Chern-Simons formulation of noncommutative gravity in three dimensions

M. Bañados and O. Chandía
Facultad de Física, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile

N. Grandi,* F. A. Schaposnik,† and G. A. Silva*
Departamento de Física, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina
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We formulate noncommutative three-dimensional (3D) gravity by making use of its connection with 3D Chern-Simons theory. In the Euclidean sector, we consider the topology $T^2 \times R$ and show that the 3D black hole solves the noncommutative equations. We then consider the black hole on a constant U(1) background and show that the black hole charges (mass and angular momentum) are modified by the presence of this background.

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I. INTRODUCTION

Recently, field theories in noncommutative spaces have attracted much attention, partly in connection with string theory. More specifically, it has been shown that noncommutative U(N) gauge theory emerges in a certain low energy limit of a system of Dp-branes in a constant Neveu-Schwarz B field background [1–3]. In general, gauge theories can be formulated in noncommutative spaces starting from Lagrangians written in terms of ordinary fields multiplied using the Moyal * product. It should be noted that consistency requires that the gauge group has to be U(N) [or certain subgroups of U(N) [4–6]].

It is then natural to analyze whether noncommutative extensions can be also constructed for gravity. There have been several investigations on this issue that basically start by gauging, instead of the SO(d) Lorentz group, the U(1,d -1) [7–12] (or some orthogonal and symplectic subalgebras of unitary groups [4–6]) and then define the theory in terms of vielbeins and spin connection to be multiplied using the * product.

It is well known that in three-dimensional space-time, (ordinary) gravity can be formulated as a Chern-Simons theory [13,14]. Many aspects, both at the classical and quantum levels, have been understood using this connection since, through field redefinitions, it simplifies the equations and introduces a rich mathematical structure. The construction of a black hole in $2+1$ space-time with a negative cosmological constant [the so called Bañados-Teitelboim-Zanelli (BTZ) blackhole [15,16]] also enhanced the interest in 3D gravity, particularly in view of the role it plays in string theory [17].

The goal of this work is to use the Chern-Simons formulation of three-dimensional (3D) gravity to give a definition for 3d noncommutative gravity. We will rely on the fact that many classical and quantum aspects of noncommutative Chern-Simons theory are well understood [18–31] to define the noncommutative 3D gravity action in terms of the corresponding noncommutative Chern-Simons action (NCCS).[1]

The paper is organized as follows. We start by describing in Sec. II the NCCS theory for the group GL$(2,C)$, the one that will be relevant for the formulation of noncommutative 3D gravity. Then, in Sec. III we establish the connection between gauge fields and gravitational variables (triad and spin connection) so that the noncommutative "Einstein equations," and their corresponding action, can be obtained. We also work out the metric formulation of the equations. In Sec. IV we study gravitational solutions for the particular topology $M_3 = T^2 \times \mathfrak{R}$. After showing the chiral character of these solutions, we construct the corresponding metric and explore its conformal properties and relate it to the corresponding commutative solutions. In Sec. V we couple the chiral solution to a constant Abelian field and discuss how noncommutative effects determine the properties of the resulting black hole solution.

II. NONCOMMUTATIVE CHERN-SIMONS THEORY

Noncommutative Chern-Simons theory can be defined by the equations of motion

$$\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + \mathcal{A}_\mu * \mathcal{A}_\nu - \mathcal{A}_\nu * \mathcal{A}_\mu = 0, \tag{1}$$

which are invariant under the noncommutative gauge transformations

$$\mathcal{A}'_\mu = U^{-1} * \mathcal{A}_\mu * U + U^{-1} * \partial_\mu U. \tag{2}$$

Here the $*$ product of two functions $f(x)$ and $g(x)$ is defined as

$$(f*g)(x) = \exp\left|\left(\frac{i}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}\right) f(x) g(y)\right|_{y=x} \tag{3}$$

with $\theta_{\mu\nu}$ a constant antisymmetric matrix.

*Associated with CONICET.
†Associated with CICBA.

[1] There is another kind of noncommutative field theories, namely, the so called q-deformed theories. In this context, a q-deformed 3D gravity theory has been discussed using the CS connection [32].

It is important to note that noncommutative Chern-Simons theory is not invariant under diffeomorphisms. In particular, the known relation between diffeomorphisms and gauge transformations, discussed in Ref. [14], breaks down. In other words, the group of gauge transformations does not include the diffeomorphisms, as in the commutative case. We also mention that in the applications to general relativity the relevant gauge group is noncompact and therefore the quantization is nontrivial. We shall restrict our discussion to classical considerations.

The Seiberg-Witten [3] map provides a powerful method to find solutions to Eq. (1). In fact, the key property of this map is that $\hat{\mathcal{A}}+\hat{\delta}\hat{\mathcal{A}}=\hat{\mathcal{A}}(\mathcal{A}+\delta\mathcal{A})$ (with $\mathcal{A}$ and $\hat{\mathcal{A}}$ gauge fields for spaces with different values of $\theta_{\mu\nu}$). Thus, if $\mathcal{A}$ is a solution to the commutative equation $d\mathcal{A}+\mathcal{A}\mathcal{A}=0$, it follows that $\hat{\mathcal{A}}$ is a solution to the noncommutative equation.

Euclidean gravity, which will be our main interest here, can be formulated as a Chern-Simons theory for the group $\mathrm{SL}(2,C)$. It is well known, however, that in the noncommutative case this group is not closed with respect to the Moyal product and thus we are forced to consider $\mathrm{GL}(2,C)$. The gauge field $\mathcal{A}\in\mathrm{GL}(2,C)$ can be expanded in the basis $\{J_a,i\}$,

$$\mathcal{A}_\mu=A^a_\mu J_a+b_\mu i, \tag{4}$$

where $J_1=(i/2)\sigma_1, J_2=-(i/2)\sigma_2, J_3=(i/2)\sigma_3$ are anti-Hermitian (σ_a are the Pauli matrices). Since A^a and b are complex, we define a second field

$$\bar{\mathcal{A}}_\mu=\bar{A}^a_\mu J_a+\bar{b}_\mu i \tag{5}$$

which satisfies the Chern-Simons equations as well. It is conventional to use the same basis $\{J_a,i\}$ for both fields and thus $\bar{\mathcal{A}}_\mu$ is not the complex conjugate of $\mathcal{A}_\mu$.

The Abelian field b can be set equal to zero in the commutative case because it decouples from A^a. This is no longer true in the noncommutative theory, although solutions with $b=0$ do exist.

The full set of equations for $\mathcal{A}$ is

$$F^a[A]=-i(A^a*b+b*A^a),$$

$$db=-i(b*b+(1/4)A^a*A_a) \tag{6}$$

with $F^a[A]=dA^a+(1/2)\epsilon^a{}_{bc}A^b*A^c$. The right-hand side terms are zero at $\theta=0$ showing that A^a and b are decoupled in the commutative limit. For future reference, we mention that "flat" solutions with $F^a=0$ exist provided

$$A^a*b+b*A^a=0. \tag{7}$$

Analogous equations can be written for $\bar{\mathcal{A}}$.

III. THREE-DIMENSIONAL NONCOMMUTATIVE GRAVITY

A. Connection representation

Consider a GL(2,C) gauge field $\mathcal{A}$, satisfying two copies of Eq. (1):

$$d\mathcal{A}+\mathcal{A}*\mathcal{A}=0, \tag{8}$$

$$d\bar{\mathcal{A}}+\bar{\mathcal{A}}*\bar{\mathcal{A}}=0. \tag{9}$$

(Here, the wedge symbol has been omitted.) Now we define the combinations

$$e=\frac{l}{2i}(\mathcal{A}-\bar{\mathcal{A}}),$$

$$w=\frac{1}{2}(\mathcal{A}+\bar{\mathcal{A}}), \tag{10}$$

where $e=e^aJ_a+e^4 i$ and $w=w^aJ_a+w^4 i$. These relations are the natural noncommutative generalization of

$$e^a=\frac{l}{2i}(A^a-\bar{A}^a),$$

$$w^a=\frac{1}{2}(A^a+\bar{A}^a). \tag{11}$$

Adding and subtracting the Chern-Simons equations, it is direct to prove that e and w satisfy the noncommutative "Einstein equations":

$$dw+w*w-\frac{1}{l^2}e*e=0, \tag{12}$$

$$de+w*e+e*w=0. \tag{13}$$

These equations can be derived from the noncommutative "Einstein-Hilbert" action

$$I[e,w]=\int \mathrm{Tr}\left(R*e-\frac{1}{3l^2}e*e*e\right), \tag{14}$$

where $R=dw+w*w$. The variation with respect to the triad yields Eq. (12) while the variation with respect to w yields the noncommutative torsion condition (13). In deriving the equations of motion from Eq. (14) one has to take into account surface terms which arise in handling Moyal products (and are absent in the ordinary commutative case). This terms vanish for the choice of $\theta_{\mu\nu}$ that will be done below (see Sec. IV).

Despite the similarities between the action (14) and the usual Einstein-Hilbert action, it should be kept in mind that, in the former, the Abelian fields b and $\bar{b}$ are coupled to e^a and w^a in a nontrivial way. The full action (14) depends on all fields,

$$I=I[e^a,w^a,b,\bar{b}]. \tag{15}$$

The couplings between b and the gravitational variables are proportional to θ. We define noncommutative three-dimensional gravity by this action.

If we set the Abelian fields equal to zero, Eqs. (12) and (13) become

$$R^a - \frac{1}{l^2}\epsilon^a{}_{bc}e^b * e^c = 0, \tag{16}$$

$$de^a + (1/2)\epsilon^a{}_{bc}w^b * e^c + (1/2)\epsilon^a{}_{bc}e^b * w^c = 0, \tag{17}$$

where $R^a = dw^a + (1/2)\epsilon^a{}_{bc}w^b * w^c$. The first equation can be regarded as a noncommutative constant curvature condition, written in terms of connections. The second equation is the analogous to a torsion condition. This equation, however, does not imply that the affine connection is symmetric.

Equations (16) and (17) are valid provided the equations for the Abelian field are satisfied with $b = \bar{b} = 0$. This implies,

$$\frac{1}{l^2}e^a * e_a - w^a * w_a = 0, \tag{18}$$

$$e^a * w_a + w_a * e^a = 0 \tag{19}$$

(which are identically satisfied at $\theta = 0$). We shall display below explicit solutions fulfilling these conditions.

B. Metric representation

Equations (16) and (17) have the same form of Einstein equations in the triad formalism, where all products of functions have been replaced by the $*$ product. It is now natural to ask whether there exists a metric formulation for them.

We shall assume that the constraints (19) are satisfied and try to write Eq. (16) in terms of the metric and affine connection. (See Refs. [8–11] for other approaches to this problem in four dimensions.)

We define the metric and affine connection as[2]

$$g_{\mu\nu} = e^a{}_\mu * e^b_\nu\, \eta_{ab}, \tag{20}$$

$$\Gamma^\mu_{\lambda\rho} = \epsilon_{abc}e^{\mu a} * w^b{}_\rho * e^c{}_\lambda + e^\mu{}_a * \partial_\rho e^a{}_\lambda . \tag{21}$$

In other words, $g_{\mu\nu}$ and $\Gamma^\rho{}_{\mu\nu}$ represent, as usual, the metric and connection in the coordinate basis. Given e^a and w^a, the above formulas completely determines g and Γ. If e^a and w^a satisfy the Chern-Simons equations, we would like to find the differential equation satisfied by g and Γ.

The curvature in the coordinate basis is

$$R^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\sigma * \Gamma^\sigma{}_\nu \qquad (\Gamma^\mu{}_\nu = \Gamma^\mu{}_{\nu\sigma}dx^\sigma), \tag{22}$$

and it is related to R^a by the formula

$$R^\mu{}_\nu = \epsilon_{abc}\, e^{\mu a} * R^b * e^c{}_\nu . \tag{23}$$

This follows by direct replacement of Eq. (21) into Eq. (22), and it expresses the fact that the curvature is a tensor. Since R^a satisfies Eq. (16) we find the "Einstein" equation

$$R^\mu{}_{\nu\ \alpha\beta} = -\frac{1}{l^2}(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu}) + E^\mu{}_{\nu\ \alpha\beta}, \tag{24}$$

where $g_{\mu\nu}$ is defined in Eq. (20), and

$$E^\mu{}_{\nu\ \alpha\beta} = \frac{1}{2l^2}e^\mu{}_a * (e^a{}_{[\alpha} * e^b{}_{\beta]} - e^b{}_{[\beta} * e^a{}_{\alpha]}) * e_{b\nu}. \tag{25}$$

The first term in Eq. (24) is the usual contribution from the cosmological constant to the Einstein equations. Recall, however, that in this theory the metric is not symmetric. The second term (E) is a purely noncommutative effect, depending on the commutator of triads with respect to the Moyal product, and cannot be expressed in terms of the metric only.

To summarize, given e^a and w^a satisfying the Chern-Simons equations of motion then the metric (20) and affine connection (21) satisfy the "Einstein" equation (24). We shall exhibit below a family of solutions satisfying these equations.

IV. SOLUTIONS

Before discussing the gravitational solutions, we shall make some general remarks on the solutions to the Chern-Simons equations. All solutions considered here live on the topology $M_3 = T^2 \times \Re$. We shall not consider the generalization to other topologies with higher genus. The local coordinates on T^2 are $\{z, \bar{z}\}$ and $\rho \in \Re$. The components of the gauge field are then $\mathcal{A}_\mu = \{\mathcal{A}_z, \mathcal{A}_{\bar{z}}, \mathcal{A}_\rho\}$. We shall take $\theta_{\rho z} = \theta_{\rho\bar{z}} = 0$ while the noncommutative coordinates satisfy

$$[z, \bar{z}] = \theta. \tag{26}$$

This means that, to first order in θ,

$$f * g = f\,g + \frac{\theta}{2}(\partial f \bar{\partial} g - \bar{\partial} f \partial g) + \mathcal{O}(\theta^2) \tag{27}$$

with $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$. In particular, we find the Moyal representation of Eq. (26), $z * \bar{z} - \bar{z} * z = \theta$. We shall not consider the generalization to other topologies with higher genus.

The choice of manifold M_3 and nontrivial component of $\theta_{\mu\nu}$ ensures that when varying the CS action one can use the cyclic property of the $*$ product without worrying about surface terms. The boundary condition $\mathcal{A}_{\bar{z}} = 0$ is required in order to have well defined functional derivatives of the CS action.

It should be clear that the 3D black hole [15,16] (the Euclidean three-dimensional black hole has been studied in

[2]The definition of the affine connection can be motivated by the gauge invariance of the action. Under gauge transformations the spin connection transforms as $w \to w' = U^{-1} * w * U + U^{-1} * dU$. Let $w' = \Gamma^\rho{}_{\lambda\sigma}$ be the connection in a coordinate basis related to the tangent basis via the matrix $U = e^a{}_\mu$. The new connection $\Gamma^\rho{}_{\lambda\sigma}$ becomes (21). This equation can also be expressed as $\partial_\rho e^a{}_\lambda + \epsilon^a{}_{bc}w^b{}_\rho * e^c{}_\lambda - e^a{}_\mu * \Gamma^\mu{}_{\lambda\rho} = 0$, i.e., the full covariant derivative of $e^a{}_\mu$ is zero.

Ref. [33], and Euclidean anti–de Sitter space in Ref. [34]) is a solution to the full noncommutative equations simply because this field has two Killing vectors ∂_z and $\partial_{\bar{z}}$, which effectively reduce the Moyal product to the usual one.

In order to explore the noncommutative structure, we need to look at more general solutions. We shall start by looking at solutions to the noncommutative Chern-Simons equations.

A. The chiral solution

Let us rewrite the first of Eqs. (6) in the form

$$
\begin{aligned}
&F^a_{\rho z}[A]+i[b_\rho,A^a_z]+i[A^a_\rho,b_z]=0,\\
&F^a_{\rho \bar{z}}[A]+i[b_\rho,A^a_{\bar{z}}]+i[A^a_\rho,b_{\bar{z}}]=0,\\
&F^a_{z\bar{z}}[A]+i[b_z,A^a_{\bar{z}}]+i[A^a_z,b_{\bar{z}}]=0,
\end{aligned}
\tag{28}
$$

where $[A,b]=A*b-b*A$. Now, fixing the gauge to

$$A_\rho=iJ^3,\quad b_\rho=0 \tag{29}$$

the first two equations (28) become

$$
\begin{aligned}
&\partial_\rho A^a_z+i\,\delta^3{}_b\varepsilon^{ab}{}_c A^c_z=0,\\
&\partial_\rho A^a_{\bar{z}}+i\,\delta^3{}_b\varepsilon^{ab}{}_c A^c_{\bar{z}}=0
\end{aligned}
\tag{30}
$$

with solution

$$
\begin{aligned}
&A_z=d^{-1}\tilde{A}_z(z,\bar{z})d,\\
&A_{\bar{z}}=d^{-1}\tilde{A}_{\bar{z}}(z,\bar{z})d,
\end{aligned}
\tag{31}
$$

where

$$d=e^{i\rho J^3}. \tag{32}$$

Now, the boundary condition $A_{\bar{z}}|_{\partial M}=0$ implies $\tilde{A}_{\bar{z}}=0$, this resulting in $A_{\bar{z}}=0$. Finally, replacing this solution in the last equation in Eq. (28), we obtain

$$\partial_{\bar{z}}A^a_z+i[b_{\bar{z}},A^a_z]=D_{\bar{z}}[b]A^a_z=0. \tag{33}$$

Let us now study the last equation in Eq. (6)

$$
\begin{aligned}
&\partial_\rho b_z-\partial_z b_\rho+i[b_\rho,b_z]+\frac{i}{4}[A^a_\rho,A_{za}]=0,\\
&\partial_\rho b_{\bar{z}}-\partial_{\bar{z}} b_\rho+i[b_\rho,b_{\bar{z}}]+i\frac{i}{4}[A^a_\rho,A_{\bar{z}a}]=0,\\
&\partial_z b_{\bar{z}}-\partial_{\bar{z}} b_z+i[b_z,b_{\bar{z}}]+i\frac{i}{4}[A^a_z,A_{\bar{z}a}]=0.
\end{aligned}
\tag{34}
$$

Using $A_{\bar{z}}=0$ and the gauge condition (29), Eq. (34) reads

$$
\begin{aligned}
&\partial_\rho b_z=0,\\
&\partial_\rho b_{\bar{z}}=0,\\
&\partial_z b_{\bar{z}}-\partial_{\bar{z}}b_z+i[b_z,b_{\bar{z}}]=0.
\end{aligned}
\tag{35}
$$

One then sees that $b_z,b_{\bar{z}}$ must be independent of ρ. Being the boundary condition $b_{\bar{z}}|_{\partial M}=0$, this implies that $b_{\bar{z}}=0$ everywhere. The remaining equation is

$$\partial_{\bar{z}}b_z=0 \tag{36}$$

and then $b_z=b_z(z)$. With this solution for the $U(1)$ field, the Eq. (33) simplifies to

$$\partial_{\bar{z}}A^a_z=0, \tag{37}$$

which implies $A_z=A_z(z)$.

Then, the general solution to Eqs. (6) with boundary conditions $A_{\bar{z}}|_{\partial M}=b_{\bar{z}}|_{\partial M}=0$, closely related to the 3D black hole, is chiral,

$$
\begin{aligned}
&A_z=d^{-1}\tilde{A}_z(z)d,\\
&A_{\bar{z}}=0,\\
&A_\rho=iJ_3=d^{-1}\partial_\rho d,\\
&b_z=b_z(z),\\
&b_\rho=b_{\bar{z}}=0
\end{aligned}
\tag{38}
$$

with $\tilde{A}_z(z),b_z(z)$ arbitrary Lie algebra-valued functions of z. This configuration solves both, the commutative and noncommutative equations. It can also be checked that it is a fixed point under the Seiberg-Witten map [3]. A similar analysis can be done for the second complex field $\bar{\mathcal{A}}$ leading to a solution analogous to Eq. (38) but with $A_z(z)\to\bar{A}_{\bar{z}}(\bar{z})$, $b_z(z)\to\bar{b}_{\bar{z}}(\bar{z})$ and $d\to d^{-1}$.

A gauge transformation (with group element d^{-1}) brings the solution to the simpler form

$$
\begin{aligned}
&A_z=A_z(z),\\
&A_{\bar{z}}=A_\rho=0,\\
&b_z=b_z(z),\\
&b_{\bar{z}}=b_\rho=0.
\end{aligned}
\tag{39}
$$

An important property of Eq. (39) is its Kac-Moody symmetry under holomorphic gauge transformations. To see this, let us specialize to the case $b_z=0$ and note that the configuration (39) is form invariant under gauge transformations which only depend on z. Let $\lambda=\lambda(z)$. We act with the noncommutative transformation (2) and find

$$\delta A_z=\partial_z\lambda+A_z*\lambda-\lambda*A_z=\partial_z\lambda+A_z\lambda-\lambda A_z, \tag{40}$$

$$\delta A_{\bar{z}}=0, \tag{41}$$

$$\delta A_\rho = 0. \tag{42}$$

The $*$ product has been eliminated because the whole solution only depends on z. This symmetry of the space of solutions (39) is generated by a Kac-Moody algebra and play an important role in various approaches to understand the 3d black hole entropy as well as the Brown-Henneaux conformal symmetry.

B. The metric

Let us construct the metric corresponding to the solution found above. We start from Eq. (20) with the vierbeins e_μ contructed according to Eq. (10) which, for the affine solution takes the form

$$e_z^a J_a = \frac{l}{2i} d^{-1}\tilde{A}(z) d, \qquad e_{\bar z}^a J_a = -\frac{l}{2i} d\,\tilde{\bar{A}}(\bar z) d^{-1},$$
$$e_\rho^a J_a = l J_3. \tag{43}$$

Defining

$$\tilde{A} = \frac{i}{2}\begin{pmatrix} A^3 & A^+ \\ A^- & -A^3 \end{pmatrix}, \qquad \tilde{\bar{A}} = \frac{i}{2}\begin{pmatrix} \bar{A}^3 & \bar{A}^+ \\ \bar{A}^- & -\bar{A}^3 \end{pmatrix} \tag{44}$$

then, the symmetric (arc length) part of the associated metric is $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$,

$$ds^2 = l^2 d\rho^2 - \frac{l^2}{4}({A^3}^2 + A^+A^-)dz^2 - \frac{l^2}{4}({\bar{A}^3}{}^2 + \bar{A}^+\bar{A}^-)d\bar{z}^2$$
$$+\frac{l^2}{8}(2\{A^3,\bar{A}^3\}_+ + \{A^-,\bar{A}^+\}_+ e^{-2\rho}$$
$$+\{A^+,\bar{A}^-\}_+ e^{2\rho})dz d\bar{z} + il^2\bar{A}^3 d\bar{z} d\rho - il^2 A^3 dz d\rho. \tag{45}$$

At this point, we are interested in determining the conditions to be imposed on the gauge fields in order to have an asymptotically AdS metric. To this end, we follow Ref. [35] extended to the noncommutative case. The nondiagonal components should be absent. This can be achieved taking $A^3 = \bar{A}^3 = 0$, conditions that extend to the noncommutative case the first Polyakov reduction condition. The resulting metric is

$$ds^2 = l^2 d\rho^2 - \frac{l^2}{4}A^+A^- dz^2 - \frac{l^2}{4}\bar{A}^+\bar{A}^- d\bar{z}^2$$
$$+\frac{l^2}{8}(\{\bar{A}^+, A^-\}_+ e^{-2\rho} + \{A^+,\bar{A}^-\}_+ e^{2\rho})dz d\bar{z} \tag{46}$$

which has an asymptotic $(\rho\to\infty)$ form

$$ds^2 = l^2 d\rho^2 + \frac{l^2}{8}\{A^+,\bar{A}^-\}_+ e^{2\rho} dz d\bar{z}. \tag{47}$$

Then, to match with the AdS form we need to impose the condition

$$\{A^+,\bar{A}^-\}_+ = 8. \tag{48}$$

Taking the derivatives with respect to z and $\bar{z}$ we obtain the relations (remember that A^+ is holomorphic and $\bar{A}^-$ is antiholomorphic)

$$\{\partial_z A^+, \bar{A}^-\}_+ = 0, \qquad \{A^+, \partial_{\bar z}\bar{A}^-\}_+ = 0. \tag{49}$$

In the usual commutative case these relations will imply constants $A^+,\bar{A}^-$. To test this in the noncommutative case, let us first observe that following Ref. [36], one can write

$$f(z) * g(\bar z) = e^{(\theta/2)\partial\bar\partial} f(z) g(\bar z),$$
$$g(\bar z) * f(z) = e^{-(\theta/2)\partial\bar\partial} f(z) g(\bar z) \tag{50}$$

which implies

$$\frac{1}{2}\{f,g\} = \frac{1}{2}(e^{(\theta/2)\partial\bar\partial} + e^{-(\theta/2)\partial\bar\partial}) f(z) g(\bar z)$$
$$= \cosh\left(\frac{\theta}{2}\partial\bar\partial\right) f(z) g(\bar z). \tag{51}$$

Using this, Eqs. (49) can be rewritten as

$$\cosh\left(\frac{\theta}{2}\partial\bar\partial\right)(\partial_z A^+ \bar{A}^-) = 0, \quad \cosh\left(\frac{\theta}{2}\partial\bar\partial\right)(A^+ \partial_{\bar z}\bar{A}^-) = 0. \tag{52}$$

Calling ψ_λ and λ the eigenfunctions and eigenvalues of $\partial\bar\partial$ and assuming that $\{\psi_\lambda\}$ is complete, one can write $\cosh[(\theta/2)\partial\bar\partial] = \Sigma_\lambda \cosh[(\theta/2)\lambda]|\psi_\lambda\rangle\langle\psi_\lambda|$. This ensures that $\cosh[(\theta/2)\partial\bar\partial]$ has no zero modes and then one has, from Eq. (52)

$$\partial_z A^+ \bar{A}^- = 0, \quad A^+ \partial_{\bar z}\bar{A}^- = 0 \tag{53}$$

this implies that $A^+,\bar{A}^-$ should be constants. Then we have found the second reduction condition

$$A^+ = 2, \quad \bar{A}^- = 2. \tag{54}$$

We conclude that in order to have an asymptotic AdS form in the noncommutative case, one needs to impose just the usual Polyakov reduction conditions, previously discussed in Ref. [35]. In this case, Eqs. (44) take the form

$$A_z = i\begin{pmatrix} 0 & e^\rho \\ \frac{1}{2l}T(z)e^{-\rho} & 0 \end{pmatrix}, \tag{55}$$

$$\bar{A}_{\bar z} = i\begin{pmatrix} 0 & \frac{1}{2l}\bar{T}(\bar z)e^{-\rho} \\ e^\rho & 0 \end{pmatrix}, \tag{56}$$

$$A_\rho = -\bar{A}_\rho = iJ_3,$$
$$A_{\bar{z}} = \bar{A}_z = b = \bar{b} = 0. \tag{57}$$

With this, the symmetric metric as defined in Eq. (45) becomes

$$ds^2 = l^2 d\rho^2 - \frac{l}{2} T dz^2 - \frac{l}{2} \bar{T} d\bar{z}^2 + \frac{1}{8}(\{\bar{T},T\}_+ e^{-2\rho} + 8l^2 e^{2\rho}) dz d\bar{z}. \tag{58}$$

We see that the only component of the symmetric metric affected by noncommutativity is $g^S_{z\bar{z}}$. Using Eq. (50), this component can be written as

$$g^S_{z\bar{z}} = \cosh\left(\frac{\theta}{2}\partial\bar{\partial}\right)\tilde{g}_{z\bar{z}}, \tag{59}$$

$\tilde{g}$ being the metric constructed in Ref. [37] for the commutative case. The operator $\cosh[(\theta/2)\partial\bar{\partial}]$ acts similar to the identity when applied to the other components of the metric (all derivative terms vanish),

$$g^S_{zz} = \cosh\left(\frac{\theta}{2}\partial\bar{\partial}\right)\tilde{g}_{zz}$$
$$g^S_{\bar{z}\bar{z}} = \cosh\left(\frac{\theta}{2}\partial\bar{\partial}\right)\tilde{g}_{\bar{z}\bar{z}} \tag{60}$$

so that the relation between the commutative and the (symmetric) noncommutative solutions can be compactly written as

$$g^S_{\mu\nu} = \cosh\left(\frac{\theta}{2}\partial\bar{\partial}\right)\tilde{g}_{\mu\nu}. \tag{61}$$

The full metric $g_{\mu\nu} = g^S_{\mu\nu} + g^A_{\mu\nu}$, where $g^A_{\mu\nu}$ is the antisymmetric part, satisfies the "Einstein" equation (24). Note that $g^A_{\mu\nu}$ is in fact nonzero. Its nonzero contributions come from

$$g_{z\bar{z}} = \exp\left(\frac{\theta}{2}\partial\bar{\partial}\right)\tilde{g}_{z\bar{z}}, \quad g_{\bar{z}z} = \exp\left(-\frac{\theta}{2}\partial\bar{\partial}\right)\tilde{g}_{\bar{z}z} \tag{62}$$

which imply

$$g^A_{z\bar{z}} = \sinh\left(\frac{\theta}{2}\partial\bar{\partial}\right)\tilde{g}_{z\bar{z}}. \tag{63}$$

Recall that the deviation of Eq. (24) from the ordinary Einstein equations is encoded in the combination $E^\mu{}_{\nu\alpha\beta}$ which depends on the commutator $[e^a{}_\alpha, e^b{}_\beta]$. In the present case the only nonvanishing contribution to this commutator is the $(\alpha=z, \beta=\bar{z})$ component, and it is proportional to the commutator $[T,\bar{T}] = 2\sinh[(\theta/2)\partial\bar{\partial}]T(z)\bar{T}(\bar{z})$. For future use, let us end this section rewriting the solution (55),(56) in the $A_\rho = 0$ gauge

$$A_z = i\begin{pmatrix} 0 & 1 \\ \frac{1}{2l}T(z) & 0 \end{pmatrix}, \tag{64}$$

$$\bar{A}_{\bar{z}} = i\begin{pmatrix} 0 & \frac{1}{2l}\bar{T}(\bar{z}) \\ 1 & 0 \end{pmatrix}. \tag{65}$$

V. CONSTANT ABELIAN BACKGROUND

We consider in this section the chiral solution considered in the last section coupled to a constant Abelian field of magnitude $F_{z\bar{z}} = i\alpha$. We shall see that the black hole field with constant values of T and $\bar{T}$ will feel the Abelian field due to noncommutative effects.

In order to fix the value of the Abelian field we add to the action the term $-2i\int \mathrm{Tr}(\alpha\mathcal{A})$ where α is a fixed 2-form $\alpha = \alpha dz \wedge d\bar{z}$. This is a term of the kind introduced in Ref. [23]. The equations of motion (1) are replaced by

$$\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + \mathcal{A}_\mu * \mathcal{A}_\nu - \mathcal{A}_\nu * \mathcal{A}_\mu = \alpha_{\mu\nu}\, i. \tag{66}$$

α is a number and it contributes only to the Abelian curvature.[3]

The generalization of the chiral solution satisfying Eq. (66) in the $\mathcal{A}_\rho = 0$ gauge is simply

$$\mathcal{A}_z = A(z) - i\,\alpha\,\bar{z}$$
$$\mathcal{A}_{\bar{z}} = 0, \tag{67}$$
$$\mathcal{A}_\rho = 0.$$

Since the extra term only contributes to the Abelian field, one could naively conclude that the black hole solution has not changed. However, this field depends on both coordinates and noncommutative effects do take place.

The point is that, the noncommutative structure of the gauge transformations changes the affine algebra and, as a result, Polyakov's reduction conditions needs to be modified. Let $\lambda = \lambda(z)$ and compute the noncommutative gauge transformation (2) acting on Eq. (67). The components $\mathcal{A}_\rho$ and $\mathcal{A}_{\bar{z}}$ are left invariant while the transformation for $\mathcal{A}_z$ yields

$$\delta\mathcal{A}_z = \partial_z\lambda + (A_z - i\alpha\bar{z}) * \lambda - \lambda * (A_z - i\alpha\bar{z}),$$
$$= (1 + i\theta\alpha)\partial_z\lambda + A_z\lambda - \lambda A_z. \tag{68}$$

The extra term proportional to θ comes from the Moyal formula $\bar{z} * f - f * \bar{z} = -\theta\,\partial f$. The solution (67) still has an affine holomorphic Kac-Moody symmetry but its form has changed.

[3]A constant noncommutative Abelian field has been studied in detail in Ref. [38].

Even though the extra term $\alpha\theta$ in Eq. (68) does not affect the gauge symmetries in any significant way,[4] it does change the definition of global charges. We shall see that the mass and angular momentum of the black hole are modified by the presence of α.

The point is that under the transformation (68), the reduction condition $A_z^+=2$ is not consistent, and does not yield the Virasoro algebra. The correct reduction conditions are

$$A_z^3=0,\qquad A_z^+=2(1+i\alpha\,\theta),\tag{69}$$

and the Virasoro charge is $T(z)=A^-/(2+2i\alpha\theta)$. The reduced field is then

$$A_z=i(1+i\alpha\theta)\begin{pmatrix}0 & 1\\ \dfrac{T(z)}{2l} & 0\end{pmatrix}.\tag{70}$$

In order to match the boundary conditions (keeping the periodicity of the torus fixed) with the solution (64) we perform a constant gauge transformation on A_z with a group element $g=e^{iaJ_3}$ and $a=\log(1+i\alpha\theta)$. The field (70) is transformed into

$$A_z=i\begin{pmatrix}0 & 1\\ (1+i\alpha\theta)^2\dfrac{T(z)}{2l} & 0\end{pmatrix}\tag{71}$$

[4]In fact one could define $A=(1+\alpha\theta)A'$ and A' would transform in the usual way. This corresponds to the Seiberg-Witten map [3] applied to this particular situation.

which is of the form (64). The antiholomorphic field can be constructed in a similar way and one finds

$$\overline{A_{\bar z}}=i\begin{pmatrix}0 & (1-i\overline{\alpha}\theta)^2\dfrac{\overline{T}(\overline{z})}{2l}\\ 1 & 0\end{pmatrix}.\tag{72}$$

For constant values of T and $\overline{T}$ this field represent a black hole. However, the relation between the mass and angular momentum and the Virasoro charges have changed,

$$Ml=(1+i\alpha\theta)^2T+(1-i\overline{\alpha}\theta)^2\overline{T},\tag{73}$$

$$iJ=(1+i\alpha\theta)^2T-(1-i\overline{\alpha}\theta)^2\overline{T}.\tag{74}$$

It is instructive to expand these relations to first order in θ,

$$lM=lM_0+2\,\theta\alpha\,J_0,\tag{75}$$

$$J=J_0-2\,\theta\alpha\,lM_0,\tag{76}$$

where M_0 and J_0 are the values of M and J at $\alpha=0$. For example one can start at $\alpha=0$ with a nonrotating black hole ($J_0=0$). Then we turn on the Abelian field with $\alpha\neq 0$ and find that the corresponding black hole will have a nonzero angular momentum.

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