Gravity on branes

Nathalie Deruelle
Département d'Astrophysique Relativiste et de Cosmologie, UMR 8629 du Centre National de la Recherche Scientifique, Observatoire de Paris, 92195 Meudon, France
and Institut des Hautes Etudes Scientifiques, 91440, Bures-sur-Yvette, France
and Centre for Mathematical Sciences, DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, England

Joseph Katz
Racah Institute of Physics, Hebrew University, 91904 Jerusalem, Israel
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We consider the four-dimensional discontinuity generated by two identical pieces of a five-dimensional space pasted along their edge (that is a "brane" in a "Z_2 symmetric" "bulk"). Using a four plus one decomposition of the Riemann tensor, we write the equations for gravity on the brane and recover in a simple manner a number of known "brane world" scenarios. We study under which conditions these equations reduce, exactly or approximately, to the four-dimensional Einstein equations. We conclude that if the bulk is imposed to be only an Einstein space near the brane, Einstein's equations can be recovered approximately on the brane, but if it is imposed to be strictly anti–de Sitter space then the Einstein equations cannot hold, even approximately, on a quasi-Minkowskian brane, unless matter obeys a very contrived equation of state.

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I. INTRODUCTION

Since their covariant description by Israel (clarifying earlier treatments by, e.g., Lanczos and Darmois) [1], thin shells have been intensively used as a model for matter in general relativity. However, a question is rarely asked when considering a four-dimensional spacetime, to wit whether or not matter on the shell obeys the three-dimensional Einstein equations, as there is no (experimental) reason why it should.

The situation changed recently with the increasing interest in gravity theories within spacetimes with large extra dimensions and the advent of the idea that our universe may be a four dimensional singular hypersurface, or "brane," in a five-dimensional spacetime, or "bulk" [2]. Indeed, in this new context, it becomes crucial to recover Einstein gravity for realistic matter on the brane, at least to some approximation compatible with the present experiments.

The Randall Sundrum scenario [3], where our universe is a four-dimensional quasi-Minkowskian edge of a double-sided perturbed anti–de Sitter spacetime, was the first explicit model where the linearized Einstein equations were claimed to hold on the brane. This claim was substantiated by further analyses and the corrections to Newton's law calculated [4]. Soon after followed the building of cosmological models, where the brane is taken to be a Robertson–Walker spacetime, which can tend at late times to the standard big-bang scenario [5,6]. The perturbations of the geometry and matter content of these models, in the view of calculating the microwave background anisotropies, are currently being studied [7]. More sophisticated models, including, e.g., two branes or curvature squared corrections in the bulk gravity equations are also being considered [8,9].

In most papers the issue of whether or not Einstein's equations can be recovered on the brane is slightly confused for the following reason: some authors take a brane world point of view, that is they ignore the bulk as much as they can, which shows up in the gravity equations on the brane as some extra radiation fluid or seeds; whereas other authors impose a geometry for the bulk (to be, e.g., perturbed anti–de Sitter spacetime) and see how this geometry influences the equations for gravity in the brane. This divide can be seen in the techniques used; the first category of authors tends to use Gaussian normal coordinates, which are well adapted to the brane, whereas the second tends to use coordinates adapted to the bulk (e.g., conformally Minkowskian or Schwarzschild-like coordinates). In this paper we shall adopt the first point of view. We make the junction with the bulk point of view in an accompanying paper [10].

The issue, however, can be described in a coordinate independent way as follows. Start with an $(N+1)$-dimensional "generating" spacetime $\mathcal{V}_{(N+1)}$ (that one can visualize as a surface embedded in a higher-dimensional space). Assume that $\mathcal{V}_{(N+1)}$ satisfies Einstein's equations, i.e., that the Einstein tensor of $\mathcal{V}_{(N+1)}$ is linearly related to some (smooth) stress-energy tensor $\mathcal{T}_{AB}$. Consider in $\mathcal{V}_{(N+1)}$ an N-dimensional hypersurface $\mathcal{M}_N$. Cut $\mathcal{V}_{(N+1)}$ along $\mathcal{M}_N$ into two parts, $\mathcal{V}^1_{(N+1)}$ and $\mathcal{V}^2_{(N+1)}$, and keep, say, $\mathcal{V}^1_{(N+1)}$, which now has a boundary $\mathcal{M}_N$. Then make a copy of $\mathcal{V}^1_{(N+1)}$ and paste these two identical pieces along $\mathcal{M}_N$; call the new spacetime with a discontinuity $\mathcal{M}_{(N+1)}$.

This cutting, copying, and pasting procedure is the geometrical expression of the so-called Z_2 symmetry.[1] In brane cosmology language, $\mathcal{M}_{(N+1)}$ is the bulk and $\mathcal{M}_N$ is the brane. Since $\mathcal{M}_{(N+1)}$ has a deltalike curvature singularity at

[1]There are two distinct ways of pasting two identical pieces together along a cut: one into another (double-sided space) and one to another (single-sided space). There is no way to distinguish these two constructs by the intrinsic and extrinsic curvatures of their discontinuity because they are the same for double-sided or single-sided spaces.

its edge $\mathcal{M}_N$, it can satisfy Einstein's equations only if a deltalike tensor is added to $\mathcal{T}_{AB}$. This new tensor (which can be visualized as the "glue" necessary to paste the two copies of $\mathcal{V}^1_{(N+1)}$) is the stress-energy tensor of matter in the brane $\mathcal{M}_N$. It is given by the integrated Einstein equations across $\mathcal{M}_N$, also called the Lanczos–Darmois–Israel equation, and is linearly related to the extrinsic curvature of $\mathcal{M}_N$ in the generating space $\mathcal{V}_{(N+1)}$.

A first remark is that any brane in any bulk cannot be kept as a candidate to represent our universe since conditions (such as energy conditions or an equation of state) must be imposed on the stress-energy tensor of the brane, i.e., through the Lanczos-Darmois-Israel equation, on its extrinsic curvature.

The conditions on the bulk and the brane are even more stringent if we impose that the brane $\mathcal{M}_N$ itself satisfies Einstein's equations (or an approximate version of those) because this implies highly nontrivial relations between the extrinsic and intrinsic curvatures of the brane.

In this paper we show how the equations for gravity on a brane depend only on the geometry of the bulk *near* the brane, but do so crucially (we shall be more precise about what we mean by "near" below). We will first see that if the geometry of the bulk can be chosen at will near the brane, then the Einstein equations can always be recovered on the brane, whatever the matter we choose on it. Second, we will see that if the bulk is imposed to be an Einstein space near the brane then the Einstein equations can also be recovered on the brane, under the condition, however, that terms quadratic in the stress-energy tensor of matter can be neglected (this result is already well-known in the context of brane cosmologies [5,6]); otherwise, matter must satisfy a very special equation of state (typically $P=-\frac{1}{3}\rho$). Finally, we will see that if the bulk is imposed to be maximally symmetric near the brane, then the Einstein equations cannot in general be recovered on the brane, even when terms quadratic in the stress-energy tensor of matter can be neglected (an exception being the case when the brane is a Robertson-Walker spacetime). In particular, we will see that the linearized Einstein equations cannot hold on a quasi-Minkowskian brane at the edge of a strictly anti–de Sitter bulk, unless matter obeys a very contrived equation of state. Our approach, which is based on a four plus one decomposition of the bulk Riemann tensor and an identification of the extrinsic curvature of the brane with its stress-energy tensor (thanks to the Lanczos–Darmois–Israel equations) is similar to that of Ref. [11] and our results are an extension of those presented there.

The paper is organized as follows: in Sec. II we express, in Gaussian normal coordinates, the metric near the brane in terms of the stress-energy tensor of matter on the brane and an extra, "seed" tensor. We also write the equations for gravity on the brane in terms of these two tensors and show that if the geometry of the bulk near the brane can be chosen at will, then the exact Einstein equations can be recovered on the brane. In Sec. III we restrict our attention to bulks that are Einstein spaces near the brane, and in Sec. IV to maximally symmetric bulks. Section V draws a few conclusions and we relegate to the Appendix the procedure to obtain, by iteration, the metric in the whole bulk, when matter in the bulk is known everywhere, not only near the brane.

II. THE EQUATIONS FOR GRAVITY ON A BRANE

Consider an arbitrary smooth five-dimensional "generating" space $\mathcal{V}_5$ that we foliate (at least locally) by a family of timelike hypersurfaces Σ_y. In Gaussian normal coordinates, the metric of $\mathcal{V}_5$ reads

$$ds^2 \equiv \gamma_{AB} dx^A\, dx^B = dy^2 + \gamma_{\mu\nu}(x^\rho,y)dx^\mu\, dx^\nu, \quad (2.1)$$

where x^ρ are four coordinates (one timelike, three spacelike) parametrizing the hypersurfaces Σ_y and where $x^5\equiv y$ $=$const are the equations of Σ_y. We introduce the extrinsic curvature of an hypersurface Σ_y and its trace

$$\mathcal{K}_{\mu\nu} \equiv -\frac{1}{2}\frac{\partial \gamma_{\mu\nu}}{\partial y}, \quad \mathcal{K}\equiv \gamma^{\rho\sigma}\mathcal{K}_{\rho\sigma} \quad (2.2)$$

as well as the Lanczos tensor

$$\mathcal{L}_{\mu\nu} \equiv \mathcal{K}_{\mu\nu} - \gamma_{\mu\nu}\mathcal{K} \quad (2.3)$$

that we decompose in terms of a "τ tensor" as

$$\mathcal{L}_{\mu\nu} \equiv \frac{1}{2}\lambda\,\gamma_{\mu\nu} + \frac{\kappa}{2}\tau_{\mu\nu}, \quad (2.4)$$

λ being a "tension" and κ a coupling constant.

We now single out the hypersurface $\Sigma_0 \equiv \mathcal{M}_4$. Near $\mathcal{M}_4$ the metric can be expanded in a Taylor series as

$$\gamma_{\mu\nu}(x^\rho,y) = g_{\mu\nu}(x^\rho) + k_{\mu\nu}(x^\rho)y + \tfrac{1}{2} l_{\mu\nu}(x^\rho)y^2 + \mathcal{O}(y^3), \quad (2.5)$$

where $g_{\mu\nu}$ is the metric on $\mathcal{M}_4$. We write the expansion of the τ tensor as

$$\tau_{\mu\nu} = T_{\mu\nu}(x^\rho) + \Theta_{\mu\nu}(x^\rho)y + \mathcal{O}(y^2), \quad (2.6)$$

where $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ can be expressed in terms of $k_{\mu\nu}$ and $l_{\mu\nu}$ (see the Appendix). Conversely $k_{\mu\nu}$ and $l_{\mu\nu}$ can be expressed in terms of $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ so that the metric near the brane can be written as Eq. (1) with

$$\begin{aligned}\gamma_{\mu\nu} = {} & g_{\mu\nu}(1+\tfrac{1}{3}\lambda y + \tfrac{1}{18}\lambda^2 y^2) - \kappa y(1+\tfrac{1}{6}\lambda y)(T_{\mu\nu} - \tfrac{1}{3}T g_{\mu\nu}) \\ & - \tfrac{1}{2}\kappa y^2[(\Theta_{\mu\nu} - \tfrac{1}{3}\Theta g_{\mu\nu}) - \tfrac{1}{3}\kappa(g_{\mu\nu}T_{\rho\sigma}T^{\rho\sigma} - T T_{\mu\nu})] \\ & + \mathcal{O}(y^3) \end{aligned} \quad (2.7)$$

traces being defined by means of $g^{\mu\nu}$.

We now assume that $\mathcal{V}_5$ satisfies the five-dimensional Einstein equations

$$\mathcal{G}_{AB} = \tfrac{1}{6}\lambda^2\gamma_{AB} + \kappa\mathcal{T}_{AB}, \quad (2.8)$$

where $\mathcal{G}_{AB}$ is its Einstein tensor, and $\mathcal{T}_{AB}$ a smooth stress-energy tensor. If we use the standard four plus one decomposition of the five dimensional Riemann tensor $\mathcal{R}_{ABCD}$,

$$\mathcal{R}_{y\mu y\nu}=\frac{\partial}{\partial y}\mathcal{K}_{\mu\nu}+\mathcal{K}_{\rho\mu}\mathcal{K}^{\rho}_{\nu},$$
$$\mathcal{R}_{y\mu\nu\rho}=\nabla_{\nu}\mathcal{K}_{\mu\rho}-\nabla_{\rho}\mathcal{K}_{\mu\nu}, \tag{2.9}$$
$$\mathcal{R}_{\mu\nu\rho\sigma}={}^{4}\mathcal{R}_{\mu\nu\rho\sigma}+\mathcal{K}_{\mu\sigma}\mathcal{K}_{\nu\rho}-\mathcal{K}_{\mu\rho}\mathcal{K}_{\nu\sigma},$$

where ∇_μ and ${}^4\mathcal{R}_{\mu\nu\rho\sigma}$ are the covariant derivative and Riemann tensor associated with the metric $\gamma_{\mu\nu}(x^\rho,y)|_{y=\mathrm{const}}$, we can rewrite the five-dimensional Einstein equations (8) in terms of the quantities introduced previously, at zeroth order in y, i.e., on $\mathcal{M}_4$, as

$$G_{\mu\nu}=-\frac{\kappa\lambda}{6}T_{\mu\nu}-\frac{\kappa}{2}\Theta_{\mu\nu}-\frac{\kappa^2}{2}\left[T_{\mu\rho}T^{\rho}_{\nu}-\frac{1}{6}TT_{\mu\nu}+\frac{g_{\mu\nu}}{4}\left(T_{\rho\sigma}T^{\rho\sigma}-\frac{1}{3}T^2\right)\right]+\kappa\mathcal{T}_{\mu\nu}|_{y=0}, \tag{2.10a}$$

$$D_\nu T^{\nu}_{\mu}=-2\mathcal{T}_{5\mu}|_{y=0}, \tag{2.10b}$$

$$-R=-\frac{\kappa\lambda}{6}T+\frac{\kappa^2}{4}\left(T_{\rho\sigma}T^{\rho\sigma}-\frac{1}{3}T^2\right)+2\kappa\mathcal{T}_{55}|_{y=0}, \tag{2.10c}$$

where $G_{\mu\nu}$ is the Einstein tensor of the metric $g_{\mu\nu}$,$(-R)$ its trace, and D_μ the covariant derivative associated with $g_{\mu\nu}$. A consequence of Eq. (10) is

$$\Theta-2\mathcal{T}^{\rho}_{\rho}+4\mathcal{T}_{55}=\kappa\left(-\frac{5}{2}T_{\rho\sigma}T^{\rho\sigma}+\frac{2}{3}T^2\right). \tag{2.11}$$

Equations (10) can be seen as an ''initial value'' problem (with inverted commas because $\mathcal{M}_4$ is a timelike hypersurface) given a metric and its first derivative in y on $\mathcal{M}_4$, i.e., with the notations used here, given a metric $g_{\mu\nu}$ on $\mathcal{M}_4$ and a tensor $T_{\mu\nu}$, satisfying the constraints (10b) and (10c), then Eq. (10a) gives $\Theta_{\mu\nu}$, that is the second y derivative of the metric on the hypersurface. One then knows the metric and its first derivative on a neighboring hypersurface $y=\epsilon$ and, by iteration, one can get, in principle, the metric in the whole spacetime if $\mathcal{T}_{AB}$ is known everywhere (see the Appendix for an illustration of such a procedure).

The reason for decomposing the five-dimensional Einstein equations (8) in terms of $T_{\mu\nu}$ rather than the extrinsic curvature of $\mathcal{M}_4$ as is usual, is that, in brane cosmology, $T_{\mu\nu}$ is the stress-energy tensor of ordinary matter on the brane. Indeed, as recalled in the Introduction, the bulk $\mathcal{M}_5$ is obtained by cutting $\mathcal{V}_5$ into two pieces along $\mathcal{M}_4$, by making a copy of the $y\geqslant 0$ piece, say, and pasting it along $\mathcal{M}_4$, which hence becomes a singular hypersurface, or ''brane.'' The metric of $\mathcal{M}_5$ is continuous across $\mathcal{M}_4$ and reads

$$\overline{\gamma}_{AB}=\gamma_{AB}(x^\rho,y)\quad\text{for}\quad y\geqslant 0,$$
$$\overline{\gamma}_{AB}=\gamma_{AB}(x^\rho,-y)\quad\text{for}\quad y\leqslant 0. \tag{2.12}$$

The stress-energy tensor $\overline{\mathcal{T}}_{AB}$ is defined similarly. The extrinsic curvature of $\mathcal{M}_4$ in $\mathcal{M}_5$ is $-(k_{\mu\nu}/2)$ when $y\to 0_+$ and $+(k_{\mu\nu}/2)$ when $y\to 0_-$. The Einstein tensor of $\mathcal{M}_5$ therefore exhibits a deltalike singularity at $\mathcal{M}_4$ and satisfies the following equations:

$$\overline{\mathcal{G}}_{AB}=\tfrac{1}{6}\lambda^2\overline{\gamma}_{AB}+\kappa\overline{\mathcal{T}}_{AB}+\kappa\overline{T}_{AB}\delta(y), \tag{2.13}$$

where $\overline{T}_{AB}$ is the stress-energy tensor of matter on the brane. Integrating Eq. (13) [using Eq. (9)] across $y=0$, yields the Lanczos-Darmois-Israel equations [1]

$$\overline{T}_{A5}=0,\quad \kappa\overline{T}_{\mu\nu}=2\mathcal{L}_{\mu\nu}|_{y=0}=\lambda g_{\mu\nu}+\kappa T_{\mu\nu}, \tag{2.14}$$

which amounts to identifying the tensor $T_{\mu\nu}$, which we introduced with the stress-energy tensor of ordinary matter on the brane. Equations (10) therefore, become the equations for gravity in the brane.

As for the seed tensor $\Theta_{\mu\nu}$, which is related to the y derivative of the extrinsic curvature of $\mathcal{M}_4$, it encapsulates the influence of the geometry of the bulk near (rather than on) the brane, and can be expressed in terms of the Weyl $\mathcal{C}_{ABCD}$ tensor as in Ref. [11]. [More precisely we have $\Theta_{\mu\nu}=(2/\kappa)E_{\mu\nu}-(\kappa/2)(T_{\mu\rho}T^{\rho}_{v}+g_{\mu\nu}T_{\rho\sigma}T^{\rho\sigma}-\frac{1}{3}T^2g_{\mu\nu})$, where $E_{\mu\nu}\equiv\mathcal{C}_{5\mu 5v}$.]

Now, it is clear that, if the geometry of the bulk near the brane can be chosen at will, then the four-dimensional Einstein equations

$$G_{\mu\nu}=8\pi G_N T_{\mu\nu} \tag{2.15a}$$

with

$$8\pi G_N\equiv-\tfrac{1}{6}\kappa\lambda, \tag{2.15b}$$

G_N being Newton's constant, can be exactly recovered on the brane. Indeed one simply has to impose

$$\mathcal{T}_{5\mu}|_{y=0}=0, \tag{2.16a}$$

$$8\mathcal{T}_{55}|_{y=0}=\kappa(\tfrac{1}{3}T^2-T_{\rho\sigma}T^{\rho\sigma}), \tag{2.16b}$$

$$2\mathcal{T}_{\mu\nu}|_{y=0}-\Theta_{\mu\nu}=\kappa\left[T_{\mu\rho}T^{\rho}_{\nu}-\tfrac{1}{6}TT_{\mu\nu}+\frac{g_{\mu\nu}}{4}(T_{\rho\sigma}T^{\rho\sigma}-\tfrac{1}{3}T^2)\right]. \tag{2.16c}$$

If one wishes, however, that $\mathcal{T}_{AB}$ describes some ''realistic'' matter, then conditions (16) may not be fulfilled for a realistic $T_{\mu\nu}$. Indeed, consider, e.g., the case when matter in the bulk is a massless scalar field Φ and the brane is a Robertson-Walker spacetime. Equation (16b) then reads

$$\psi^2+\dot{\phi}^2=-\frac{\kappa}{6}\rho(\rho+3P), \tag{2.17}$$

where $\psi\equiv\partial\Phi/\partial y|_{y=0}$, $\dot{\phi}\equiv\partial\Phi/\partial t|_{y=0}$ (t being cosmic time), and where ρ and P are the energy density and pressure of matter in the brane. For the matter satisfying $P>-\rho/3$ (and $\kappa\rho>0$) Eq. (17) has no solution. (Defining $8\pi G_N\equiv$

$-(\alpha/6)\kappa\lambda, \alpha$ being an arbitrary constant, does not relax this constraint—nor the others we shall encounter.)

Let us summarize this section: the metric near the brane is given in terms of the metric of the brane, the stress-energy tensor of its matter content and a seed tensor by Eq. (7); gravity on the brane is described by Eqs. (10), $\mathcal{T}_{AB}$ being the stress-energy tensor of matter in the bulk; these equations reduce to the four dimensional Einstein equations if conditions (16) are satisfied.

III. THE CASE OF AN EINSTEIN BULK

When $\mathcal{T}_{AB}|_{y=0}=0$ the bulk is an Einstein space near the brane. Introducing the seed tensor

$$\Sigma_{\mu\nu}\equiv\Theta_{\mu\nu}+\kappa\left[T_{\mu\rho}T^{\rho}_{\nu}-\frac{1}{6}TT_{\mu\nu}+\frac{g_{\mu\nu}}{4}\left(T_{\rho\sigma}T^{\rho\sigma}-\frac{1}{3}T^2\right)\right], \tag{3.1}$$

Eqs. (2.10) for gravity in the brane are then equivalent to

$$G_{\mu\nu}=8\pi G_N T_{\mu\nu}-\frac{\kappa}{2}\Sigma_{\mu\nu} \tag{3.2a}$$

with the seed tensor $\Sigma_{\mu\nu}$ restricted to satisfy

$$\Sigma=\frac{\kappa}{2}\left(\frac{1}{3}T^2-T_{\rho\sigma}T^{\rho\sigma}\right), \tag{3.2b}$$

$$D_\mu\Sigma^{\mu}_{\nu}=0. \tag{3.2c}$$

Note that Eqs. (2b) and (2c) define $\Sigma_{\mu\nu}$ (and hence $\Theta_{\mu\nu}$) up to a conserved and traceless (i.e., radiationlike) tensor. As for the metric near the brane, it is given by Eq. (2.7), the tensor $\Theta_{\mu\nu}$ being constrained to satisfy conditions (2b) and (2c) [with the definition (1)].

The Einstein equations will hold on the brane if, first

$$\Sigma_{\mu\nu}=0\Leftrightarrow\Theta_{\mu\nu}=-\kappa\left[T_{\mu\rho}T^{\rho}_{\nu}-\frac{1}{6}TT_{\mu\nu}+\frac{g_{\mu\nu}}{4}\left(T_{\rho\sigma}T^{\rho\sigma}-\frac{1}{3}T^2\right)\right], \tag{3.3}$$

and if matter on the brane satisfies the constraint

$$\tfrac{1}{3}T^2-T_{\rho\sigma}T^{\rho\sigma}=0. \tag{3.4}$$

Outside matter, $T_{\mu\nu}=0$. Constraint (4) is hence satisfied, so that the Einstein equations can hold on the brane if we choose $\Theta_{\mu\nu}=0$. The bulk metric near the brane is then given by Eq. (2.7) with $T_{\mu\nu}=\Theta_{\mu\nu}=0$, i.e.,

$$\gamma_{\mu\nu}=g_{\mu\nu}[1+\tfrac{1}{3}\lambda y+\tfrac{1}{18}\lambda^2y^2+\mathcal{O}(y^3)] \tag{3.5}$$

with $g_{\mu\nu}$ a Ricci flat metric. We recognize in Eq. (5) the expansion of $\gamma_{\mu\nu}=g_{\mu\nu}\exp\lambda y/3$, the metric studied in Ref. [12], which is obtained by iteration of Eqs. (2.8) when $\mathcal{T}_{AB}$ is imposed to be zero everywhere, and not only on the brane (see the Appendix).

Inside matter, $T_{\mu\nu}\neq 0$. However, at linear order in $T_{\mu\nu}$, constraint (4) is still approximately satisfied. Einstein's equations can therefore still hold approximately on the brane if we choose $\Theta_{\mu\nu}=0$. The bulk metric near the brane is given in that case by Eq. (2.7) with $\Theta_{\mu\nu}=0$ and the terms that are quadratic in $T_{\mu\nu}$ are neglected.

Now, if terms that are quadratic in the stress-energy tensor cannot be neglected, then Eq. (4) becomes a restriction on the matter allowed on the brane. In the case of a perfect fluid, $T_{\mu\nu}=(\rho+P)u_\mu u_\nu+Pg_{\mu\nu}$ Eq. (4) yields

$$P=-\tfrac{1}{3}\rho. \tag{3.6}$$

In conclusion, when the bulk is an Einstein space in the vicinity of the brane, the Einstein equations can be recovered on the brane, at least at linear order in $T_{\mu\nu}$, by choosing $\Theta_{\mu\nu}=0$, whatever the equation of state for the matter. However, when quadratic corrections are taken into account, the equations for gravity on the brane differ from Einstein's, unless matter satisfies condition (4) (or 6).

These results generalize known results that can be found in the literature when the brane is taken to be a spatially flat Robertson–Walker spacetime [5,6]. Indeed, in that case the metric $g_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ are supposed to be of the form

$$\begin{aligned} &g_{tt}=-1,\quad g_{ti}=0,\quad g_{ij}=a^2(t)\delta_{ij},\\ &T_{tt}=\rho(t),\quad T_{ti}=0,\quad T_{ij}=a^2P(t)\delta_{ij}. \end{aligned} \tag{3.7}$$

The solution of the equations for gravity on the brane is obtained by integrating either Eqs. (3.2), or Eqs. (2.10) with $\mathcal{T}_{AB}=0$. Equation (2.10b), e.g., is the standard conservation law

$$\dot\rho+3\frac{\dot a}{a}(\rho+P)=0. \tag{3.8}$$

As for Eq. (2.10c) it reads

$$\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}=-\frac{\kappa\lambda}{36}(\rho-3P)-\frac{\kappa^2\rho}{36}(\rho+3P), \tag{3.9}$$

which is equivalent, whatever the equation of state, to

$$\frac{\dot a^2}{a^2}=\frac{\kappa}{36}\rho(\kappa\rho-2\lambda)+\frac{c}{a^4} \tag{3.10}$$

with c a constant of integration. We recognize in Eq. (10) the evolution equation for the scale factor a first obtained in Ref. [5]. Finally Eq. (2.10a) gives the seed tensor $\Theta_{\mu\nu}$ [and hence the bulk metric near the brane to second order in y, see Eq. (2.7)] as

$$\begin{aligned} &\Theta_{00}=\frac{\kappa\rho}{6}(5\rho+6P)-\frac{6c}{\kappa a^4},\\ &\Theta_{ij}=-a^2\delta_{ij}\left[\frac{\kappa}{6}(3P^2+6P\rho+2\rho^2)+\frac{2c}{\kappa a^4}\right]. \end{aligned} \tag{3.11}$$

First, in order for these equations to reduce to the standard Friedmann equation, conditions (2.16) (with $\mathcal{T}_{AB}=0$) must be fulfilled: Eq. (2.16b) implies that, as we have already seen, $P=-\rho/3$ [which renders Eq. (9) linear in ρ]; hence $\rho \propto a^{-2}$; Eq. (2.16c), together with Eq. (11) then imposes $c/a^4=-\kappa^2\rho^2/36$ [which renders Eq. (10) equivalent to the Friedmann equation].

Second, when terms that are quadratic in $T_{\mu\nu}$ can be neglected, that is at late time, and when $\Theta_{\mu\nu}\approx 0$, i.e., for $c=0$, then Eq. (10) tends, as expected, to the Friedmann equation. The observational consequences, in particular on nucleosynthesis, of the nonstandard evolution of the scale factor at early times have been thoroughly analyzed in Refs. [7–9].

IV. THE CASE OF AN ANTI–DE SITTER BULK

Suppose now that the bulk is maximally symmetric near the brane, i.e., that its Riemann tensor $\mathcal{R}_{ABCD}$ is such that

$$\mathcal{R}_{ABCD}|_{y=0}=-\frac{\lambda^2}{36}(\gamma_{AC}\gamma_{BD}-\gamma_{AD}\gamma_{BC})|_{y=0}. \tag{4.1}$$

Using again the standard four plus one decomposition [see Eq. (2.9)] as well as the quantities introduced in Sec. II, this equation can be rewritten as

$$\Theta_{\mu\nu}=-\frac{\kappa}{2}\left[T_{\mu\rho}T^{\rho}_{\nu}+g_{\mu\nu}\left(T_{\rho\sigma}T^{\rho\sigma}-\frac{1}{3}T^2\right)\right], \tag{4.2a}$$

$$0=D_\nu T_{\mu\rho}-D_\rho T_{\mu\nu}-\tfrac{1}{3}(g_{\mu\rho}\partial_\nu T-g_{\mu\nu}\partial_\rho T), \tag{4.2b}$$

$$\begin{aligned}R_{\mu\nu\rho\sigma}=&-\frac{\kappa T}{36}(2\lambda+\kappa T)(g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma})-\frac{\kappa^2}{4}(T_{\mu\sigma}T_{\nu\rho}\\&-T_{\mu\rho}T_{\nu\sigma})+\frac{\kappa}{12}(\lambda+\kappa T)(g_{\mu\sigma}T_{\nu\rho}-g_{\mu\rho}T_{\nu\sigma}\\&+T_{\mu\sigma}g_{\nu\rho}-T_{\mu\rho}g_{\nu\sigma}),\end{aligned} \tag{4.2c}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the brane metric $g_{\mu\nu}$. These equations that describe gravity on the brane are more constraining than Eqs. (3.2). For example, they imply that, outside matter ($T_{\mu\nu}=0$): $R_{\mu\nu\rho\sigma}=0$, which means that the brane is necessarily flat [and not only a solution of Eq. (3.2) with $T_{\mu\nu}=0$ as is the case when the bulk is only imposed to be an Einstein space]. Outside matter, we also have $\Theta_{\mu\nu}=0$, so that the metric near the brane is, see Eq. (2.7)

$$\gamma_{\mu\nu}=\eta_{\mu\nu}\left[1+\frac{\lambda}{3}y+\frac{\lambda^2}{18}y^2+\mathcal{O}(y^3)\right], \tag{4.3}$$

which is nothing but the lower-order expansion of the anti–de Sitter metric in the Randall-Sundrum coordinates: $ds^2=dy^2+\eta_{\mu\nu}\exp(\lambda y/3)$, a metric that can be obtained by iteration of Eqs. (2) when the bulk is imposed to be anti–de Sitter spacetime everywhere and not only near the brane.

Inside matter, Eq. (2a) gives $\Theta_{\mu\nu}$ in terms of the metric of the brane and its matter content; Eq. (2b) is a constraint on the matter on the brane (which includes the conservation law $D_\mu T^\mu_\nu=0$, but is in general more constraining than that) and Eq. (2c) replaces the Einstein equations and describes gravity on the brane.

As an example, consider a spatially flat Robertson-Walker brane, where the metric $g_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ are given by Eq. (3.7). For such an ansatz, Eq. (2b) turns out to be equivalent to the conservation law (3.8). As for Eq. (2c) it is equivalent to Eq. (3.10) with

$$c=0. \tag{4.4}$$

Finally, Eq. (2a) gives $\Theta_{\mu\nu}$ as Eq. (3.11) with $c=0$, or, equivalently, the bulk metric near the brane to second order in y, which turns out to be the expansion at leading orders of the anti–de Sitter metric in the Gaussian normal coordinates introduced in Ref. [5]. Therefore, when one considers Robertson-Walker branes, the difference is quite tenuous between imposing the bulk to be just an Einstein space or maximally symmetric near the brane; in the latter case the constant c must be zero, in the former it is arbitrary and, when $P=-\rho/3$, can be chosen in such a way that the Friedmann equations hold exactly. And in both cases, i.e., for $c=0$ or c arbitrary, the terms that are quadratic in $T_{\mu\nu}$ become negligible at late time and the evolution of the brane tends to Friedmann's.

For less symmetric branes, however, the Einstein equations cannot, in general, be recovered, even when terms that are quadratic in $T_{\mu\nu}$ are negligible, as we shall now see.

In order to compare and contrast the brane gravity equations (2) with the four-dimensional Einstein equations, let us compute the brane Einstein tensor from Eq. (2c). We obtain

$$\begin{aligned}G_{\mu\nu}=&-\frac{\kappa\lambda}{6}T_{\mu\nu}+\frac{\kappa^2}{4}\left[-T_{\mu\rho}T^{\rho}_{\nu}+\frac{1}{3}TT_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\left(T_{\rho\nu}T^{\rho\sigma}\right.\right.\\&\left.\left.-\frac{1}{3}T^2\right)\right],\end{aligned} \tag{4.5}$$

which, inside matter, can never exactly reduce to the Einstein equations (as we already saw in the case of a Roberston-Walker brane). Now, at linear order in $T_{\mu\nu}$, and with the identification $8\pi G_N=-\kappa\lambda/6$, Eqs. (5) reduce to the four-dimensional Einstein equations. However, one must not forget that they are not equivalent to the linear version of Eqs. (2), i.e.,

$$\Theta_{\mu\nu}\approx 0, \tag{4.6a}$$

$$0\approx D_\nu T_{\mu\rho}-D_\rho T_{\mu\nu}-\tfrac{1}{3}(g_{\mu\rho}\partial_\nu T-g_{\mu\nu}\partial_\rho T), \tag{4.6b}$$

$$\begin{aligned}R_{\mu\nu\rho\sigma}\approx&-\frac{\kappa\lambda}{18}T(g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma})+\frac{\kappa\lambda}{12}(g_{\mu\sigma}T_{\nu\rho}-g_{\mu\rho}T_{\nu\sigma}\\&+T_{\mu\sigma}g_{\nu\sigma}-T_{\mu\rho}g_{\nu\sigma})\end{aligned} \tag{4.6c}$$

but only a consequence of those, and one must check that the chosen solution of Einstein's equations satisfies all Eqs. (6). This is the case, as we have already seen, when the brane is a Robertson-Walker spacetime. But consider now the case of an almost flat brane.

At zeroth order in $T_{\mu\nu}$, the brane must be flat and the metric can be taken to be $g_{\mu\nu}=\eta_{\mu\nu}$. We decompose the stress-energy tensor at first order, as is usual, into

$$T_{00}=\rho, \quad T_{0i}=-\partial_i v-v_i,$$
$$T_{ij}=\delta_{ij}(P-\tfrac{1}{3}\Delta\Pi)+\partial_{ij}\Pi+\partial_i\Pi_j+\partial_j\Pi_i+\Pi_{ij}, \tag{4.7}$$

where $\partial_i v^i=\partial_i\Pi^i=\partial_i\Pi^{ij}=\Pi^i_i=0$, and where all components tend to zero at spatial infinity and at $t\to\pm\infty$. Equations (6b) then read

$$0\approx\partial_i(\dot v+\tfrac{2}{3}\rho+P)+\dot v_i,$$
$$0\approx\tfrac{1}{3}\delta_{ij}(\dot\rho-\Delta\dot\Pi)+\partial_{ij}(\ddot\Pi+v)+\partial_i\dot\Pi_j+\partial_j\dot\Pi_i+\partial_j v_i+\dot\Pi_{ij},$$
$$0\approx\partial_i v_j-\partial_j v_i, \tag{4.8}$$
$$0\approx\tfrac{1}{3}\delta_{jk}\partial_i(\rho-\Delta\Pi)-\tfrac{1}{3}\delta_{ji}\partial_k(\rho-\Delta\Pi)+\partial_j(\partial_i\Pi_k-\partial_k\Pi_i)+\partial_i\Pi_{jk}-\partial_k\Pi_{ij},$$

which include the conservation laws $\partial_\mu T^\mu_\nu\approx0$, i.e.,

$$\dot\rho+\Delta v\approx0, \quad \dot v+P+\tfrac{2}{3}\Delta\Pi\approx0, \quad \dot v_i+\Delta\Pi_i\approx0 \tag{4.9}$$

but also impose matter to obey the following, very contrived, equation of state:

$$\rho\approx\Delta\Pi, \quad v\approx-\dot\Pi, \quad P\approx\ddot\Pi-\tfrac{2}{3}\Delta\Pi, \quad v_i\approx\Pi_i\approx\Pi_{ij}\approx0. \tag{4.10}$$

Matter being described solely in terms of the anisotropic stress Π by Eqs. (9) and (10), Eq. (6c) gives the Riemann tensor of the brane as

$$R_{\mu\nu\rho\sigma}\approx\frac{\kappa\lambda}{12}(\eta_{\mu\sigma}\partial^2_{\nu\rho}+\eta_{\nu\rho}\partial^2_{\mu\sigma}-\eta_{\mu\rho}\partial^2_{\nu\sigma}-\eta_{\nu\sigma}\partial^2_{\mu\rho})\Pi, \tag{4.11}$$

which defines uniquely the geometry of the brane as the conformally flat metric

$$g_{\mu\nu}\approx(1+\tfrac{1}{6}\kappa\lambda\Pi)\eta_{\mu\nu}. \tag{4.12}$$

Finally Eq. (2.7), together with Eqs. (6a) and Eqs. (9) and (10) gives the metric near the brane as

$$\gamma_{\mu\nu}\approx\eta_{\mu\nu}(1+\tfrac{1}{3}\lambda y+\tfrac{1}{18}\lambda^2y^2)-\kappa y(1+\tfrac{1}{6}\lambda y)\partial_{\mu\nu}\Pi+\mathcal{O}(y^3). \tag{4.13}$$

This metric describes, by construction, a strictly anti–de Sitter bulk near the brane, and can, therefore, be cast into the form (3) by a mere change of coordinates. However, this change of coordinates changes the equation giving the position of the brane that is no longer given by $y=0$. The last term hence describes, in Gaussian normal coordinates, the so-called ''brane-bending'' effect (see also Ref. [13]).

V. CONCLUSIONS

The main result of this paper is that the question of whether or not Einstein's equations are recovered on a brane depends crucially on the geometry of the bulk near the brane. If the bulk is an Einstein space near the brane, then the Einstein equations, at least at linear order in the stress-energy tensor $T_{\mu\nu}$, can be recovered, *whatever* the equation of state of the matter on the brane. However, when quadratic terms in $T_{\mu\nu}$ cannot be neglected, Einstein's equations hold only if the equation of state for matter is $P=-\rho/3$ (for a perfect fluid). If, now, the bulk is imposed to be strictly anti–de Sitter space near the brane, then the brane *must* be flat outside matter. Moreover the Einstein equations can never be recovered when terms quadratic in $T_{\mu\nu}$ are important. Finally, when terms quadratic in $T_{\mu\nu}$ can be neglected, the linearized Einstein equations can hold on a quasi-Minkowskian brane, but only for very contrived matter.

This last result does not by any means imply that the linearized Einstein equations cannot be recovered in the Randall-Sundrum scenario. Indeed, in that scenario, the bulk is a *perturbed* anti–de Sitter space, i.e., an Einstein space. The results of Sec. III then tell us that if we choose, at zeroth order in $\lambda T_{\mu\nu}$ and $\Theta_{\mu\nu}$ the flat solution of the brane equation for gravity (3.2a), then, at linear order, the gravity on the brane is governed by the equation

$$\tilde G_{\mu\nu}\approx8\pi G_N T_{\mu\nu}-\frac{\kappa}{2}\Theta_{\mu\nu}, \tag{5.1}$$

where $\tilde G_{\mu\nu}$ is the Einstein tensor of the metric $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ at linear order in $h_{\mu\nu}$ and where $\kappa\Theta_{\mu\nu}$ describes a radiationlike fluid which, *a priori*, can contribute as much as $G_N T_{\mu\nu}$ to $\tilde G_{\mu\nu}$, but which can also be chosen to be zero. Finally the metric near the brane reads

$$\gamma_{\mu\nu}\approx(\eta_{\mu\nu}+h_{\mu\nu})(1+\tfrac{1}{3}\lambda y+\tfrac{1}{18}\lambda^2y^2)-\kappa y(1+\tfrac{1}{6}\lambda y)(T_{\mu\nu}-\tfrac{1}{3}T\eta_{\mu\nu})-\tfrac{1}{2}\kappa y^2(\Theta_{\mu\nu}-\tfrac{1}{3}\Theta\eta_{\mu\nu})+\mathcal{O}(y^3), \tag{5.2}$$

which is not simply, as in Eq. (4.13), anti–de Sitter metric in disguise.

We leave to another work [10] the comparison of the ''brane world'' point of view developed here with the bulk point of view, where the bulk is imposed to be a perturbed anti–de Sitter space *everywhere* and not only near the brane, and where the perturbations are imposed to satisfy boundary conditions that may restrict $\Theta_{\mu\nu}$ and/or $T_{\mu\nu}$.

We also leave to further study the following, delicate, issue that would allow observational tests outside the realm of cosmology. Suppose the bulk is an Einstein space with known boundary conditions that determine $\Theta_{\mu\nu}$. How much does gravity differ from Einstein's in a realistic situation when strong fields are present as in, e.g., a collapsing star?

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APPENDIX

Consider a five-dimensional spacetime $\mathcal{V}_5$ in Gaussian normal coordinates $x^A=(x^\rho,y)$,

$$ds^2=\gamma_{AB}dx^A\,dx^B=\epsilon\,dy^2+\gamma_{\mu\nu}(x^\rho,y)dx^\mu\,dx^\nu, \quad \text{(A1)}$$

with $\epsilon=\pm1$ and expand the metric coefficients $\gamma_{\mu\nu}(x^\rho,y)$ near the surface $y=0$ as

$$\gamma_{\mu\nu}(x^\rho,y)=g_{\mu\nu}(x^\rho)+k_{\mu\nu}(^\rho)y+\tfrac{1}{2}l_{\mu\nu}(x^\rho)y^2+\tfrac{1}{6}m_{\mu\nu}(x^\rho)y^3+\mathcal{O}(y^4). \quad \text{(A2)}$$

The extrinsic curvature of the surface $y=$const and its y derivative are given by

$$\mathcal{K}_{\mu\nu}\equiv-\frac{1}{2}\frac{\partial\gamma_{\mu\nu}}{\partial y}=-\frac{1}{2}\left(k_{\mu\nu}+l_{\mu\nu}y+\frac{1}{2}m_{\mu\nu}y^2\right)+\mathcal{O}(y^3),$$
$$\frac{\partial\mathcal{K}_{\mu\nu}}{\partial y}=-\frac{1}{2}(l_{\mu\nu}+m_{\mu\nu}y)+\mathcal{O}(y^2). \quad \text{(A3)}$$

The Riemann tensor of the metric (A1) reads

$$\mathcal{R}_{y\mu y\nu}=\frac{\partial}{\partial y}\mathcal{K}_{\mu\nu}+\mathcal{K}_{\rho\mu}\mathcal{K}^\rho_\nu,$$
$$\mathcal{R}_{y\mu\nu\rho}=\boldsymbol{\nabla}_\nu\mathcal{K}_{\mu\rho}-\boldsymbol{\nabla}_\rho\mathcal{K}_{\mu\nu}, \quad \text{(A4)}$$
$$\mathcal{R}_{\mu\nu\rho\sigma}={}^4\mathcal{R}_{\mu\nu\rho\sigma}+\epsilon(\mathcal{K}_{\mu\sigma}\mathcal{K}_{\nu\rho}-\mathcal{K}_{\mu\rho}\mathcal{K}_{\nu\sigma}),$$

where $\boldsymbol{\nabla}_\mu$ and ${}^4\mathcal{R}_{\mu\nu\rho\sigma}$ are the covariant derivative and Riemann tensor associated with the metric $\gamma_{\mu\nu}(x^\rho,y)|_{y=\text{const}}$. Expanding Eq. (A4) to first order in y, it is a straightforward calculation to obtain the Einstein tensor of the metric (A1) as

$$\mathcal{G}_{yy}=-\frac{\epsilon}{2}R+\frac{1}{8}(k^2-k.k)+\frac{1}{4}y[2\epsilon(\Box k-D.k+k.R)+kl-k.l-k(k.k)+k.k.k]+\mathcal{O}(y^2), \quad \text{(A5a)}$$

where R and D are the scalar curvature and covariant derivative of the metric $g_{\mu\nu}$, the traces are defined by means of $g^{\mu\nu}$, $\Box\equiv D_\rho D^\rho$, and $a.b\equiv a_{\mu\nu}b^{\mu\nu}$, $a.b.c\equiv a_{\mu\nu}b^{\nu\rho}c^\mu_\rho$;

$$\mathcal{G}_{y\mu}=\tfrac{1}{2}(D_\nu k^\nu_\mu-\partial_\mu k)-\tfrac{1}{2}y[\partial_\mu l-D_\nu l^\nu_\mu-\tfrac{1}{2}k^\nu_\mu\partial_\nu k+D_\nu(k^{\nu\rho}k_{\rho\mu})-\tfrac{3}{4}\partial_\mu(k.k)]+\mathcal{O}(y^2). \quad \text{(A5b)}$$

Finally

$$\mathcal{G}_{\mu\nu}=G_{\mu\nu}+\frac{\epsilon}{2}\left[g_{\mu\nu}l-l_{\mu\nu}+k^\rho_\mu k_{\rho\nu}-\frac{1}{2}kk_{\mu\nu}+\frac{1}{4}g_{\mu\nu}(k^2-3k.k)\right]+\frac{\epsilon}{2}y\left[mg_{\mu\nu}-m_{\mu\nu}+k_{\mu\rho}l^\rho_\nu+k_{\nu\rho}l^\rho_\mu-kl_{\mu\nu}-\frac{1}{2}lk_{\mu\nu}+\frac{1}{2}g_{\mu\nu}(kl-2k.l)\right]-\frac{\epsilon}{2}y\left\{k_{\mu\rho}k^{\rho\lambda}k_{\lambda\nu}+\frac{1}{4}(k.k-k^2)k_{\mu\nu}+\frac{1}{2}[k(k.k)-3k.k.k]\right\}+\frac{1}{2}y[D_{\rho\mu}k^\rho_\nu+D_{\rho\nu}k^\rho_\mu-\Box k_{\mu\nu}-D_{\mu\nu}k-g_{\mu\nu}(D.k-\Box k-k.R)-Rk_{\mu\nu}], \quad \text{(A5c)}$$

where $G_{\mu\nu}$ is the Einstein tensor of the metric $g_{\mu\nu}$.

If $\mathcal{V}_5$ is now imposed to be an Einstein space, $\mathcal{G}_{AB}=\Lambda\gamma_{AB}$, not only on the brane as in the main text but at linear order in y, then

$$\mathcal{G}_{yy}=\epsilon\Lambda,\quad \mathcal{G}_{y\mu}=0,\quad \mathcal{G}_{\mu\nu}=\Lambda(g_{\mu\nu}+yk_{\mu\nu})+\mathcal{O}(y^2). \quad \text{(A6)}$$

Suppose now that we are given a metric and its y derivative at $y=0$, i.e., that we know $g_{\mu\nu}$ and $k_{\mu\nu}$ satisfying the constraints

$$D_\nu k^\nu_\mu-\partial_\mu k=0,\quad -\epsilon R+\tfrac{1}{4}(k^2-k.k)=2\epsilon\Lambda. \quad \text{(A7)}$$

Then Eqs. (A5a) and (A5b) with Eq. (A6) are satisfied at zeroth order in y, and Eq. (A5c) [with Eq. (A6)] gives $l_{\mu\nu}$ in terms of $g_{\mu\nu}$ and $k_{\mu\nu}$ as

$$l_{\mu\nu}-g_{\mu\nu}l=2\epsilon G_{\mu\nu}+k^\rho_\mu k_{\rho\nu}-\tfrac{1}{2}kk_{\mu\nu}+\tfrac{1}{4}g_{\mu\nu}(k^2-3k.k)-2\epsilon\Lambda g_{\mu\nu}. \quad \text{(A8)}$$

Hence the zeroth-order Einstein equations give us the metric near the brane at quadratic order in y.

It is then straightforward to see that Eqs. (A5a) and (A5b) together with Eq. (A6) are satisfied at linear order in y. As for Eq. (A5c) together with Eq. (A6) it gives $m_{\mu\nu}$, and hence the metric at cubic order in y.

Iterating this procedure, assuming that $\mathcal{V}_5$ is an Einstein space up to higher and higher order in y, should give the metric of $\mathcal{V}_5$ everywhere (or at least in a finite region near $y=0$).

To make the connection with the main text, first take $\epsilon=+1$ and $\Lambda=\lambda^2/6$ and introduce the "τ" tensor

$$\frac{k}{2}\tau_{\mu\nu}\equiv\mathcal{K}_{\mu\nu}-\gamma_{\mu\nu}\mathcal{K}-\frac{1}{2}\lambda\gamma_{\mu\nu} \quad \text{(A9)}$$

and expand it as

$$\tau_{\mu\nu}=T_{\mu\nu}+y\Theta_{\mu\nu}+\tfrac{1}{2}y^2\mathcal{H}_{\mu\nu}+\mathcal{O}(y^3). \quad \text{(A10)}$$

Using Eq. (A2) we have

$$\frac{\kappa}{2}T_{\mu\nu}=-\frac{1}{2}[k_{\mu\nu}+(\lambda-k)g_{\mu\nu}],$$
$$\frac{\kappa}{2}\Theta_{\mu\nu}=-\frac{1}{2}[l_{\mu\nu}+(\lambda-k)k_{\mu\nu}-(l-k.k)g_{\mu\nu}], \quad \text{(A11)}$$
$$\frac{\kappa}{2}\mathcal{H}_{\mu\nu}=-\frac{1}{2}[m_{\mu\nu}+(\lambda-k)l_{\mu\nu}-2(l-k.k)k_{\mu\nu}-(m-3k.l+2k.k.k)g_{\mu\nu}].$$

Consider now the particularly simple example where, instead of knowing $g_{\mu\nu}$ and $k_{\mu\nu}$, (or equivalently $T_{\mu\nu}$) we are given

$$T_{\mu\nu}=\Theta_{\mu\nu}=0. \tag{A12}$$

Then we first get from Eq. (A11),

$$k_{\mu\nu}=\frac{\lambda}{3}g_{\mu\nu} \text{ and } l_{\mu\nu}=\frac{\lambda^2}{9}g_{\mu\nu}. \tag{A13}$$

Equations (A5) together with (A6) then give, at zeroth order in y,

$$G_{\mu\nu}=0 \tag{A14}$$

and, at linear order in y,

$$m_{\mu\nu}=\frac{\lambda^3}{27}g_{\mu\nu} \text{ and } \mathcal{H}_{\mu\nu}=0. \tag{A15}$$

The metric near the brane is then the expansion, up to cubic order in y of the metric $\gamma_{\mu\nu}=g_{\mu\nu}\exp(\lambda y/3)$, with $g_{\mu\nu}$ being a Ricci flat metric. We also have that $\tau_{\mu\nu}$ is zero up to quadratic order in y. Iterating the procedure, with the condition that $\mathcal{V}_5$ is an Einstein space everywhere would yield (at least in a finite region near $y=0$)

$$G_{\mu\nu}=0, \quad \gamma_{\mu\nu}=g_{\mu\nu}\exp(\lambda_y/3), \quad \tau_{\mu\nu}=0, \tag{A16}$$

i.e., the metric discussed in Ref. [12].

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