Hamiltonian formulation of general relativity in the teleparallel geometry

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We establish a Hamiltonian formulation of the teleparallel equivalent of general relativity, without fixing the time gauge condition, by rigorously performing a Legendre transform. The time gauge condition, previously considered, restricts the teleparallel geometry to the three-dimensional spacelike hypersurface. Geometrically, the teleparallel geometry is now extended to four-dimensional space-time. The resulting Hamiltonian formulation is structurally different from the standard Arnowitt-Deser-Misner formulation in many aspects, the main one being that the dynamics is now governed by the Hamiltonian constraint H_0 and a set of primary constraints. The vector constraint H_i is *derived* from the Hamiltonian constraint. The vanishing of the latter implies the vanishing of the vector constraint.

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I. INTRODUCTION

Hamiltonian formulations, when consistently established, not only guarantee that field quantities have a well defined time evolution, but also allow us to understand physical theories from a different perspective. We have learned from the work of Arnowitt, Deser and Misner (ADM) [1] that a Hamiltonian analysis of Einstein's general relativity reveals the intrinsic structure of the theory: the time evolution of field quantities is determined by the Hamiltonian and vector constraints. Thus four of the ten Einstein's equations acquire a prominent status in the Hamiltonian framework. Ultimately this is an essential feature for a canonical approach to the quantum theory of gravity.

It is the case in general relativity that two distinct Lagrangian formulations that yield Einstein's equations lead to completely different Hamiltonian constructions. An important example in this respect is the reformulation of the ordinary variational principle, based on the Hilbert-Einstein action, in terms of self-dual connections that define Ashtekar variables [2]. Under a Palatini type variation of an action integral constructed out of these field quantities one precisely obtains Einstein's equations. Interesting features of this approach reside in the Hamiltonian domain.

Einstein's general relativity can also be reformulated in the context of the teleparallel (Weitzenböck) geometry [3]. In this geometrical setting the dynamical field quantities correspond to orthornormal tetrad fields $e^a{}_\mu$ [a and μ are SO(3,1) and space-time indices, respectively]. These fields allow the construction of the Lagrangian density of the teleparallel equivalent of general relativity (TEGR) [4–12], which offers an alternative geometrical framework for Einstein's equations. The Lagrangian density for the tetrad field in the TEGR is given by a sum of quadratic terms in the torsion tensor $T^a{}_{\mu\nu}=\partial_\mu e^a{}_\nu-\partial_\nu e^a{}_\mu$, which is related to the antisymmetric part of Cartan's connection $\Gamma^\lambda_{\mu\nu}=e^{a\lambda}\partial_\mu e_{a\nu}$. The curvature tensor constructed out of the latter vanishes identically. This connection defines a space with teleparallelism, or absolute parallelism [13].

In a space-time with an underlying tetrad field two vectors at distant points are called parallel [4] if they have identical components with respect to the local tetrads at the points considered. Thus we consider a vector field $V^\mu(x)$. At the point x^λ its tetrad components are given by $V^a(x)=e^a{}_\mu(x)V^\mu(x)$. For the tetrad components $V^a(x+dx)$ it is easy to show that $V^a(x+dx)=V^a(x)+DV^a(x)$, where $DV^a(x)=e^a{}_\mu(\nabla_\lambda V^\mu)dx^\lambda$. The covariant derivative ∇ is constructed out of Cartan's connection $\Gamma^\lambda_{\mu\nu}=e^{a\lambda}\partial_\mu e_{a\nu}$. Therefore the vanishing of such covariant derivative defines a condition for absolute parallelism in space-time. Hence in the teleparallel geometry tetrad fields transform under the global SO(3,1) group. Teleparallel geometry is less restrictive than Riemannian geometry [14]. For a given Riemaniann geometry there are many ways to construct the teleparallel geometry, since one Riemaniann geometry corresponds to a whole equivalence class of teleparallel geometries.

In the framework of the TEGR it is possible to make definite statements about the energy and momentum of the gravitational field. This fact constitutes the major motivation for considering this theory. In the $3+1$ formulation of the TEGR [12], and by imposing Schwinger's time gauge condition [15], we find that the Hamiltonian and vector constraints contain each one a divergence in the form of scalar and vector densities, respectively, that can be identified with the energy and momentum *densities* of the gravitational field [16].

In this paper we carry out a Hamiltonian formulation of the TEGR without imposing the time gauge condition, by rigorously performing a Legendre transform. We have not found it necessary to establish a $3+1$ decomposition for the tetrad field. We only assume $g^{00}\neq 0$, a condition that ensures that $t=$const hypersurfaces are spacelike. The Lagrange multipliers are given by the zero components of the tetrads, e_{a0}. The constraints corresponding to the Hamiltonian (H_0) and vector (H_i) constraints are obtained in the form $C^a=0$. The dynamical evolution of the field quantities is completely determined by H_0 and by a set of primary constraints Γ^{ik} and Γ^k, as we will show. A surprising feature is that if $H_0=0$ in the subspace of the phase space determined by $\Gamma^{ik}=\Gamma^k=0$, then it follows that $H_i=0$. As we will see, H_i can be obtained from the very definition of H_0. Furthermore by calculating Poisson brackets we show that the constraints consti-

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tute a first class set. Hence the theory is well defined regarding time evolution.

As a consequence of this analysis, we arrive at a scalar density that transforms as a 4-vector in the SO(3,1) space, again arising in the expression of the constraints of the theory, and whose zero component is related to the energy of the gravitational field. In analogy with previous investigations, we interpret the constraint equations $C^a=0$ as energy-momentum equations for the gravitational field.

The analysis developed here is similar to that developed in Ref. [17], in which the Hamiltonian formulation of the TEGR in null surfaces was established. The $3+1$ formulation of the TEGR was already considered in Ref. [10]. There are several differences between the latter analysis and the present analysis. The investigation in Ref. [10] pointed out neither the emergence of the scalar densities mentioned above nor the relationship between H_0 and H_i. Our approach is different, and allows us to proceed further in the understanding of the constraint structure of the theory.

[Notation: space-time indices $\mu,\nu,\ldots$ and SO(3,1) indices $a,b,\ldots$ run from 0 to 3. Time and space indices are indicated according to $\mu=0,i$ and $a=(0),(i)$. The tetrad field $e^a\,_\mu$ yields the definition of the torsion tensor: $T^a\,_{\mu\nu}=\partial_\mu e^a\,_\nu-\partial_\nu e^a\,_\mu$. The flat, Minkowski space-time metric is fixed by $\eta_{ab}=e_{a\mu}e_{b\nu}g^{\mu\nu}=(-+++)$.]

II. LAGRANGIAN FORMULATION

In order to carry out the $3+1$ decomposition we need a first order differential formulation of the Lagrangian density of the TEGR. For this purpose we introduce an auxiliary field quantity $\phi_{abc}=-\phi_{acb}$ that will be related to the torsion tensor. The first order differential Lagrangian formulation in empty space-time reads

$$L(e,\phi)=ke\Lambda^{abc}(\phi_{abc}-2T_{abc}), \tag{1}$$

where $T_{abc}=e_b\,^\mu e_c\,^\nu T_{a\mu\nu}$, $e=\det(e^a\,_\mu)$ and $k=1/16\pi$. Λ^{abc} is defined by

$$\Lambda^{abc}=\frac{1}{4}(\phi^{abc}+\phi^{bac}-\phi^{cab})+\frac{1}{2}(\eta^{ac}\phi^b-\eta^{ab}\phi^c), \tag{2}$$

and $\phi_b=\phi^a\,_{ab}$. The Lagrangian density [Eq. (1)] is invariant under coordinate and global SO(3,1) transformations.

Variation of the action constructed out of Eq. (1) with respect to ϕ^{abc} yields an equation that can be reduced to $\phi_{abc}=T_{abc}$. This equation can be split into two equations:

$$\phi_{a0k}=T_{a0k}=\partial_0 e_{ak}-\partial_k e_{a0}, \tag{3a}$$

$$\phi_{aik}=T_{aik}=\partial_i e_{ak}-\partial_k e_{ai}. \tag{3b}$$

The variation of the action integral with respect to $e_{a\mu}$ yields the field equation

$$\frac{\delta L}{\delta e^{a\mu}}=e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu})-e\left(\Sigma^{b\nu}\,_a T_{b\nu\mu}-\frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}\right)=0. \tag{4}$$

The tensor Σ^{abc} is defined in terms of T^{abc} exactly like Λ^{abc} in terms of ϕ^{abc}. By explicit calculations [12] it is verified that these equations are equivalent to Einstein's equations in tetrad form:

$$\frac{\delta L}{\delta e^{a\mu}}\equiv\frac{1}{2}e\left\{R_{a\mu}(e)-\frac{1}{2}e_{a\mu}R(e)\right\}.$$

We note finally that by substituting Eqs. (3a) and (3b) into Eq. (1), the Lagrangian density reduces to

$$L(e_{a\mu})=-k\,e\Sigma^{abc}T_{abc}=-k\,e\left(\frac{1}{4}T^{abc}T_{abc}+\frac{1}{2}T^{abc}T_{bac}-T^aT_a\right).$$

III. LEGENDRE TRANSFORM AND THE 3+1 DECOMPOSITION

The Hamiltonian density will be obtained by a standard prescription $L=p\dot{q}-H_0$ and by properly identifying primary constraints. We have not found it necessary to establish any kind of $3+1$ decomposition for the tetrad fields. Therefore in the following both $e_{a\mu}$ and $g_{\mu\nu}$ are space-time fields. Here we will follow the procedure presented in Ref. [17].

The Lagrangian density [Eq. (1)] can be expressed as

$$L(e,\phi)=-4ke\Lambda^{a0k}\dot{e}_{ak}+4ke\Lambda^{a0k}\partial_k e_{a0}-2ke\Lambda^{aij}T_{aij}+ke\Lambda^{abc}\phi_{abc}, \tag{5}$$

where the dot indicates a time derivative, and $\Lambda^{a0k}=\Lambda^{abc}e_b\,^0e_c\,^k$, $\Lambda^{aij}=\Lambda^{abc}e_b\,^ie_c\,^j$. Therefore the momentum canonically conjugated to e_{ak} is given by

$$\Pi^{ak}=-4ke\Lambda^{a0k}, \tag{6}$$

In terms of Eq. (6) expression (5) reads

$$\begin{aligned}L&=\Pi^{ak}\dot{e}_{ak}-\Pi^{ak}\partial_k e_{a0}-2ke\Lambda^{aij}T_{aij}+ke\Lambda^{abc}\phi_{abc}\\&=\Pi^{ak}\dot{e}_{ak}-\Pi^{ak}\partial_k e_{a0}-ke\Lambda^{aij}(2T_{aij}-\phi_{aij})\\&\quad+2ke\Lambda^{a0k}\phi_{a0k}.\end{aligned} \tag{7}$$

The last term on the right hand side of Eq. (7) is identified as $2ke\Lambda^{a0k}\phi_{a0k}=-\frac{1}{2}\Pi^{ak}\phi_{a0k}$.

The Hamiltonian formulation is established once we rewrite the Lagrangian density [Eq. (7)] in terms of e_{ak}, Π^{ak} and further nondynamical field quantities. This is carried out in two steps. First, we take Eq. (3b) into account in Eq. (7), so that half of the auxiliary fields ϕ_{aij} are eliminated from the Lagrangian by means of the identification

$$\phi_{aij}=T_{aij}.$$

As a consequence we have

$$-ke\Lambda^{aij}(2T_{aij}-\phi_{aij})=-ke\Lambda^{aij}T_{aij}=-ke\left(\frac{1}{4}g^{im}g^{nj}T^{a}{}_{mn}T_{aij}+\frac{1}{2}g^{nj}T^{i}{}_{mn}T^{m}{}_{ij}-g^{ik}T^{j}{}_{ji}T^{n}{}_{nk}\right)$$
$$+ke\left(-\frac{1}{2}g^{0i}g^{jk}\phi^{a}{}_{0k}T_{aij}-\frac{1}{2}g^{jk}\phi^{i}{}_{0k}T^{0}{}_{ij}+\frac{1}{2}g^{0j}\phi^{i}{}_{0k}T^{k}{}_{ij}-g^{0k}\phi^{j}{}_{0j}T^{i}{}_{ik}+g^{ik}\phi^{0}{}_{0i}T^{j}{}_{jk}\right).$$

The last five terms of the expression above may be rewritten as

$$-\frac{1}{2}ke\,\phi_{a0k}[g^{0i}g^{kj}T^{a}{}_{ij}-e^{ai}(g^{0j}T^{k}{}_{ij}-g^{kj}T^{0}{}_{ij})+2(e^{ak}g^{0i}-e^{a0}g^{ki})T^{j}{}_{ji}].$$

Therefore we have

$$L(e_{ak},\Pi^{ak},e_{a0},\phi_{a0k})=\Pi^{ak}\dot{e}_{ak}+e_{a0}\partial_k\Pi^{ak}-\partial_k(e_{a0}\Pi^{ak})-ke\left(\frac{1}{4}g^{im}g^{nj}T^{a}{}_{mn}T_{aij}+\frac{1}{2}g^{nj}T^{i}{}_{mn}T^{m}{}_{ij}-g^{ik}T^{j}{}_{ji}T^{n}{}_{nk}\right)$$
$$-\frac{1}{2}\phi_{a0k}\{\Pi^{ak}+ke[g^{0i}g^{kj}T^{a}{}_{ij}-e^{ai}(g^{0j}T^{k}{}_{ij}-g^{kj}T^{0}{}_{ij})+2(e^{ak}g^{0i}-e^{a0}g^{ki})T^{j}{}_{ji}]\}. \tag{8}$$

The second step consists of expressing the remaining auxiliary field quantities, the "velocities" ϕ_{a0k}, in terms of the momenta Π^{ak}. This is the nontrivial step of the Legendre transform.

We need to consider the full expression of Π^{ak}. It is given by Eq. (6),

$$\Pi^{ak}=ke\{g^{00}(-g^{kj}\phi^{a}{}_{0j}-e^{aj}\phi^{k}{}_{0j}+2e^{ak}\phi^{j}{}_{0j})+g^{0k}(g^{0j}\phi^{a}{}_{0j}+e^{aj}\phi^{0}{}_{0j})+e^{a0}(g^{0j}\phi^{k}{}_{0j}+g^{kj}\phi^{0}{}_{0j})-2(e^{a0}g^{0k}\phi^{j}{}_{0j}+e^{ak}g^{0j}\phi^{0}{}_{0j})-g^{0i}g^{kj}T^{a}{}_{ij}+e^{ai}(g^{0j}T^{k}{}_{ij}-g^{kj}T^{0}{}_{ij})-2(g^{0i}e^{ak}-g^{ik}e^{a0})T^{j}{}_{ji}\}, \tag{9}$$

where we have already identified $\phi_{aij}=T_{aij}$. Denoting $(\cdots)$ and $[\cdots]$ as the symmetric and antisymmetric parts of field quantities, respectively, we decompose Π^{ak} into irreducible components

$$\Pi^{ak}=e^{a}{}_{i}\Pi^{(ik)}+e^{a}{}_{i}\Pi^{[ik]}+e^{a}{}_{0}\Pi^{0k}, \tag{10}$$

where

$$\Pi^{(ik)}=ke\{g^{00}(-g^{kj}\phi^{i}{}_{0j}-g^{ij}\phi^{k}{}_{0j}+2g^{ik}\phi^{j}{}_{0j})+g^{0k}(g^{0j}\phi^{i}{}_{0j}+g^{ij}\phi^{0}{}_{0j}-g^{0i}\phi^{j}{}_{0j})+g^{0i}(g^{0j}\phi^{k}{}_{0j}+g^{kj}\phi^{0}{}_{0j}-g^{0k}\phi^{j}{}_{0j})-2g^{ik}g^{0j}\phi^{0}{}_{0j}+\Delta^{ik}\}, \tag{11a}$$

$$\Delta^{ik}=-g^{0m}(g^{kj}T^{i}{}_{mj}+g^{ij}T^{k}{}_{mj}-2g^{ik}T^{j}{}_{mj})-(g^{km}g^{0i}+g^{im}g^{0k})T^{j}{}_{mj}, \tag{11b}$$

$$\Pi^{[ik]}=ke\{-g^{im}g^{kj}T^{0}{}_{mj}+(g^{im}g^{0k}-g^{km}g^{0i})T^{j}{}_{mj}\}, \tag{12}$$

$$\Pi^{0k}=-2ke(g^{kj}g^{0i}T^{0}{}_{ij}-g^{0k}g^{0i}T^{j}{}_{ij}+g^{00}g^{ik}T^{j}{}_{ij}). \tag{13}$$

The crucial point in this analysis is that only the symmetrical components $\Pi^{(ij)}$ depend on the "velocities" ϕ_{a0k}. The other six components $\Pi^{[ij]}$ and Π^{0k} depend solely on T_{aij}. Therefore we can express only six of the "velocity" fields ϕ_{a0k} in terms of the components $\Pi^{(ij)}$. With the purpose of finding out which components of ϕ_{a0k} can be inverted in terms of the momenta we decompose ϕ_{a0k} identically as

$$\phi^{a}{}_{0j}=e^{ai}\,\psi_{ij}+e^{ai}\,\sigma_{ij}+e^{a0}\lambda_j, \tag{14}$$

where $\psi_{ij}=\frac{1}{2}(\phi_{i0j}+\phi_{j0i})$, $\sigma_{ij}=\frac{1}{2}(\phi_{i0j}-\phi_{j0i})$, $\lambda_j=\phi_{00j}$, and $\phi_{\mu 0j}=e^{a}{}_{\mu}\phi_{a0j}$ (like ϕ_{abc}, the components ψ_{ij}, σ_{ij} and λ_j are also auxiliary field quantities). Next we substitute Eq. (14) into Eq. (11a). By defining

$$P^{ik}=\frac{1}{ke}\Pi^{(ik)}-\Delta^{ik}, \tag{15}$$

we find that P^{ik} depends only on ψ_{ij}:

$$P^{ik}=-2g^{00}(g^{im}g^{kj}\psi_{mj}-g^{ik}\psi)+2(g^{0i}g^{km}g^{0j}+g^{0k}g^{im}g^{0j})\psi_{mj}-2(g^{ik}g^{0m}g^{0j}\psi_{mj}+g^{0i}g^{0k}\psi), \tag{16}$$

where $\psi=g^{mn}\psi_{mn}$.

We can now invert ψ_{mj} in terms of P^{ik}. After a number of manipulations we arrive at

$$\psi_{mj}=-\frac{1}{2g^{00}}\left(g_{im}g_{kj}P^{ik}-\frac{1}{2}g_{mj}P\right), \tag{17}$$

where $P=g_{ik}P^{ik}$.

At last we need to rewrite the second line of the Lagrangian density [Eq. (8)] in terms of canonical variables. By making use of Eqs. (9), (14) and (17) we can rewrite

$$-\frac{1}{2}\phi_{a0k}\{\Pi^{ak}+ke[g^{0i}g^{kj}T^a{}_{ij}-e^{ai}(g^{0j}T^k{}_{ij}-g^{kj}T^0{}_{ij})+2(e^{ak}g^{0i}-e^{a0}g^{ki})T^j{}_{ji}]\}$$

in the form

$$\frac{1}{4g^{00}}ke\left(g_{ik}g_{jl}P^{ij}P^{kl}-\frac{1}{2}P^2\right).$$

Thus we finally obtain the primary Hamiltonian density $H_0=\Pi^{ak}\dot{e}_{ak}-L$:

$$H_0(e_{ak},\Pi^{ak},e_{a0})=-e_{a0}\partial_k\Pi^{ak}-\frac{1}{4g^{00}}ke\left(g_{ik}g_{jl}P^{ij}P^{kl}-\frac{1}{2}P^2\right)+ke\left(\frac{1}{4}g^{im}g^{nj}T^a{}_{mn}T_{aij}+\frac{1}{2}g^{nj}T^i{}_{mn}T^m{}_{ij}-g^{ik}T^j{}_{ji}T^n{}_{nk}\right). \quad (18)$$

We may now write the total Hamiltonian density. For this purpose we have to identify the primary constraints. They are given by expressions (12) and (13), which represent relations between e_{ak} and the momenta Π^{ak}. Thus we define

$$\Gamma^{ik}=-\Gamma^{ki}=\Pi^{[ik]}-ke\{-g^{im}g^{kj}T^0{}_{mj}+(g^{im}g^{0k}-g^{km}g^{0i})T^j{}_{mj}\}, \quad (19)$$

$$\Gamma^k=\Pi^{0k}+2ke(g^{kj}g^{0i}T^0{}_{ij}-g^{0k}g^{0i}T^j{}_{ij}+g^{00}g^{ik}T^j{}_{ij}). \quad (20)$$

Therefore the total Hamiltonian density is given by

$$H(e_{ak},\Pi^{ak},e_{a0},\alpha_{ik},\beta_k)=H_0+\alpha_{ik}\Gamma^{ik}+\beta_k\Gamma^k+\partial_k(e_{a0}\Pi^{ak}), \quad (21)$$

where α_{ik} and β_k are Lagrange multipliers.

IV. SECONDARY CONSTRAINTS

Since the momenta $\{\Pi^{a0}\}$ vanish identically, they also constitute primary constraints that induce the secondary constraints

$$C^a\equiv\frac{\delta H}{\delta e_{a0}}=0. \quad (22)$$

In order to obtain the expression of C^a we have only to vary H_0 with respect to e_{a0}, because variations of Γ^{ik} and Γ^k with respect to e_{a0} yield the constraints themselves:

$$\frac{\delta\Gamma^{ik}}{\delta e_{a0}}=-\frac{1}{2}(e^{ai}\Gamma^k-e^{ak}\Gamma^i), \quad (23a)$$

$$\frac{\delta\Gamma^k}{\delta e_{a0}}=-e^{a0}\Gamma^k. \quad (23b)$$

In Eqs. (23a) and (23b), we make use of variations like $\delta e^{b\mu}/\delta e_{a0}=-e^{a\mu}e^{b0}$. In the process of obtaining C^a we need a variation of P^{ij} with respect to e_{a0}. This reads

$$\frac{\delta P^{ij}}{\delta e_{a0}}=-e^{a0}P^{ij}+\gamma^{aij},$$

with γ^{aij} defined by

$$\gamma^{aij}=-\frac{1}{2ke}(e^{ai}\Gamma^j+e^{aj}\Gamma^i)-e^{ak}[g^{00}(g^{jm}T^i{}_{km}+g^{im}T^j{}_{km}+2g^{ij}T^m{}_{mk})+g^{0m}(g^{0j}T^i{}_{mk}+g^{0i}T^j{}_{mk})-2g^{0i}g^{0j}T^m{}_{mk}+(g^{jm}g^{0i}+g^{im}g^{0j}-2g^{ij}g^{0m})T^0{}_{mk}]. \quad (24)$$

Note that γ^{aij} satisfies $e_{a0}\gamma^{aij}=0$.

After a long calculation we arrive at an expression for C^a:

$$C^a=-\partial_k\Pi^{ak}+e^{a0}\left[-\frac{1}{4g^{00}}ke\left(g_{ik}g_{jl}P^{ij}P^{kl}-\frac{1}{2}P^2\right)+ke\left(\frac{1}{4}g^{im}g^{nj}T^b{}_{mn}T_{bij}+\frac{1}{2}g^{nj}T^i{}_{mn}T^m{}_{ij}-g^{ik}T^m{}_{mi}T^n{}_{nk}\right)\right]-\frac{1}{2g^{00}}ke\left(g_{ik}g_{jl}\gamma^{aij}P^{kl}-\frac{1}{2}g_{ij}\gamma^{aij}P\right)-kee^{ai}(g^{0m}g^{nj}T^b{}_{ij}T_{bmn}+g^{nj}T^0{}_{mn}T^m{}_{ij}+g^{0j}T^n{}_{mj}T^m{}_{ni}-2g^{0k}T^m{}_{mk}T^n{}_{ni}-2g^{jk}T^0{}_{ij}T^n{}_{nk}). \quad (25)$$

In spite of the fact that expression above is somehow intricate, we immediately note that

$$e_{a0}C^a=H_0. \quad (26)$$

Therefore, the total Hamiltonian becomes

$$H(e_{ak},\Pi^{ak},e_{a0},\alpha_{ik},\beta_k)=e_{a0}C^a+\alpha_{ik}\Gamma^{ik}+\beta_k\Gamma^k+\partial_k(e_{a0}\Pi^{ak}). \quad (27)$$

We observe that $\{e_{a0}\}$ arise as Lagrange multipliers [see Eq. (50) below].

Before closing this section we remark that the Hamiltonian formulation described here is different from that developed in Ref. [10], the difference residing in the definition of the canonical momentum. In the latter reference the canonical momentum is not defined by taking the variation of L with respect to $\dot{e}_{ak}$. Instead, it is defined by

$$\pi_a{}^k=\frac{\delta L}{\delta(N^\perp T^a{}_{\perp k})}=\frac{\delta L}{\delta(T^a{}_{0k}-N^iT^a{}_{ik})},$$

where $N^\perp$ and N^i are the usual lapse and shift functions. As a consequence, three of the six primary constraints of Ref. [10] are different from the corresponding constraints obtained here. The expression for the components $\tau^{[ik]}$ and $\tau_\perp{}^k$ of Ref. [10], equivalent to $\Pi^{[ik]}$ and Π^{0k}, respectively, given by Eqs. (12) and (13), read, in our notation,

$$\tau^{[ik]}=-e\{g^{im}g^{kj}T^0{}_{ij}+N^j(g^{im}g^{0k}-g^{km}g^{0i})T^0{}_{mj}\},$$

$$\tau_{\perp}{}^{k}=\frac{1}{2k}N^{\perp}\Pi^{0k}.$$

The Hamiltonian and vector constraints of the above mentioned reference are parametrized in terms of the lapse and shift functions. In the present work we have parametrized the set of four constraints according to Eq. (26), and identified the Lagrange multipliers as e_{a0}. The final expression of C^a acquires the total divergence $-\partial_k\Pi^{ak}$. This divergence is different from the one that appears in the expression of the total Hamiltonian density of gravitational fields for asymptotically flat space-times, either in the metric [18] or in the tetrad formulation [see, for example, Eq. (3.17) of Ref. [10] or Eq. (27) above; it is possible to show that the latter expressions are exactly the same field quantities]. We finally note that the constraint algebra to be presented in the coming section has not been evaluated in Ref. [10].

V. SIMPLIFICATION OF THE CONSTRAINTS AND POISSON BRACKETS

The first two terms of the expression of C^a yield the primary Hamiltonian in the form $e^{a0}H_0$. This fact can be easily verified by expressing the first term of Eq. (25) as

$$-\partial_k\Pi^{ak}=e^{a0}(-e_{b0}\partial_k\Pi^{bk})+e^{aj}(-e_{bj}\partial_k\Pi^{bk}).$$

The second term considered above is the collection of terms in Eq. (25) multiplied by e^{a0}. Substituting definitions (11b) and (24) for Δ^{ij} and γ^{aij}, respectively, into Eq. (25), after a long calculation we obtain a simplified form for C^a,

$$C^a=e^{a0}H_0+e^{ai}F_i\,, \tag{28}$$

with the following definitions:

$$F_i=H_i+\Gamma^m T_{0mi}+\Gamma^{lm}T_{lmi}+\frac{1}{2g^{00}}\left(g_{ik}g_{jl}P^{kl}-\frac{1}{2}g_{ij}P\right)\Gamma^j, \tag{29}$$

$$H_i=-e_{bi}\partial_k\Pi^{bk}-\Pi^{bk}T_{bki}\,. \tag{30}$$

We denote H_0 the Hamiltonian constraint. H_i is the vector constraint. This amounts to a SO(3,1) version of the vector constraint of Ref. [12]. The true constraints of the theory are C^a, Γ^{ik}, and Γ^k. Dispensing with the surface term the total Hamiltonian reads

$$H=e_{a0}C^a+\alpha_{ik}\Gamma^{ik}+\beta_k\Gamma^k. \tag{31}$$

The Poisson bracket between two quantities F and G is defined by

$$[F,G]=\int d^3x\left(\frac{\delta F}{\delta e_{ai}(x)}\frac{\delta G}{\delta\Pi^{ai}(x)}-\frac{\delta F}{\delta\Pi^{ai}(x)}\frac{\delta G}{\delta e_{ai}(x)}\right),$$

by means of which we can write down the evolution equations. The first set of Hamilton's equations is given by

$$\dot{e}_{aj}(x)=\{e_{aj}(x),\mathbf{H}\}=\int d^3y\frac{\delta}{\delta\Pi^{aj}(x)}[H_0(y)+\alpha_{ik}(y)\Gamma^{ik}(y)+\beta_k(y)\Gamma^k(y)], \tag{32}$$

where $\mathbf{H}$ is the total Hamiltonian. This equation can be worked to yield

$$T_{a0j}=-\frac{1}{2g^{00}}e_a{}^k\left(g_{ik}g_{jm}P^{im}-\frac{1}{2}g_{kj}P\right)+e_a{}^i\alpha_{ij}+e_a{}^0\beta_j\,, \tag{33}$$

from which we obtain

$$\frac{1}{2}(T_{i0j}+T_{j0i})=\psi_{ij}=-\frac{1}{2g^{00}}\left(g_{ik}g_{mj}P^{km}-\frac{1}{2}g_{ij}P\right), \tag{34a}$$

$$\frac{1}{2}(T_{i0j}-T_{j0i})=\sigma_{ij}=\alpha_{ij}\,, \tag{34b}$$

$$T_{00j}=\lambda_j=\beta_j\,, \tag{34c}$$

according to the definitions in Eq. (14). Thus the Lagrange multipliers in Eq. (31) acquire a well defined meaning. Expression (34a) is in total agreement with Eq. (17). Consequently we can obtain an expression for $\Pi^{(ij)}$ in terms of velocities via Eqs. (15) and (16). The dynamical evolution of the field quantities is completed with Hamilton's equations for $\Pi^{(ij)}$,

$$\dot{\Pi}^{(ij)}(x)=\{\Pi^{(ij)}(x),\mathbf{H}\}=\int d^3y\left(\frac{\delta\Pi^{(ij)}(x)}{\delta e_{ak}(y)}\frac{\delta\mathbf{H}}{\delta\Pi^{ak}(y)}-\frac{\delta\Pi^{(ij)}(x)}{\delta\Pi^{ak}(y)}\frac{\delta\mathbf{H}}{\delta e_{ak}(y)}\right), \tag{35}$$

together with

$$\Gamma^{ik}=\Gamma^k=0. \tag{36}$$

The calculations of the Poisson brackets between these constraints are exceedingly complicated. Here we will just present the results. Instead of considering $C^a(x)$ in the calculations below, we found it more appropriate to consider $H_0(x)$ and $H_i(x)$. The constraint algebra is given by

$$\{H_0(x),H_0(y)\}=0, \tag{37}$$

$$\{H_0(x),H_i(y)\}=-H_0(x)\frac{\partial}{\partial y^i}\delta(x-y)-H_0e^{a0}\partial_ie_{a0}\delta(x-y)-F_je^{aj}\partial_ie_{a0}\delta(x-y), \tag{38}$$

$$\{H_j(x),H_k(y)\} = -H_k(x)\frac{\partial}{\partial x^j}\delta(x-y) - H_j(y)\frac{\partial}{\partial y^k}\delta(x-y), \tag{39}$$

$$\{\Gamma^i(x),\Gamma^j(y)\} = 0, \tag{40}$$

$$\{\Gamma^{ij}(x),\Gamma^k(y)\} = (g^{0j}\Gamma^{ki} - g^{0i}\Gamma^{kj})\,\delta(x-y), \tag{41}$$

$$\{\Gamma^{ij}(x),\Gamma^{kl}(y)\} = \frac{1}{2}(g^{il}\Gamma^{jk} + g^{jk}\Gamma^{il} - g^{ik}\Gamma^{jl} - g^{jl}\Gamma^{ik})\,\delta(x-y), \tag{42}$$

$$\{H_0(x),\Gamma^{ij}(y)\} = \bigg[\frac{1}{2g^{00}}P^{kl}\bigg(\frac{1}{2}g_{kl}g_{mn} - g_{km}g_{nl}\bigg) \times (g^{mi}\Gamma^{nj} - g^{mj}\Gamma^{ni}) + \frac{1}{2} \times (\Gamma^{nj}e^{ai} - \Gamma^{ni}e^{aj})\,\partial_n e_{a0}\bigg]\,\delta(x-y), \tag{43}$$

$$\{H_0(x),\Gamma^i(y)\} = \bigg[g^{0i}H_0 + \frac{1}{g^{00}}P^{kl}\bigg(\frac{1}{2}g_{kl}g_{jm} - g_{kj}g_{ml}\bigg) \times g^{0j}\Gamma^{mi} + (\Gamma^{ni}e^{a0} + \Gamma^n e^{ai})\,\partial_n e_{a0} + \frac{1}{2}\Gamma^{mn}T^i{}_{nm} + 2\,\partial_n\Gamma^{ni} + g^{in}(H_n - \Gamma^j T_{0nj} - \Gamma^{mj}T_{mnj})\bigg] \times \delta(x-y) + \Gamma^{ni}(x)\frac{\partial}{\partial x^n}\delta(x-y), \tag{44}$$

$$\{H_i(x),\Gamma^j(y)\} = \delta^j_i\Gamma^n(y)\frac{\partial}{\partial y^n}\delta(x-y) + \Gamma^j(x)\frac{\partial}{\partial x^i}\delta(x-y) - \Gamma^j e^{a0}\partial_i e_{a0}\,\delta(x-y), \tag{45}$$

$$\{H_k(x),\Gamma^{ij}(y)\} = \Gamma^{ij}(x)\frac{\partial}{\partial x^k}\delta(x-y) + (\delta^j_k\Gamma^{ni}(y) - \delta^i_k\Gamma^{nj}(y))\frac{\partial}{\partial x^n}\delta(x-y) + \frac{1}{2}(e^{aj}(x)\Gamma^i(x) - e^{ai}(x)\Gamma^j(x))\frac{\partial}{\partial x^k}e_{a0}(x)\,\delta(x-y). \tag{46}$$

It is clear from the constraint algebra above that H_0, H_i, Γ^{ik}, and Γ^k constitute a set of first class constraints. Now it is easy to conclude that C^a, Γ^{ik}, and Γ^k also constitute a first class set. By means of Eq. (28) we have $\{C^a(x),C^b(y)\} = e^{a0}(x)\{H_0(x),H_0(y)\}e^{b0}(y) + H_0(x) \times\{e^{a0}(x),H_0(y)\}e^{b0}(y) + \cdots$. On the right hand side of this Poisson bracket as well as of the brackets $\{C^a(x),\Gamma^{ik}(y)\}$ and $\{C^a(x),\Gamma^k(y)\}$, there will always appear a combination of the constraints $H_0 = e_{a0}C^a$, Γ^{ik}, Γ^k and

$$H_i = e_{ai}C^a - \Gamma^m T_{0mi} - \Gamma^{lm}T_{lmi} - \frac{1}{2g^{00}} \times\bigg(g_{ik}g_{jl}P^{kl} + \frac{1}{2}g_{ij}P\bigg)\Gamma^j. \tag{47}$$

The expression above follows from Eq. (29). All constraints of the theory are first class, and therefore the theory is well defined regarding time evolution.

The Hamiltonian density [Eq. (31)] determines the time evolution of any field quantity $f(x)$:

$$\dot f(x) = \int d^3y\{f(x),H(y)\}\big|_{\Gamma^{ik}=\Gamma^k=0}. \tag{48}$$

Physical quantities take values in the subspace of the phase space $\mathbf{P}_\Gamma$ defined by Eq. (36). In this subspace the constraints C^a become

$$C^a = e^{a0}H_0 + e^{ai}H_i. \tag{49}$$

Restricting considerations to $\mathbf{P}_\Gamma$ we note that if H_0 vanishes, then $e_{a0}C^a$ also vanishes. Since $\{e_{a0}\}$ are arbitrary, it follows that $C^a = 0$. In order to arrive at this conclusion we note that the constraints C^a are independent of e_{a0}. From the orthogonality relation $e_{a\mu}e^{a\lambda} = \delta^\lambda_\mu$ we obtain $\delta e^{b\mu}/\delta e_{a0} = -e^{a\mu}e^{b0}$. Using this variational relation and Eqs. (22) and (49), it is possible to show that

$$\begin{aligned}\frac{\delta C^a}{\delta e_{b0}} &= \frac{\delta}{\delta e_{b0}}(e^{a0}H_0 + e^{ai}H_i) = -e^{b0}e^{a0}H_0 + e^{a0}\frac{\delta H_0}{\delta e_{b0}} - e^{bi}e^{a0}H_i\\ &= -e^{b0}e^{a0}H_0 + e^{a0}(e^{b0}H_0 + e^{bi}H_i) - e^{bi}e^{a0}H_i = 0.\end{aligned} \tag{50}$$

H_i does not depend explicitly or implicitly on e_{a0}. We remark that by taking the variation with respect to e_{b0} of both sides of Eq. (26), $H_0 = e_{a0}C^a$, we arrive at

$$C^b = C^b = e_{a0}\frac{\delta C^a}{\delta e_{b0}},$$

from which follows the general result $e_{a0}(\delta C^a/\delta e_{b0}) = 0$. Taking into account the arbitrariness of e_{a0} in the latter equation, we are led to Eq. (50).

Therefore the vanishing of the Hamiltonian constraint H_0 implies the vanishing of C^a, and ultimately of the vector constraint H_i. Moreover we observe from Eqs. (47) and (49) that H_i can be obtained from H_0 in $\mathbf{P}_\Gamma$ according to

$$e_{ai}\frac{\delta}{\delta e_{a0}}H_0 = e_{ai}C^a = H_i. \tag{51}$$

Thus H_i is *derived* from H_0. In the complete phase space the vanishing of H_i is a consequence of the vanishing of H_0, Γ^{ik} and Γ^k.

Finally we would like to remark that the Hamiltonian formulation of the theory can be described more succinctly in terms of the constraints H_0, Γ^{ik} and Γ^k, by the Hamiltonian density in the form

$$H(e_{ak},\Pi^{ak},e_{a0},\alpha_{ik},\beta_k)=H_0+\alpha_{ik}\Gamma^{ik}+\beta_k\Gamma^k. \quad (52)$$

The Poisson brackets between these constraints are given by Eqs. (37) and (40)–(44). They constitute a first class set, except for the fact that on the right hand side of Eq. (44) there appears the constraint H_i. However, this poses no problem for the consistency of the constraints provided H_0, Γ^{ik} and Γ^k are taken to vanish at the initial time $t=t_0$. Let $\phi(x^i,t)$ represent any of the latter constraints. At the initial time we have $\phi(x^i,t_0)=0$. At $t_0+\delta t$ we find $\phi(x^i,t_0+\delta t)=\phi(x^i,t_0)+\dot{\phi}(x^i,t_0)\delta t$ such that $\dot{\phi}(x^i,t_0)=\{\phi(x^i,t_0),\mathbf{H}\}$. Since the vanishing of H_i at an instant of time is a consequence of the vanishing of H_0, Γ^{ik} and Γ^k at the same time, the consistency of the constraints is guaranteed at any $t>t_0$.

VI. DISCUSSION

The Weitzenböck space-time allows a consistent description of the Hamiltonian formulation of the gravitational field. Although the underlying geometry is not Riemannian, the Lagrangian field equations (4) assure that the theory determined by Eq. (1) is equivalent to Einstein's general relativity. To our knowledge there does not exist any impediment based on experimental facts that rules out the teleparallel geometry in favor of the Riemannian geometry for the description of the physical space-time. The natural geometrical setting for teleparallel gravity is the teleparallel geometry. The Hamiltonian formulation of the TEGR in the Riemannian geometry, with local SO(3,1) symmetry, requires the introduction of a large number of field variables that renders an intricate constraint structure [19].

We have shown that the vector constraint H_i can be obtained from the Hamiltonian constraint H_0 by means of a functional derivative of H_0, making use of the orthogonality properties of the tetrads in the reduced phase space $\mathbf{P}_\Gamma$. However, it is an independent constraint. In contrast, in the ADM formulation the Hamiltonian and vector constraints are not mutually related, and in practice one has to consider both constraints for the dynamical evolution via Hamilton equations.

The number of degrees of freedom may be counted as the total number of canonical variables, e_{ak} and Π^{ak}, minus twice the number of first class constraints. Therefore, we have $24-20=4$ degrees of freedom in the phase space, as expected. Since the constraints Γ^{ik} and Γ^k are first class, they act on e_{ak}, and Π^{ak}, and generate symmetry transformations. In particular, for $e_{a\mu}$ we have

$$\delta e_{ak}(x)=\int d^3z[\varepsilon_{ij}(z)\{e_{ak}(x),\Gamma^{ij}(z)\}+\varepsilon_j(z)\times\{e_{ak}(x),\Gamma^j(z)\}]=\int d^3z\left[\varepsilon_{ij}(z)\frac{\delta\Gamma^{ij}(z)}{\delta\Pi^{ak}(x)}+\varepsilon_j(z)\frac{\delta\Gamma^j(z)}{\delta\Pi^{ak}(x)}\right]=\varepsilon_{ik}e_a{}^i+\varepsilon_k e_a{}^0, \quad (53)$$

where $\varepsilon_{ij}(x)=-\varepsilon_{ji}(x)$ and $\varepsilon_j(x)$ are infinitesimal parameters. Note that these transformations do *not* act on e_{a0}. This issue has not been completely analyzed. The physical implications of these symmetries to the theory are currently under investigation.

In an analysis of a theory described by a Lagrangian density similar to Eq. (1), Møller pointed out that some supplementary conditions on the tetrads are needed. He suggested that these conditions arise from suitable boundary conditions for the field equations, possibly in the form of an antisymmetric tensor. These supplementary conditions would uniquely determine a *tetrad lattice* [4], apart from a constant rotation of the tetrads in the lattice. The problem of consistently defining these supplementary conditions is likely to be related to the symmetry transformation determined by Eq. (53).

The Hamiltonian density [Eq. (52)] determines the time evolution of field quantities via Eq. (48), and in particular of the metric tensor g_{ij} of three-dimensional spacelike hypersurfaces. This property might simplify approaches to a canonical, nonperturbative quantization of gravity provided we manage to construct the reduced phase space determined by Eq. (36).

After implementing the primary constraints via Eq. (36), the first term of C^a is given by $-\partial_i\Pi^{ai}$, with Π^{ai} defined by Eq. (9). From our previous experience (cf. Ref. [16]) we are led to conclude that this term is related to energy and momentum of the gravitational field. In the present case we also interpret equations $C^a=0$ as energy-momentum equations for the gravitational field. According to this interpretation, the integral form of the constraint equation $C^{(0)}=0$ can be written in the form $E-\mathcal{H}=0$. Integration of $-\partial_i\Pi^{ai}$ over the whole three-dimensional space yields the ADM energy. A complete analysis of this issue will be presented elsewhere.

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[1] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: an Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[2] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).

[3] R. Weitzenböck, *Invarianten Theorie* (Nordhoff, Groningen, 1923).

[4] C. Møller, *Tetrad Fields and Conservation Laws in General Relativity*, Proceedings of the International School of Physics

"Enrico Fermi," edited by C. Møller (Academic Press, London, 1962); *Conservation Laws in the Tetrad Theory of Gravitation*, Proceedings of the Conference on Theory of Gravitation, Warszawa and Jablonna 1962 (Gauthier-Villars, Paris, and PWN-Polish Scientific Publishers, Warszawa, 1964) (NORDITA Publication No. 136).

[5] F. W. Hehl, in *Proceedings of the 6th School of Cosmology and Gravitation on Spin, Torsion, Rotation and Supergravity*, Erice, 1979, edited by P. G. Bergmann and V. de Sabbata (Plenum, New York, 1980); F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, Phys. Rep. **258**, 1 (1995).

[6] K. Hayashi, Phys. Lett. **69B**, 441 (1977); K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979); **24**, 3312 (1981).

[7] W. Kopczyński, J. Phys. A **15**, 493 (1982); Ann. Phys. (N.Y.) **203**, 308 (1990).

[8] F. Müller-Hoissen and J. Nitsch, Phys. Rev. D **28**, 718 (1983); Gen. Relativ. Gravit. **17**, 747 (1985).

[9] P. Baekler and E. W. Mielke, Fortschr. Phys. **36**, 549 (1988); E. W. Mielke, Phys. Rev. D **42**, 3388 (1990); Ann. Phys. (N.Y.) **219**, 78 (1992).

[10] J. M. Nester, Int. J. Mod. Phys. A **4**, 1755 (1989).

[11] V. C. de Andrade and J. G. Pereira, Phys. Rev. D **56**, 4689 (1997).

[12] J. W. Maluf, J. Math. Phys. **35**, 335 (1994).

[13] J. A. Schouten, *Ricci Calculus*, 2nd ed. (Springer-Verlag, London, 1954), p. 142.

[14] J. M. Nester, J. Math. Phys. **33**, 910 (1992).

[15] J. Schwinger, Phys. Rev. **130**, 1253 (1963).

[16] J. W. Maluf, J. Math. Phys. **36**, 4242 (1995).

[17] J. W. Maluf and J. F. da Rocha-Neto, Gen. Relativ. Gravit. **31**, 173 (1999).

[18] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974); L. D. Faddeev, Sov. Phys. Usp. **25**, 130 (1982) [Usp. Fiz. Nauk. **136**, 435 (1982)].

[19] M. Blagojević and I. A. Nikolić, Phys. Rev. D **62**, 024021 (2000).