flat branes. Incidentally, the technique proposed can also be used to construct static five-dimensional solutions which on the brane reduce to four-dimensional black holes.

The authors of [41,42] have argued that the self-tuning mechanism proposed in [18] is actually a fine tuning in disguise due to the presence of the singularity in the bulk. For the brane worlds studied in this paper, however, we find that whenever $k^2<3$ the four-dimensional effective theory is well defined, in the sense that the four-dimensional Planck scale is finite and the consistency condition of Ref. [42] is satisfied without adding extra sources due to the ''on-shell'' identity

$$\frac{2}{3}\Lambda\int_{-\infty}^{\infty}d\chi\sqrt{-\mathcal{G}}e^{-(2/3)k\phi(\chi,x)}+\frac{1}{3}\sqrt{-g}[\lambda(\varphi)+\rho_0 e^{4b\varphi(x)}]=0, \tag{31}$$

where we have used the fact that for our family of solutions $L_{\text{matter}}=-\rho_0$. Actually, Eq. (31) is automatically enforced by the Israel junction conditions and the matching condition for the dilaton, which in turn determine both the energy-momentum tensor on the brane and $\lambda(\varphi)$. If $k^2\geqslant 3$ the divergence in the four-dimensional Planck scale can be cut off, provided a second ''hidden'' vacuum brane is located at some $\chi_0>0$. The coordinate χ is then restricted to the interval $[0,\chi_0]$ by assuming $\mathbf{Z}_2$ reflection symmetry around the location of the second brane. Imposing the junction conditions at χ_0 we find $\lambda(\phi)_{\text{hidden}}=-\lambda(\phi)$, where $\lambda(\phi)$ is given by Eq. (22) with $\rho_0=0$. This value of the vacuum energy for the second brane implies again that the model satisfies the consistency condition of [42] for any value of k.

In order to illustrate the physics of the family of brane worlds considered, we have analyzed a number of explicit examples of physical relevance. For Bianchi type-I and all the FRW models we find that the time dependence of the scale factor(s) is controlled only by the constant k measuring the slope of the potential of the scalar field in the bulk. The result is that no inflation occurs on the brane for any value of k, whereas there is a region in parameter space for which inflation takes place in the bulk. The difference between the behavior on the brane and in the bulk is mainly due to differences in the character of the velocity of the fluid flow as derived from the scalar field on the brane and in the bulk. While the five-velocity in the bulk is not orthogonal to the hypersurfaces of constant time, its projection onto the brane is.

In the analysis presented here we have assumed that the brane world is trapped at an orbifold point and it does not move in the bulk. It can be easily seen that this is a consistent assumption for the self-tuning brane cosmologies studied in Sec. IV. Once $\mathbf{Z}_2$ symmetry is relaxed in general the brane will move and the problem of the dynamical stability of the brane world arises. An especially interesting way to relax reflection symmetry in our case is to consider bulk space-times in which ζ takes different values at the two sides of the brane. If we regard the dilaton potential as arising from some kind of phase transition, the brane world plays the role of a (infinitely thin) domain wall separating two regions of space with different values for the cosmological constant, in the spirit of the scenarios discussed in [43]. In this case the possible motion of the brane in the bulk will affect the gravitational dynamics on the brane world [44,45]. A detailed study of these nonsymmetric brane worlds and their stability will be presented elsewhere.

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[1] K. Akama, in *Gauge Theory and Gravitation*, edited by K. Kikkawa, N. Nakanishi, and H. Nariai (Springer-Verlag, Berlin 1983), hep-th/0001113; V. Rubakov and M. Shaposhnikov, Phys. Lett. **125B**, 136 (1983).
[2] P. Hořava and E. Witten, Nucl. Phys. **B460**, 506 (1995).
[3] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *ibid.* **436**, 257 (1998).
[4] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).
[5] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000).
[6] R. Sundrum, Phys. Rev. D **59**, 085009 (1999); N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *ibid.* **59**, 086004 (1999); H. Davoudias, J. L. Hewett, and T. G. Rizzo, Phys. Rev. Lett. **84**, 2080 (2000); I. I. Kogan, S. Mouslopoulos, A. Papazoglou, G. G. Ross, and J. Santiago, Nucl. Phys. **B584**, 313 (2000); S. B. Bae and H. S. Lee, Phys. Lett. B **487**, 299 (2000); A.-C. Davis, S. C. Davis, W. B. Perkins, and I. R. Vernon, *ibid.* **504**, 254 (2001); K. Cheung, Phys. Rev. D **63**, 056007 (2001).
[7] R. Maartens, D. Wands, B. A. Basset, and I. Heard, Phys. Rev. D **62**, 041301 (2000).
[8] E. J. Copeland, A. R. Liddle, and J. E. Lidsey, Phys. Rev. D **64**, 023509 (2001).
[9] A. Lukas, B. A. Ovrut, and D. Waldram, Phys. Rev. D **60**, 086001 (1999); **61**, 023506 (2000); G. Dvali and S. H. Henry Tye, Phys. Lett. B **450**, 72 (1999); P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000); E. E. Flanagan, S. H. Henry Tye, and I. Wasserman, Phys. Rev. D **62**, 024011 (2000); A. Kehagias and E. Kiritsis, J. High Energy Phys. **11**, 022 (1999); D. Ida, *ibid.* **09**, 014 (2000); C. Barceló and M. Visser, Phys. Lett. B **482**, 183 (2000); S. Nojiri, O. Obregón, and S. D. Odintsov, Phys. Rev. D **62**, 104003 (2000); D. Youm, Mod. Phys. Lett. A **16**, 937 (2001); P. Binetruy, C. Deffayet, and D. Langlois, "The Radion in Brane Cosmology," hep-th/0101234; A. Hebecker and J. March-Russell, "Randall-Sundrum II Cosmology, AdS/CFT, and the Bulk Black Hole," hep-ph/0103214; J. Khoury, B. A. Ovrut, P. Steinhardt, and N. Turok, "The Ekpyrotic Universe: Colliding Branes and the Origin of the Hot Big-Bang," hep-th/0103239; R. Kallosh, L. Kofman, and A. Linde, "Pyrotechnic Universe," hep-th/0104073; A. Kehagias and K. Tamvakis, "A note on Brane Cosmology," hep-ph/0104195.
[10] H. A. Chamblin and H. S. Reall, Nucl. Phys. **B562**, 133 (1999); R. M. Hawkins and J. E. Lidsey, Phys. Rev. D **63**, 041301 (2001).
[11] R. Maartens, V. Sahni, and T. D. Saini, Phys. Rev. D **63**, 063509 (2001).
[12] M. G. Santos, F. Vernizzi, and P. G. Ferreira, "Isotropization and Instability of the Brane," hep-ph/0103112.
[13] G. T. Horowitz, I. Low, and A. Zee, Phys. Rev. D **62**, 086005 (2000).
[14] A. V. Frolov, "Kasner-AdS Spacetime and Anisotropic Brane-World Cosmology," gr-qc/0102064.
[15] A. Campos and C. F. Sopuerta, Phys. Rev. D (to be published), hep-th/0105100.
[16] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989); "The Cosmological Constant Problems," talk at Dark Matter 2000, hep-ph/0005265.
[17] N. Straumann, "On the Mystery of the Cosmic Vacuum Energy Density," talk at DARK 2000, astro-ph/0009386.
[18] S. Kachru, M. Schulz, and E. Silverstein, Phys. Rev. D **62**, 045021 (2000).
[19] S. Kachru, M. Schulz, and E. Silverstein, Phys. Rev. D **62**, 085003 (2000).
[20] N. Arkani-Hamed, S. Dimpopoulos, N. Kaloper, and R. Sundrum, Phys. Lett. B **480**, 193 (2000).
[21] S. M. Carroll and L. Mersini, "Can we Live in a Selftuning Universe?," hep-th/0105007.
[22] K. Maeda and D. Wands, Phys. Rev. D **62**, 124009 (2000).
[23] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).
[24] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
[25] A. Mennim and R. A. Battye, Class. Quantum Grav. **18**, 2171 (2001).
[26] R. Maartens, "Geometry and Dynamics of the Brane World," talk at the 2000 Spanish Relativity Meeting, gr-qc/0101059.
[27] R. A. Battye, B. Carter, A. Mennim, and J.-P. Uzan, Phys. Rev. D (to be published), hep-th/0105091.
[28] W. Israel, Nuovo Cimento B **44**, 1 (1966).
[29] O. Fonarev, Class. Quantum Grav. **12**, 1739 (1995).
[30] J. E. Lidsey, Class. Quantum Grav. **17**, L39 (2000).
[31] N. Alonso-Alberca, B. Janssen, and P. J. Silva, Class. Quantum Grav. **17**, L163 (2000).
[32] A. Lukas, B. A. Ovrut, K. S. Stelle, and D. Waldram, Phys. Rev. D **59**, 086001 (1999).
[33] A. Feinstein and M. A. Vázquez-Mozo, Nucl. Phys. **B568**, 405 (2000).
[34] A. Chamblin, S. W. Hawking, and H. S. Reall, Phys. Rev. D **61**, 065007 (2000).
[35] H. Ochiai and K. Sato, Phys. Lett. B **503**, 404 (2001).
[36] D. Youm, Phys. Rev. D **63**, 047503 (2001); Y. S. Myung and H. W. Lee, *ibid.* **63**, 064034 (2001); C. Barceló and M. Visser, J. High Energy Phys. **10**, 019 (2000).
[37] G. Veneziano, in *The Primordial Universe*, Proceedings of the Les Houches Summer School of Theoretical Physics, 1999, edited by P. Binetruy, R. Schaeffer, J. Silk, and F. David (Springer-Verlag, Berlin, 2001), hep-th/0002094.
[38] M. P. Dąbrowski, Phys. Lett. B **474**, 226 (2000).
[39] A. Feinstein, K. E. Kunze, and M. A. Vázquez-Mozo, Class. Quantum Grav. **17**, 3599 (2000).
[40] C. Gómez, B. Janssen, and P. Silva, J. High Energy Phys. **04**, 024 (2000).
[41] S. Förste, Z. Lalak, S. Lavignac, and H. P. Nilles, Phys. Lett. B **481**, 360 (2000).
[42] S. Förste, Z. Lalak, S. Lavignac, and H. P. Nilles, J. High Energy Phys. **09**, 034 (2000).
[43] B. Carter and J.-P. Uzan, "Reflection Symmetry Breaking Scenarios with Minimal Gauge Form Coupling in Brane World Cosmology," gr-qc/0101010.
[44] P. Kraus, J. High Energy Phys. **12**, 011 (1999).
[45] H. Stoica, S.-H. Henry Tye, and I. Wasserman, Phys. Lett. B **482**, 205 (2000).