Chaos and gravitational waves

Neil J. Cornish

Department of Physics, Montana State University, Bozeman, Montana 59717

The gravitational waveforms of a chaotic system will exhibit a sensitive dependence on initial conditions. The waveforms of nearby orbits decohere on a time scale fixed by the largest Lyapunov exponent of the orbit. The loss of coherence has important observational consequences for systems where the Lyapunov time scale is short compared to the chirp time scale. Detectors that rely on matched filtering techniques will be unable to detect gravitational waves from these systems.

The evolution of two compact objects under their mutual gravitational attraction is a major unsolved problem in general relativity. A vast array of analytical and numerical approximations have been used to attack the problem, but much remains to be understood. Among the many outstanding questions is the degree to which the dynamics may be chaotic [1–4] or effectively chaotic [5,6]. For example, when one or more of the masses is spinning, it has been shown in test-particle [2] and post-Newtonian [3] limits that certain classes of orbits are chaotic, at least when radiation reaction is turned off [4].

Here we consider the physical and observational consequences wrought by chaotic behavior in the orbital dynamics. A key feature of chaotic systems is their sensitive dependence on initial conditions—the so-called "butterfly effect." It is intuitively obvious that a sensitive dependence in the dynamics will be reflected in the gravitational waveforms, but the precise connection has not been established until now. We find that the waveforms of nearby orbits decohere on a time scale fixed by the largest Lyapunov exponent. The observational consequences of this result depend on two time scales: the chirp time scale T_c and the Lyapunov time scale T_λ. A third time scale of interest is the gravitational wave period T_w. The wave period and chirp time scale for a system with reduced mass μ, total mass M, and orbital separation R are given by

$$T_w \simeq \pi M \left(\frac{R}{M} \right)^{3/2} \quad \text{and} \quad T_c \simeq \frac{5}{256} \frac{M^2}{\mu} \left(\frac{R}{M} \right)^4.$$

In the limit that one of the masses is very much smaller than its companion we have $\mu \ll M$ and $T_c \gg T_w$. The Lyapunov time scale has to be calculated on a case-by-case basis, but values as short as $T_\lambda \sim T_w$ are possible for very unstable orbits. Orbits whose Lyapunov time scale is short compared to the chirp time scale will produces highly unpredictable waveforms. The number of templates required to detect these waveforms is exponentially large, making them impossible to detect [3,7] with the Laser Interferometer Gravitational Observatory (LIGO).

The connection between orbital instability and waveform decoherence is easily established. It relies on the fact that the gravitational waveform $h_{\mu\nu}(t)$ can be expressed in terms of the phase space coordinates of the system, so the divergence of nearby trajectories in phase space is reflected in the divergence of nearby waveforms. The phase space coordinates we have in mind are the position and momentum of the masses. These coordinates are well defined in the test particle, post-Newtonian, and post-Minkowski approximations, but their meaning is less clear in the full theory. Indeed, the full evolution equations are a set of coupled, nonlinear partial differential equations, so the real issue is gravitational turbulence, not chaos. We will avoid this complication by assuming that the evolution equations can be approximated by some set of nonlinear ordinary differential equations for the phase space variables $X_i(t)$. Here t is taken to be the time measured by a distant observer, but the choice is unimportant. The evolution equations can be written

$$\frac{dX_i}{dt} = H_i(X_j). \tag{1}$$

Linearizing about a solution to this equation yields an equation for the perturbation,

$$\frac{d\,\delta X_i(t)}{dt} = K_{ij}(t)\,\delta X_j(t), \tag{2}$$

where

$$K_{ij}(t) = \left. \frac{\partial H_i}{\partial X_j} \right|_{X_i(t)} \tag{3}$$

is the infinitesimal evolution matrix. The solution to the linearized equations of motion can be expressed in terms of the evolution matrix $L_{ij}(t)$:

$$\delta X_i(t) = L_{ij}(t)\,\delta X_j(0). \tag{4}$$

The eigenvalues and eigenvectors of $K_{ij}(t)$ contain information about the local stability of the phase space trajectory. In a fixed coordinate system the eigenvectors may vary with time. To avoid this, a nonconstant basis can be defined that stays aligned with the eigenvectors [8]. This is done by parallel transporting the basis vectors along the trajectory $X_i(t)$. Barring degeneracies, the infinitesimal evolution matrix is diagonal in the eigenvector basis, $K_{ij}(t) = \delta_{ij}\lambda_i(t)$ (no summation on the i), where $\lambda_i(t)$ are the eigenvalues of $K_{ij}(t)$. The evolution matrix is also diagonal in the eigenvalue basis and has components

$$L_{ij}(t) = \delta_{ij} \exp\left(\int_0^t \lambda_i(t') dt' \right). \tag{5}$$

The principal Lyapunov exponent for the trajectory $X_i(t)$ is defined as

$$\lambda = \lim_{t \to \infty} \frac{1}{2t} \ln(L_{ji}^*(t) L_{ij}(t)). \tag{6}$$

The multiplicative ergodic theorems establish the existence of this limit for a large range of situations. Trajectories with positive Lyapunov exponents are unstable against small perturbations. When the unstable orbits are dense in some region of phase space, the system is said to be chaotic. A positive principal Lyapunov exponent will dominate the long term dynamics. Setting λ_1 to be the principle eigenvalue, the asymptotic form of the evolution matrix can be written

$$L_{ij}(t) \simeq e^{\lambda t} \begin{bmatrix} f(t) & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{7}$$

where $|f(t)| = 1$. If $X_1(t)$ is real, then $f(t) = 1$.

A gravitational wave can be decomposed into two polarizations with amplitudes $h^+(t)$ and $h^\times(t)$. A gravitational wave detector responds to the wave according to

$$s(t) = h^+(t) F_+ + h^\times(t) F_\times, \tag{8}$$

where F_+ and $F_\times$ are the antenna patterns for each polarization. The antenna patterns for LIGO and Laser Interferometer Space Antenna (LISA) can be found in Refs. [9] and [10] respectively. The gravitational wave amplitude $h(t)$ measured at the detector can be expressed in terms of the phase space variables $X_i(t)$. Nearby orbits have gravitational waveforms that differ by

$$\delta h(t) = \frac{\partial h(X_m)}{\partial X_i}(t) \delta X_i(t)$$

$$= \frac{\partial h(X_m)}{\partial X_i}(t) L_{ij}(t) \delta X_j(0). \tag{9}$$

Employing an eigenvector basis and using the asymptotic form [Eq. (7)] for the evolution matrix yields

$$\delta h(t) \simeq e^{\lambda t} g(t), \tag{10}$$

where

$$g(t) = \frac{\partial h(X_m)}{\partial X_1}(t) f(t) \delta X_1(0) \tag{11}$$

is an oscillatory factor with bounded amplitude. The divergence of the waveforms can be better expressed in terms of their relative phase. Writing the amplitude of the reference trajectory as $h(\Phi)$, and defining $\Phi_0(t)$ to be the phase for which $h(\Phi_0) = 0$, we find the relative phase is given by

$$\delta\Phi(t) = \frac{\delta h(\Phi_0(t))}{h'(\Phi_0(t))}. \tag{12}$$

The quantity

$$\lambda_h = \lim_{t \to \infty} \lim_{\delta\Phi(0) \to 0} \ln\left| \frac{\delta\Phi(t)}{\delta\Phi(0)} \right| \tag{13}$$

is equal to the principal Lyapunov exponent of the trajectory. In other words, the waveforms of nearby trajectories will decohere exponentially in time, on a time scale set by $T_\lambda = \lambda^{-1}$.

A couple of comments are in order. Firstly, the limit $t \to \infty$ used in Eqs. (6) and (13) will return $\lambda = \lambda_h = 0$ if we include the effects of radiation reaction. This reflects the fact that the dynamics is not chaotic in a formal sense [4]. The limit $t \to \infty$ has to be replaced by the limit $t \to T$, where $T_\lambda \ll T \ll T_c$. This is done by replacing the Lyapunov exponent by a finite-time or local Lyapunov exponent [11], and making a similar change to Eq. (13). Clearly, the whole notion of waveform decoherence only makes sense if the Lyapunov time scale T_λ is very much shorter than the chirp time scale T_c. It remains to be seen if this condition is satisfied for any realistic systems. The second comment concerns the gauge noninvariance of the Lyapunov exponents and the choice of time variable. Lyapunov exponents gained a bad reputation when they were used to study the Bianchi type IX dynamics. Different time slices yielded principal Lyapunov exponents that were positive or zero, making it difficult to decide if the dynamics was chaotic. However, this difficulty can be avoided by making relative, rather than absolute comparisons. For example, the ratio T_λ / T_c is the same in any coordinate system, so the choice of time variable is irrelevant. The confusion surrounding the Bianchi type IX dynamics could have been avoided if the Lyapunov time scale had been compared to the average time between bounces [12].

We can illustrate the connection between orbital instability and waveform decoherence with a simple example. Consider the Lagrangian for a test particle in the Schwarzschild spacetime (in units where $G = c = M = 1$)

$$\mathcal{L} = \frac{1}{2}\left(\frac{r-2}{r}\left(\frac{dt}{d\lambda}\right)^2 + \frac{r}{r-2}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 \right.$$

$$\left. + r^2 \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 \right). \tag{14}$$

The system is completely integrable (nonchaotic) as it has four generalized coordinates t, r, θ, and ϕ and four constants of motion: the mass μ; the energy E; the z component of the angular momentum, L_z; and the total angular momentum squared, L^2. Despite the lack of chaos, the dynamics does admit isolated unstable orbits that serve to illustrate the connection between orbital instability and waveform decoherence. Restricting our attention to equatorial orbits with fixed energy E allows us to make the replacement

$$\frac{d}{d\lambda} = \frac{r-2}{rE}\frac{d}{dt} \tag{15}$$

in the equations of motion

$$\dot{p}_r = -\frac{E^3 r}{(r-2)^3} - \frac{E p_r^2}{r(r-2)} + \frac{E p_\phi^2}{r^2(r-2)},$$

$$\dot{p}_\phi = 0,$$

$$\dot{r} = E p_r, \tag{16}$$

$$\dot{\phi} = \frac{E p_\phi}{r(r-2)}.$$

Our reference trajectories are circular orbits with

$$p_r = 0,$$

$$p_\phi = L_z = L,$$

$$r = r_\pm = \frac{1}{2}L(L \pm \sqrt{L^2 - 12}), \tag{17}$$

$$\theta = \omega t.$$

Perturbing the equations of motion about the reference trajectory yields the infinitesimal evolution matrix

$$K_{ij} = \begin{bmatrix} 0 & \dfrac{2EL}{r^2(r-2)} & \dfrac{2E^3(r+1)}{(r-2)^4} - \dfrac{EL^2(3r-4)}{r^3(r-2)^2} & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & \dfrac{E}{r(r-2)} & -\dfrac{2EL(r-1)}{r^2(r-2)^2} & 0 \end{bmatrix}. \tag{18}$$

The eigenvalues are

$$\lambda_1 = -\lambda_2 = \frac{\sqrt{2E^2 r^3(r+1) - L^2(3r-4)(r-2)^2}}{r^2(r-2)^2}, \tag{19}$$

$$\lambda_3 = \lambda_4 = 0.$$

The well known innermost stable circular orbit at $r=6$ has $L = 2\sqrt{3}$, $E = \sqrt{8/9}$, and $\lambda_1 = \lambda_2 = 0$. Circular orbits with larger values of L come in two varieties, those with $r = r_+$ and those with $r = r_-$; that is, setting $E = 1$ and $L = 4$ yields orbits with $r_+ = 12$ and $r_- = 4$. The outer orbit is stable, $\lambda_1 = -\lambda_2 = i\sqrt{6}/108$, while the inner orbit is unstable, $\lambda_1 = -\lambda_2 = 1/2\sqrt{2}$. The unstable orbit at $r = 4$ has eigenvectors

$$\mathbf{e}_1 = \left(\frac{1}{3\sqrt{2}}, 0, \frac{2}{3}, -\frac{1}{\sqrt{2}}\right),$$

$$\mathbf{e}_2 = \left(\frac{1}{3\sqrt{2}}, 0, -\frac{2}{3}, -\frac{1}{\sqrt{2}}\right), \tag{20}$$

$$\mathbf{e}_3 = \mathbf{e}_4 = (1,0,0,0).$$

Since the zero eigenvalues have degenerate eigenvectors, K_{ij} cannot be diagonalized, but it is enough to put K_{ij} into Jordan normal form with the transformation matrix

$$P_{ij} = \begin{bmatrix} \dfrac{7\sqrt{2}}{24} & -\dfrac{7\sqrt{2}}{24} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{7}{6} & \dfrac{7}{6} & 0 & -2 \\ -\dfrac{7\sqrt{2}}{8} & \dfrac{7\sqrt{2}}{8} & \dfrac{7}{8} & 0 \end{bmatrix}. \tag{21}$$

In terms of the transformed phase space coordinates the infinitesimal evolution matrix has components

$$K_{ij} = \begin{bmatrix} \dfrac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & -\dfrac{1}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

and the evolution matrix has components

$$L_{ij} = \begin{bmatrix} e^{t/2\sqrt{2}} & 0 & 0 & 0 \\ 0 & e^{-t/2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Taking the limit defined in Eq. (6) yields the principal Lyapunov exponent $\lambda = 1/2\sqrt{2}$. The Lyapunov time scale for this orbit, $T_\lambda = \lambda^{-1} = 2\sqrt{2}$, is actually *shorter* than the gravitational wave period $T_w = 2\pi$.

Next we need an expression for the gravitational wave amplitude in terms of the phase space variables. The correct approach would be to solve the Teukolsky equation using the method described by Hughes [5], but no analytical solutions exist in the strong field limit. Instead we will use the quad-

rupole approximation, knowing full well that it is being applied outside of its domain of validity. Since our goal is to illustrate a general phenomenon—not to produce accurate templates for LIGO data analysis—a qualitative description of the waveform's dependence on the phase space variables will suffice.

The gravitational wave amplitude can be written as $h_{\mu\nu}(t) = h^+(t)e^+_{\mu\nu} + h^\times(t)e^\times_{\mu\nu}$, where the polarization tensors have nonzero components

$$e^+_{xx} = 1, \quad e^+_{yy} = -1,$$
$$e^\times_{xy} = 1, \quad e^\times_{yx} = 1. \tag{24}$$

A detector situated at $r = R$ on the z axis will encounter a wave with plus polarization given by

$$h^+(t) = -\frac{2\mu E^2}{R}\left[\frac{2p_\phi p_r(r-3)}{(r-2)^2}\sin 2\phi + \left(\frac{E^2 r^2}{(r-2)^3}\right.\right.$$
$$\left.\left. + \frac{p_\phi^2(r+2)}{r(r-2)^2} - \frac{p_r^2(r-3)}{r-2}\right)\cos 2\phi\right], \tag{25}$$

and similarly for the cross polarization. Specializing to the orbit with $E = 1$, $L = 4$, and $r = 4$ we have

$$h^+ = -\frac{8\mu}{R}\cos 2\phi,$$

$$\frac{\partial h^+}{\partial p_r} = -\frac{2\mu}{R}\sin 2\phi,$$

$$\frac{\partial h^+}{\partial p_\phi} = -\frac{3\mu}{R}\cos 2\phi, \tag{26}$$

$$\frac{\partial h^+}{\partial r} = \frac{17\mu}{2R}\cos 2\phi,$$

$$\frac{\partial h^+}{\partial \phi} = \frac{16\mu}{R}\sin 2\phi.$$

Putting everything together in Eq. (9) yields

$$\delta h^+(t) \simeq \frac{\mu}{84R}e^{t/2\sqrt{2}}(119\cos 2\phi - 175\sqrt{2}\sin 2\phi)(3\,\delta r(0)$$
$$+ 6\,\delta p_\phi(0) + 6\sqrt{2}\delta p_r(0)), \tag{27}$$

so that

$$\delta\Phi(t) \simeq e^{t/2\sqrt{2}}\delta\Phi(0). \tag{28}$$

As promised, the phase difference grows exponentially on a time scale equal to $T_\lambda = 2\sqrt{2}$.

---

[1] N. J. Cornish and N. E. Frankel, Phys. Rev. D **56**, 1903 (1997).

[2] S. Suzuki and K. Maeda, Phys. Rev. D **55**, 4848 (1997); **61**, 024005 (2000).

[3] J. J. Levin, Phys. Rev. Lett. **84**, 3515 (2000).

[4] N. J. Cornish, Phys. Rev. Lett. **85**, 3980 (2000).

[5] S. A. Hughes, Phys. Rev. D **61**, 084004 (2000).

[6] B. Schutz, talk given at the 3rd International LISA symposium, 2000.

[7] N. J. Cornish, talk given at the Center for Gravitational Phys-

ics and Geometry, Penn. State, 1997.

[8] C. P. Dettmann and G. P. Morriss, Phys. Rev. E **53**, R5545 (1996).

[9] E. E. Flanagan, Phys. Rev. D **48**, 2389 (1993).

[10] N. J. Cornish and S. L. Larson, gr-qc/0103075.

[11] P. Grassberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988); H. E. Kandrup, B. L. Eckstein, and B. O. Bradley, Astron. Astrophys. **320**, 65 (1997).

[12] N. J. Cornish and J. J. Levin, Phys. Rev. D **55**, 7489 (1997).