Self-energy and self-force in the space-time of a thick cosmic string

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We calculate the self-energy and self-force for an electrically charged particle at rest in the background of Gott-Hiscock cosmic string space-time. We find the general expression for the self-energy which is expressed in terms of the S matrix of the scattering problem. The self-energy continuously falls down outward from the string's center with the maximum at the origin of the string. The self-force is repulsive for an arbitrary position of the particle. It tends to zero in the string's center and also far from the string and it has a maximum value at the string's surface. The plots of the numerical calculations of the self-energy and self-force are shown.

I. INTRODUCTION

Topological defects may have been formed during the expansion of the Universe due to spontaneous symmetry breaking [1]. Among these defects, cosmic strings seem to be of particular interest [2] due to their very intriguing properties compared with those associated with a nonrelativisic linear distribution of matter.

A variety of models of strings has been proposed. The first model corresponding to an infinitely thin cosmic string has been considered by Vilenkin in Ref. [3]. This space-time is locally flat but globally it is the direct product of the two-dimensional Minkowski space-time and a two-dimensional cone. The main geometrical peculiarity of this space-time is the deficit angle. The Riemann tensor is a delta function with support on the string's origin [4].

A more realistic model of a cosmic string with inner structure has been considered by Gott [5] and Hiscock [6]. The interior of this string is a constant curvature space-time and the exterior is a conical space-time as in the Vilenkin model. The deficit angle of the conical space is expressed in terms of the energy density of matter inside the string as we will see in Sec. II. There is no singularity at the origin of the string and the Riemann tensor is constant inside the string and zero outside it. The metric functions are C^1 regular at the string's surface.

The space-time of a cosmic string in the U(1) gauge theory has been considered numerically by Garfinkle in Ref. [7]. It was shown that this space-time smoothly tends to a conical space-time far from the string's origin.

The self-energy and the self-force for a (un)charged particle in infinitely thin cosmic string space-time has been investigated in Refs. [8–12]. The self-force on electric and magnetic linear sources in this space-time was also investigated in Ref. [13] and extended to the case of multiple cosmic strings [14].

It is well known [3] that a straight cosmic string produces

no gravitational force on surrounding matter. Nevertheless, there is interaction between the cosmic string and a particle due to its own field. This effect is nonlocal and depends on the deficit angle of the conical section of the cosmic string space-time. It was shown that even for a particle at rest, the electromagnetic force has only one radial component; it is repulsive [8,9,11,12] and given by

$$F_r^{el} = L_0 \frac{q^2}{2r^2}. \tag{1}$$

In the gravitational case it is attractive [10] and has the following expression:

$$F_r^{gr} = -L_0 \frac{m^2}{2r^2}. \tag{2}$$

Here, q and m are charge and mass of particle, respectively; r is a distance from particle to string.

The constant L_0 depends only on the deficit angle of the conical section of the cosmic string space-time. It is zero for zero angle deficit but for a supermassive cosmic string [9,12] it becomes very strong ($L_0 \rightarrow \infty$).

As it is seen from Eqs. (1) and (2) the self-interaction force tends to infinity at the string's origin, that is, as $r \rightarrow 0$. Obviously, real cosmic strings which may appear at phase transitions in the early Universe have nonzero thickness. For example, cosmic strings that appeared in grand unified theory (GUT) have radius $r_o \sim 10^{-29}$ cm [2]. In this case the space-time inside the string is not flat and one may expect some modification in the self-force that arises from this fact. This problem has been qualitatively discussed in Ref. [15] in the context of the study of implications of cosmic string catalysis in the process of baryon decay.

The purpose of this paper is to calculate the electromagnetic self-energy and self-force for a charged particle at rest in the background of Gott-Hiscock cosmic string space-time presented in Refs. [5,6]. In order to do the calculation we use the approach developed in Refs. [8,9]. In this method the self-potential Φ and self-energy U are proportional to the

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coincidence limit at the particle position $\vec{x}_p$ of the renormalized Green's function of the three-dimensional Laplace operator

$$\Phi(\vec{x}_p) = 4\pi q G^{ren}(\vec{x}_p|\vec{x}_p), \tag{3}$$

$$U(\vec{x}_p) = \frac{1}{2}q\Phi(\vec{x}_p), \tag{4}$$

and the self-force is the minus gradient of self-energy

$$\vec{F}(\vec{x}_p) = -\vec{\nabla}_{x_p}U(\vec{x}_p). \tag{5}$$

To renormalize the Green's function we use standard procedure [16,17] and subtract from that its singular part in Hadamard's form.

The organization of this paper is as follows. In Sec. II we introduce Gott-Hiscock space-time generated by a finite thickness cosmic string space-time and describe all geometrical characteristics we need. In Sec. III we obtain the general formulas for the self-potential for a particle at rest in this background taking into account the contributions from inner and outer parts of the string. In Sec. IV we analyze qualitatively and numerically the self-energy and self-force. We discuss our results in Sec. V. In the Appendix we obtain a uniform expansion for the self-energy and discuss the self-force.

Throughout this paper we use units $c = G = 1$.

II. THE SPACE-TIME

We consider the space-time of a straight infinite cosmic string with constant energy density $\mathcal{E}$ corresponding to the matter inside it. The solution of Einstein equations for this case has been found by Gott [5] and Hiscock [6]. The space-time may be divided by the surface of the string in two parts: interior and exterior domains. The latter is a flat conical space-time described by the line element

$$ds^2_{out} = -dt^2 + dr^2 + \frac{r^2}{\nu^2}d\varphi^2 + dz^2, \tag{6}$$

where $\varphi \in [0,2\pi]$, $r \in [r_o,\infty]$, and the parameter r_o is the radius of the string.

The interior part of the string is a constant curvature space-time with line element given by

$$ds^2_{in} = -dt^2 + d\rho^2 + \frac{\rho_o^2}{\epsilon^2}\sin^2\left(\frac{\epsilon\rho}{\rho_o}\right)d\varphi^2 + dz^2, \tag{7}$$

where $\varphi \in [0,2\pi]$ and $\rho \in [0,\rho_o]$. The inner radius of the string is ρ_o and it is related with parameter ϵ by $\rho_o/\rho_* = \epsilon$, where $\rho_* = 1/\sqrt{8\pi\mathcal{E}}$ is the "energetic" inner radius of the string. The matching conditions at the surface of the string give the connection between inner (ρ_o,ϵ) and outer (r_o,ν) parameters of the string, which reads as

$$\frac{r_o}{\rho_o} = \frac{\tan\epsilon}{\epsilon}, \quad \nu = \frac{1}{\cos\epsilon}. \tag{8}$$

In this case the surface energy-momentum tensor is absent.

The nonzero components of the Riemann and Ricci tensors and scalar curvature are the following:

$$R^{\rho\varphi}_{\cdot\cdot\ \rho\varphi} = \frac{\epsilon^2}{\rho_o^2}, \quad R^{\rho}_{\rho} = R^{\varphi}_{\varphi} = \frac{\epsilon^2}{\rho_o^2}, \quad R = \frac{2\epsilon^2}{\rho_o^2}. \tag{9}$$

The space-time described by Eqs. (6) and (7) may be covered by one map using the continuation of the radial coordinate r into the inner domain of the string through the relation

$$\sin\left(\frac{\epsilon\rho}{\rho_o}\right) = \frac{r}{r_o}\sin\epsilon. \tag{10}$$

Therefore the line element can be written as [18]

$$ds^2 = -dt^2 + P^2(r)dr^2 + \frac{r^2}{\nu^2}d\varphi^2 + dz^2, \tag{11}$$

where the function $P(r)$ is given by

$$P(r) = \begin{cases} \left(\nu^2 + \dfrac{r^2}{r_o^2}(1-\nu^2)\right)^{-1/2}, & r \leq r_o \\ 1, & r \geq r_o. \end{cases} \tag{12}$$

Sometimes we use the metric inside the string in the form given by Eq. (7) instead of Eq. (11). Connection between them is obtained from Eq. (10).

III. THE SELF-ENERGY

The electromagnetic potential A^μ in the Lorenz gauge for a particle with trajectory $x^\mu = x^\mu(\tau)$ obeys the equations

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta A^\mu + R^\mu_\nu A^\nu = -4\pi J^\mu(x)$$

$$= -4\pi q\int u^\mu(\tau)\delta^4[x-x(\tau)]\frac{d\tau}{\sqrt{-g}}. \tag{13}$$

It is possible, in principle, to obtain the self-force for an arbitrary trajectory of the particle using the same procedure of Refs. [11,12], but for simplicity we shall consider the particle at the rest with trajectory

$$x^0(\tau) = \tau, \quad x^1(\tau) = r_p, \quad x^2(\tau) = \varphi = 0, \quad x^3(\tau) = z = 0, \tag{14}$$

where r_p is the radial position of the particle. Let us consider the equation for the zero component of the vector potential A^μ. In the space-time of a cosmic string with metric (11), it reads

$$(-\partial_t^2 + \Delta)A^0 = -4\pi J^0, \tag{15}$$

where $\Delta = g^{ik}\nabla_i\nabla_k$ is the three-dimensional Laplacian. For our space-time and trajectory given by Eq. (14) we get

$$\Delta A^0 = -\frac{4\pi q}{\sqrt{g^{(3)}}} \delta(r-r_p)\delta(\varphi)\delta(z), \qquad (16)$$

where $g^{(3)} = r^2 P^2(r)/\nu^2$ is the determinant of the three-dimensional part of the metric. Therefore A^0 is the scalar Green's function of the three-dimensional Laplacian multiplied by $4\pi q$:

$$A^0(x,\varphi,z) = 4\pi q G(r,\varphi,z|r_p,0,0). \qquad (17)$$

The self-potential Φ and self-energy U, according to Refs. [8,9] are

$$\Phi(r_p) = 4\pi q G^{ren}(r_p,0,0|r_p,0,0), \qquad (18)$$

$$U(r_p) = \tfrac{1}{2} q \Phi(r_p), \qquad (19)$$

where G^{ren} is the renormalized Green's function.

Since the self-energy depends only on the radial coordinate r, the self-force will have only a radial component given by

$$F_{r_p} = -\frac{d}{dr_p} U(r_p). \qquad (20)$$

Now, let us find in a closed form the three-dimensional scalar Green's function $G(x,x')$

$$\Delta G(x,x') = -\frac{1}{\sqrt{g^{(3)}}} \delta(r-r')\delta(\varphi-\varphi')\delta(z-z'). \qquad (21)$$

In the space-time under consideration, Eq. (21) turns into

$$\left(\frac{1}{rP(r)}\frac{\partial}{\partial r}\frac{r}{P(r)}\frac{\partial}{\partial r} + \frac{\nu^2}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right) G(x,x')$$

$$= -\frac{\nu}{rP(r)}\delta(r-r')\delta(\varphi-\varphi')\delta(z-z'). \qquad (22)$$

Taking into account the cylindrical symmetry of the problem we may represent the Green's function in the following form

$$G(x,x') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk_z\, e^{ik_z(z-z')} \sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi')}\phi(r,r'), \qquad (23)$$

where the radial part of Green's function obeys the equation

$$\left(\frac{1}{rP(r)}\frac{\partial}{\partial r}\frac{r}{P(r)}\frac{\partial}{\partial r} - \frac{n^2\nu^2}{r^2} - k_z^2\right)\phi(r,r')$$

$$= -\frac{\nu}{rP(r)}\delta(r-r'). \qquad (24)$$

In order to calculate the radial Green's function we use the standard approach and the following expression for it:

$$\phi(r,r') = \theta(r-r')\psi_1(r)\psi_2(r') + \theta(r'-r)\psi_1(r')\psi_2(r), \qquad (25)$$

where θ is the step function and ψ_1, ψ_2 are two linearly independent homogeneous solutions of Eq. (24). The function $\psi_1(r)$ falls down as $r\to\infty$ and $\psi_2(r)$ is regular at the origin. Integrating Eq. (24) over r around r' one has the Wronskian normalization condition [18]

$$W(\psi_2,\psi_1) = \psi_1'(r)\psi_2(r) - \psi_1(r)\psi_2'(r) = -\frac{\nu}{r}P(r). \qquad (26)$$

Both homogeneous solutions are regular at the string's surface and they obey the set of equations

$$\psi_k(r)|_{r_o+\varepsilon} = \psi_k(r)|_{r_o-\varepsilon}, \qquad (27a)$$

$$\psi_k'(r)|_{r_o+\varepsilon} = \psi_k'(r)|_{r_o-\varepsilon}, \qquad (27b)$$

with $\varepsilon\to 0$ and $k=1,2$.

The homogeneous solutions ψ yield Bessel's equation

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{n^2\nu^2}{r^2} - k_z^2\right)\psi_{out}(r) = 0, \qquad (28)$$

outside the string and Legendre's equation

$$\left(\frac{1}{\sin\left(\frac{\epsilon\rho}{\rho_o}\right)}\frac{\partial}{\partial\rho}\sin\left(\frac{\epsilon\rho}{\rho_o}\right)\frac{\partial}{\partial\rho} - \frac{n^2\epsilon^2}{\rho_0^2\sin^2\left(\frac{\epsilon\rho}{\rho_o}\right)} - k_z^2\right)\psi_{in}(\rho) = 0, \qquad (29)$$

inside it.

Therefore the homogeneous solutions with required boundary conditions are the following:

$$\psi_1 = \begin{cases} N\cdot p_\mu^{|n|}[x] + M\cdot q_\mu^{|n|}[x], & r\leqslant r_o \\ F\cdot K_{|n|\nu}[kr], & r\geqslant r_o, \end{cases}$$

$$\psi_2 = \begin{cases} E\cdot p_\mu^{|n|}[x], & r\leqslant r_o \\ A\cdot I_{|n|\nu}[kr] + B\cdot K_{|n|\nu}[kr], & r\geqslant r_o, \end{cases} \qquad (30)$$

where $k=|k_z|$ and $x=\cos(\epsilon\rho/\rho_o) = [\nu P(r)]^{-1}$; $I_{n\nu}$ and $K_{n\nu}$ are modified Bessel functions. The functions $p_\mu^{|n|}$ and $q_\mu^{|n|}$ are expressed in terms of the Legendre functions of first and second kind by

$$p_\mu^{|n|}[x] = P_\mu^{-|n|}[x], \qquad (31)$$

$$q_\mu^{|n|}[x] = \frac{(-1)^n}{2}(Q_\mu^n[x] + Q_{-\mu-1}^n[x])$$

$$= -\frac{\pi}{2\sin\pi\mu}P_\mu^{|n|}[-x], \qquad (32)$$

where $\mu = -\frac{1}{2} + \frac{1}{2}\sqrt{1-4k^2r_o^2/(\nu^2-1)}$.

On choosing these functions, they will be real for an arbitrary value of x and the Wronskian of them will take the simple form

$$W(p_\mu^{|n|}, q_\mu^{|n|}) = \frac{1}{1 - x^2}. \tag{33}$$

$$AF = \nu, \qquad ME = 1, \tag{34}$$

The six constants in Eq. (30) can be found from Eqs. (26) and (27). The Wronskian normalization condition Eq. (26) gives two relations

and the conditions of regularity on the surface of the string given by Eq. (27) define the ratios

$$S_{\text{out}}[kr_o, |n|, \nu] \overset{\text{def}}{=} \frac{B}{A} = -\frac{\nu\left(1 - \frac{1}{\nu^2}\right)p_\mu^{|n|\,\prime}\left[\frac{1}{\nu}\right]I_{|n|\nu}[kr_o] + kr_o p_\mu^{|n|}\left[\frac{1}{\nu}\right]I'_{|n|\nu}[kr_o]}{\nu\left(1 - \frac{1}{\nu^2}\right)p_\mu^{|n|\,\prime}\left[\frac{1}{\nu}\right]K_{|n|\nu}[kr_o] + kr_o p_\mu^{|n|}\left[\frac{1}{\nu}\right]K'_{|n|\nu}[kr_o]}, \tag{35a}$$

$$S_{\text{in}}[kr_o, |n|, \nu] \overset{\text{def}}{=} \frac{N}{M} = -\frac{\nu\left(1 - \frac{1}{\nu^2}\right)q_\mu^{|n|\,\prime}\left[\frac{1}{\nu}\right]K_{|n|\nu}[kr_o] + kr_o q_\mu^{|n|}\left[\frac{1}{\nu}\right]K'_{|n|\nu}[kr_o]}{\nu\left(1 - \frac{1}{\nu^2}\right)p_\mu^{|n|\,\prime}\left[\frac{1}{\nu}\right]K_{|n|\nu}[kr_o] + kr_o p_\mu^{|n|}\left[\frac{1}{\nu}\right]K'_{|n|\nu}[kr_o]}. \tag{35b}$$

These quantities characterize the scattering on the string. The S-matrix S_{out} may be represented in the following form: $S_{\text{out}} = f_n^*(ik)/f_n(ik)$, where $f_n(ik)$ is the Jost function of the scattering problem on the imaginary axis. It has been found in Ref. [19] and obeys the relations

$$f_n(ik) = -\frac{1}{\sqrt{\nu}}\left(\frac{kr_o}{\sqrt{\nu^2 - 1}}\right)^n \left\{\nu\left(1 - \frac{1}{\nu^2}\right)p_\mu^{|n|\,\prime}\left[\frac{1}{\nu}\right]K_{|n|\nu}[kr_o]\right.$$

$$\left. + kr_o p_\mu^{|n|}\left[\frac{1}{\nu}\right]K'_{|n|\nu}[kr_o]\right\}, \tag{36}$$

$$f_n^*(ik) = +\frac{1}{\sqrt{\nu}}\left(\frac{kr_o}{\sqrt{\nu^2 - 1}}\right)^n \left\{\nu\left(1 - \frac{1}{\nu^2}\right)p_\mu^{|n|\,\prime}\left[\frac{1}{\nu}\right]I_{|n|\nu}[kr_o]\right.$$

$$\left. + kr_o p_\mu^{|n|}\left[\frac{1}{\nu}\right]I'_{|n|\nu}[kr_o]\right\}. \tag{37}$$

Therefore the radial Green's function of our problem is the following ($r > r'$)

$$\phi(r, r') = \begin{cases} \nu K_{|n|\nu}[kr'](I_{|n|\nu}[kr] + S_{\text{out}}[kr_o, |n|, \nu]K_{|n|\nu}[kr]), & r, r' \geq r_o \\ p_\mu^{|n|}[x'](q_\mu^{|n|}[x] + S_{\text{in}}[kr_o, |n|, \nu]p_\mu^{|n|}[x]), & r, r' \leq r_o, \end{cases} \tag{38}$$

where $x = \cos(\epsilon \rho / \rho_o) = [\nu P(r)]^{-1}$ and $x' = \cos(\epsilon \rho' / \rho_o) = [\nu P(r')]^{-1}$. In the limit $\nu \to 1$ this function becomes

$$\phi(r, r') = K_{|n|}[kr']I_{|n|}[kr], \tag{39}$$

as it should be in flat space-time.

Let us proceed now to calculate the self-energy given by Eq. (19). The zero component of the vector potential A^0 of a particle with trajectory given by Eq. (14) and situated outside the string is

$$A^0(r, \varphi, z) = \frac{q\nu}{\pi}\int_{-\infty}^{\infty} dk_z\, e^{ik_z z} \sum_{n=-\infty}^{\infty} e^{in\varphi}K_{|n|\nu}[kr_p]$$

$$\times (I_{|n|\nu}[kr] + S_{\text{out}}[kr_o, |n|, \nu]K_{|n|\nu}[kr]). \tag{40}$$

Taking the coincidence limit in this expression for a fixed value of the angular variable, for example, $\varphi = 0$, and changing the integration variable $k_z \to k = |k_z|$, one has

$$A^0(r, z) = \frac{2q\nu}{\pi}\int_0^{\infty} dk\,\cos(kz) \sum_{n=-\infty}^{\infty} K_{|n|\nu}[kr_p]$$

$$\times (I_{|n|\nu}[kr] + S_{\text{out}}[kr_o, |n|, \nu]K_{|n|\nu}[kr]). \tag{41}$$

This expression consists of two parts. The first one is due to the first term in brackets which is exactly the potential for an infinitely thin cosmic string. This term is divergent in the coincidence limit $z \to 0, r \to r_p$. The second term is finite in this limit and tends to zero as $\nu \to 1$ as well as $r_o \to 0$. Therefore to renormalize this potential we have to renormalize just the first term. Because the exterior is a flat space-time, in

order to do the renormalization we may subtract from it the potential in Minkowski space-time which corresponds to $\nu = 1$. Therefore the self-potential has the following form:

$$\Phi(r_p) = \lim_{z \to 0} \frac{2q}{\pi} \int_0^\infty dk \cos(kz) \sum_{n=-\infty}^{\infty} (\nu K_{|n|\nu}[kr_p] I_{|n|\nu}[kr_p]$$

$$- K_{|n|}[kr_p] I_{|n|}[kr_p])$$

$$+ \frac{2q\nu}{\pi} \int_0^\infty dk \sum_{n=-\infty}^{\infty} S_{\text{out}}[kr_o, |n|, \nu] K_{|n|\nu}^2[kr_p].$$

$$(42)$$

The first contribution may be found in closed form [8,9] using formulas 6.672(3) and 8.715(2) from Ref. [20] and we arrive at the following expression for the self-potential for a particle in the exterior of the string ($R = r_p/r_o$):

$$\Phi(r_p) = \frac{q}{r_p} L(\nu, R), \quad R \geqslant 1, \tag{43}$$

where

$$L(\nu, R) = \frac{1}{\pi} \int_0^\infty \frac{\nu \coth(\nu x) - \coth x}{\sinh x} dx$$

$$+ \frac{2\nu}{\pi} \int_0^\infty dx \sum_{n=-\infty}^{\infty} S_{\text{out}}\left[\frac{x}{R}, |n|, \nu\right] K_{|n|\nu}^2[x].$$

$$(44)$$

This formula represents an interesting relation between the self-potential with the scattering problem and the Jost function on this background. The first term is a well-known result [8,9] for an infinitely thin string. The second term is the contribution due to nonzero thickness of the string. It tends to zero as the radius of the string goes to zero ($r_o \to 0$).

Let us now consider a particle situated in the interior of the string. In this case the zero component of the vector potential reads

$$A^0(r, \varphi, z) = \frac{2q}{\pi} \int_0^\infty dk \cos kz \sum_{n=-\infty}^{\infty} e^{in\varphi} p_\mu^{|n|}[x']$$

$$\times (q_\mu^{|n|}[x] + S_{\text{in}}[kr_o, |n|, \nu] p_\mu^{|n|}[x]). \tag{45}$$

To renormalize this expression we have to subtract from it all divergences in the Hadamard form [16]. The structure of divergences of Green's function for odd-dimensional spaces is more simple than for the even-dimensional case because in the former case there is no logarithmic singularity [17]. The singular part of the Green's function in three dimensions is [17]

$$G_{sing}(x, x') = \frac{\Delta^{1/2}}{4\pi} \frac{1}{\sqrt{2\sigma}}. \tag{46}$$

Taking the coincidence limit for angular variable φ we get

$$G_{sing}(\rho, z | \rho', z') = \frac{\Delta^{1/2}}{4\pi} \frac{1}{\sqrt{(\rho - \rho')^2 + (z - z')^2}}, \tag{47}$$

where

$$\Delta^{1/2} = 1 + \frac{1}{12} \frac{\epsilon^2}{\rho_o^2} (\rho - \rho')^2 + \frac{1}{160} \frac{\epsilon^4}{\rho_o^4} (\rho - \rho')^4 + \cdots,$$

$$(48)$$

and $2\sigma = (\rho - \rho')^2 + (z - z')^2$. Let us represent the singular part of Green's function (47) in the following integral form:

$$G_{sing}(\rho, z | \rho', z') = \frac{\Delta^{1/2}}{2\pi^2} \int_0^\infty dk \cos k(z - z')$$

$$\times \sum_{n=-\infty}^{\infty} K_{|n|}[k\rho] I_{|n|}[k\rho']. \tag{49}$$

To renormalize the self-potential we subtract from Eq. (45) the above expression multiplied by $4\pi q$ and take the coincidence limit $\rho = \rho' = \rho_p, z = z'$. So, we arrive at the result

$$\Phi(r_p) = \frac{2q}{\pi} \int_0^\infty dk \sum_{n=-\infty}^{\infty} \{p_\mu^{|n|}[x_p](q_\mu^{|n|}[x_p]$$

$$+ S_{\text{in}}[kr_o, |n|, \nu] p_\mu^{|n|}[x_p]) - K_{|n|}[k\rho_p] I_{|n|}[k\rho_p]\},$$

$$(50)$$

where

$$\mu = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4k^2 r_o^2}{\nu^2 - 1}} = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4k^2 \rho_o^2}{\epsilon^2}},$$

$$(51)$$

$$x_p = [\nu P(r_p)]^{-1} = \cos\left(\frac{\epsilon \rho_p}{\rho_o}\right), \quad kr_o = k\rho_o \frac{\tan \epsilon}{\epsilon}.$$

$$(52)$$

Note that we can easily show that

$$\lim_{z \to z'} \lim_{\rho \to \rho'} \lim_{r \to r'} \int_0^\infty dk \cos k(z - z')$$

$$\times \sum_{n=-\infty}^{\infty} \{K_{|n|}[k\rho] I_{|n|}[k\rho'] - K_{|n|}[kr] I_{|n|}[kr']\}$$

$$= \int_0^\infty dk \sum_{n=-\infty}^{\infty} \{K_{|n|}[k\rho] I_{|n|}[k\rho] - K_{|n|}[kr] I_{|n|}[kr]\} = 0.$$

$$(53)$$

This is due to the fact that the singular part of Green's function given by Eqs. (47) and (49) in the coincidence limit $\rho' = \rho$ does not depend on ρ. It is simply given by $1/4\pi|z - z'|$. For this reason we may change ρ_p and r_p in the last term in Eq. (50) and the self-potential Φ given by Eqs. (43) and (50) is continuous at the string's surface.

IV. DISCUSSION

From previous results we have the following expressions for the self-energy of charged particles at the point $R = r_p/r_o$ in the thick cosmic string space-time:

$$U(r_p) = \frac{q^2}{2r_o} \begin{cases} \frac{1}{R} L(\nu, R), & R \geqslant 1 \\ H(\nu, R), & R \leqslant 1, \end{cases} \quad (54)$$

where

$$L(\nu, R)$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\nu \coth(\nu x) - \coth x}{\sinh x} dx$$

$$+ \frac{2\nu}{\pi} \int_0^\infty dx \sum_{n=-\infty}^\infty S_{\text{out}}\left[\frac{x}{R}, |n|, \nu\right] K_{|n|\nu}^2[x], \quad (55)$$

$$H(\nu, R)$$

$$= \frac{2}{\pi} \int_0^\infty dx \sum_{n=-\infty}^\infty \{ p_{\mu_x}^{|n|}[x_R](q_{\mu_x}^{|n|}[x_R]$$

$$+ S_{\text{in}}[x, |n|, \nu] p_{\mu_x}^{|n|}[x_R]) - K_{|n|}[xR] I_{|n|}[xR] \}, \quad (56)$$

$$S_{\text{out}}[x, |n|, \nu]$$

$$= -\frac{\nu\left(1 - \frac{1}{\nu^2}\right) p_{\mu_x}^{|n|\,\prime}\left[\frac{1}{\nu}\right] I_{|n|\nu}[x] + x p_{\mu_x}^{|n|}\left[\frac{1}{\nu}\right] I_{|n|\nu}'[x]}{\nu\left(1 - \frac{1}{\nu^2}\right) p_{\mu_x}^{|n|\,\prime}\left[\frac{1}{\nu}\right] K_{|n|\nu}[x] + x p_{\mu_x}^{|n|}\left[\frac{1}{\nu}\right] K_{|n|\nu}'[x]}, \quad (57)$$

$$S_{\text{in}}[x, |n|, \nu]$$

$$= -\frac{\nu\left(1 - \frac{1}{\nu^2}\right) q_{\mu_x}^{|n|\,\prime}\left[\frac{1}{\nu}\right] K_{|n|\nu}[x] + x q_{\mu_x}^{|n|\,\prime}\left[\frac{1}{\nu}\right] K_{|n|\nu}'[x]}{\nu\left(1 - \frac{1}{\nu^2}\right) p_{\mu_x}^{|n|\,\prime}\left[\frac{1}{\nu}\right] K_{|n|\nu}[x] + x p_{\mu_x}^{|n|}\left[\frac{1}{\nu}\right] K_{|n|\nu}'[x]}, \quad (58)$$

and we have introduced the following notations:

$$\mu_x = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4x^2}{\nu^2 - 1}}, \quad (59)$$

$$x_R = [\nu P(r_p)]^{-1} = \sqrt{1 - R^2\left(1 - \frac{1}{\nu^2}\right)}. \quad (60)$$

Let us analyze qualitatively the above expressions for self-energy. First of all let us consider the particle situated outside the string. The function $L(\nu, R)$ defined by Eqs. (54) and (55) can be separated into two parts according to Eq. (55) as

$$L(\nu, R) = L_0(\nu) + L_1(\nu, R), \quad (61)$$

where

$$L_0(\nu) = \frac{1}{\pi} \int_0^\infty \frac{\nu \coth(\nu x) - \coth x}{\sinh x} dx \quad (62)$$

is the contribution to the self-energy due to the infinitely thin cosmic string [8,9]. The second term,

$$L_1(\nu, R) = \frac{2\nu}{\pi} \int_0^\infty dx \sum_{n=-\infty}^\infty S_{\text{out}}\left[\frac{x}{R}, |n|, \nu\right] K_{|n|\nu}^2[x], \quad (63)$$

is the contribution from the structure of the string.

The function $S_{\text{out}}[x/R, |n|, \nu]$ is positive for arbitrary angular momentum n. For this reason the additional contribution to self-energy due to nonzero thickness of the string is positive, too. Changing the variable of integration $x \to z$ such that $x = n\nu z$ (except $n = 0$) we can represent the function $L_1(\nu, R)$ as

$$L_1(\nu, R) = \int_0^\infty dz\, F_0(z, R) + 2\sum_{n=1}^\infty \int_0^\infty dz\, F_n(z, R), \quad (64)$$

where

$$F_0(z, R) = \frac{2\nu}{\pi} S_{\text{out}}\left[\frac{z}{R}, 0, \nu\right] K_0^2[z], \quad (65)$$

$$F_n(z, R) = \frac{2n\nu^2}{\pi} S_{\text{out}}\left[\frac{z}{R} n\nu, n, \nu\right] K_{n\nu}^2[zn\nu]. \quad (66)$$

In order to estimate the behavior of F_n as a function of z we use the uniform asymptotic expansion for great index n of Bessel's functions in Ref. [21] and Legendre's function in Refs. [22,23]. With the help of those expansions one obtains the following main term of the uniform expansion for F_n:

$$F_n(z, R) \sim \frac{\nu^2 - 1}{8\pi\nu n^2 R^2} \frac{z^2}{\sqrt{1 + z^2}\left(1 + \frac{z^2}{R^2}\right)^2}$$

$$\times \exp\left\{-2n\nu\left(\eta[z] - \eta\left[\frac{z}{R}\right]\right)\right\}, \quad (67)$$

where

$$\eta[z] = \ln\frac{z}{\sqrt{1 + z^2} + 1} + \sqrt{1 + z^2}. \quad (68)$$

The function in Eq. (67) tends to zero as z^2 for $z \to 0$ and it tends to zero as $\exp\{-2n\nu z[1 - (1/R^2)]\}/z^3$ for $z \to \infty$. For this reason the function $L_1(\nu, R)$ exponentially falls down a great distance from the string $R = r_p/r_o \gg 1$ and it tends to a positive constant at the surface of the string at $R = 1$. Therefore the self-energy tends to that value for an infinitely thin cosmic string far from it. The main difference appears near the surface of the string where one has an additional positive contribution.

The self-energy at the string's origin may be analyzed by formulas (54) and (56). Taking the limit $R \to 0$ and using the

behavior of Legendre's function in the neighborhood of unit [24] one has the following expression for the self-energy in the origin

$$U_{max} = \frac{q^2}{2r_o} \frac{2}{\pi} \int_0^\infty dx \left\{ S_{in}[x,0,\nu] + \frac{1}{2} \ln \frac{x^2}{\nu^2-1} \right.$$

$$\left. - \Psi[\mu_x+1] - \frac{\pi}{2} \cot \pi \mu_x \right\}, \qquad (69)$$

which is, in fact, the height of the potential barrier. Here Ψ is the logarithmic derivative of the gamma function and

$$S_{in}[x,0,\nu]$$

$$= -\frac{\sqrt{\nu^2-1}}{x} \frac{xK_1[x]q^0_{\mu_x}\left[\frac{1}{\nu}\right] - \sqrt{\nu^2-1}K_0[x]q^1_{\mu_x}\left[\frac{1}{\nu}\right]}{\sqrt{\nu^2-1}K_1[x]p^0_{\mu_x}\left[\frac{1}{\nu}\right] + xK_0[x]p^1_{\mu_x}\left[\frac{1}{\nu}\right]}.$$

$$(70)$$

As it can be seen from the uniform expansion Eq. (67), the dependence of the self-energy on the metric coefficient $\nu = 1/\cos \epsilon$ is given mainly by the expression $(\nu - 1/\nu)$. For this reason we represent the self-energy U and the height of the barrier U_{max} as

$$U = \frac{q^2}{2r_o} \frac{\nu^2-1}{\nu} \mathcal{U}(\nu,R), \qquad (71)$$

$$U_{max} = \frac{q^2}{2r_o} \frac{\nu^2-1}{\nu} \mathcal{U}_{max}(\nu), \qquad (72)$$

where

$$\mathcal{U}_{max}(\nu) = \frac{\nu}{\nu^2-1} \frac{2}{\pi} \int_0^\infty dx \left\{ S_{in}[x,0,\nu] + \frac{1}{2} \ln \frac{x^2}{\nu^2-1} \right.$$

$$\left. - \Psi[\mu_x+1] - \frac{\pi}{2} \cot \pi \mu_x \right\}. \qquad (73)$$

The dependence of $\mathcal{U}$ on ν is weak for ν close to unit and it does not depend, in fact, on ν for $\epsilon \lesssim 0.1$. In Fig. 1, $\mathcal{U}_{max}(\nu)$ is displayed as a function of ν.

Therefore for a small deficit angle we obtain the following formula for the height of the barrier

$$U_{max} \approx 0.39 \frac{q^2}{2r_o} \left(\nu - \frac{1}{\nu} \right). \qquad (74)$$

The numerical calculation of $\mathcal{U}(\nu,R)$ as a function of $R = r_p/r_o$ for $\epsilon = 0.1$ is depicted in Fig. 2 (see the Appendix for details).

Let us now compare the self-energy in the thick cosmic string space-time with that in the infinitely thin cosmic string space-time in the limit of zero thickness of the string ($r_o \to 0$). We have to compare the self-energy in two different space-times for a particle situated at the same proper distance from the string. In the infinitely thin cosmic string back-
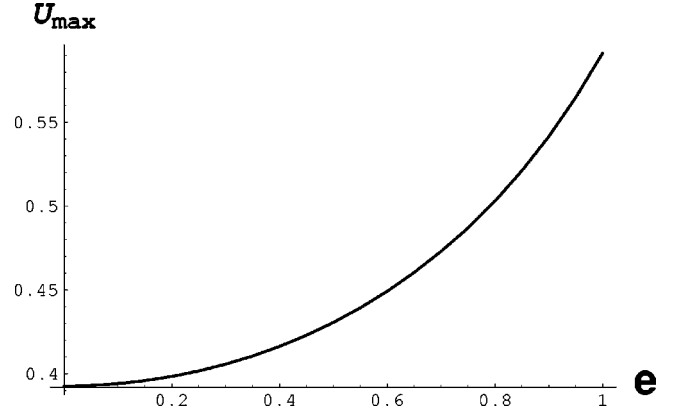

FIG. 1. The self-energy at the string's origin (height of the potential barrier) is represented as $U_{max} = (q^2/2r_o)(\nu^2-1)/\nu \mathcal{U}_{max}(\nu)$. Shown here is the plot of the dimensionless self-energy $\mathcal{U}_{max}$ as a function of ϵ. The cone parameter $\nu = 1/\cos \epsilon$. For $\epsilon \lesssim 0.1$, $\mathcal{U}_{max} \approx 0.39$.

ground the distance from the string d coincides with the coordinate of the particle, that is, $d = r_p = r_o R$. In the Gott-Hiscock cosmic string space-time the distance from the string to a particle is

$$d = \begin{cases} r_o \left(R - 1 + \dfrac{\epsilon}{\tan \epsilon} \right), & R \geq 1 \\[3mm] r_o \dfrac{\arcsin(R \sin \epsilon)}{\tan \epsilon}, & R \leq 1. \end{cases} \qquad (75)$$

Taking into account the above expression and Eqs. (54) and (55) we get for a particle situated outside the string ($D = d/r_o$) the following relation:

$$\frac{U_{thick}}{U_{inf.\ thin}} = \frac{D}{D+1-\dfrac{\epsilon}{\tan \epsilon}} \left\{ 1 + \frac{L_1\left(\nu, D+1-\dfrac{\epsilon}{\tan \epsilon}\right)}{L_0(\nu)} \right\}.$$
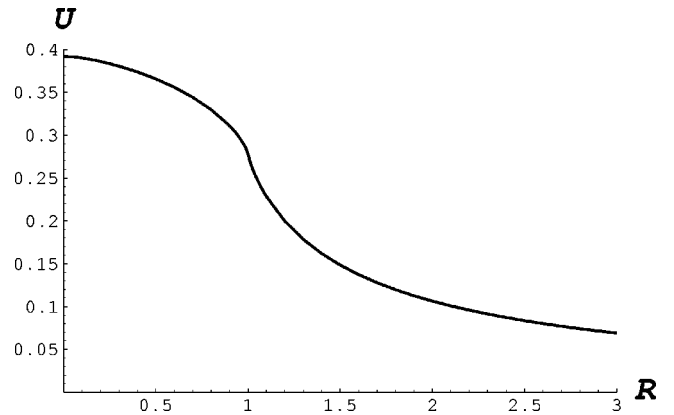
$$(76)$$

FIG. 2. The self-energy $U(\nu,R) = (q^2/2r_o)(\nu^2-1)/\nu \mathcal{U}(\nu,R)$. This is the plot of a dimensionless self-energy $\mathcal{U}$ as a function of the particle position $R = r_p/r_o$ for $\epsilon = 0.1$.
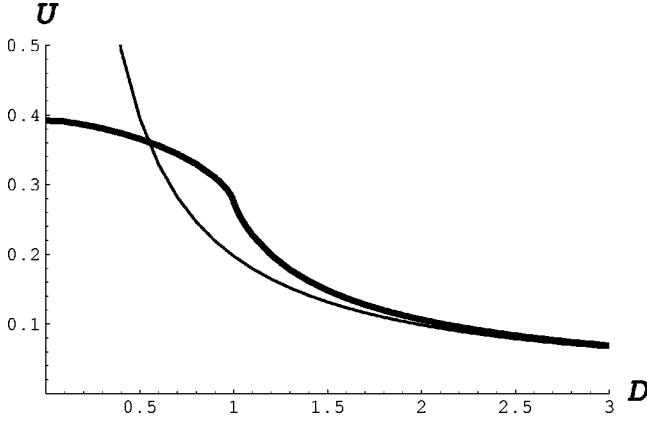
FIG. 3. The dimensionless self-energy $\mathcal{U}$ (thick line) for a particle in a thick cosmic string space-time and for a particle in an infinitely thin cosmic string space-time $\mathcal{U}_{\text{thin}}$ (thin line) as a function of the proper distance $D = d/r_{\text{o}}$ numerically calculated for $\epsilon = 0.1$.

In the limit of zero thickness $r_{\text{o}} \to 0$ $(D \to \infty)$ we obtain unit in the right-hand side and the self-energy in the Gott-Hiscock space-time tends to the same result obtained in the infinitely thin cosmic string space-time. In this case the barrier's height U_{max} given by Eq. (72) tends to infinity $\sim 1/r_{\text{o}}$.

In Fig. 3 we display the numerical calculation of the self-energy $\mathcal{U}$ given by Eq. (71) for a particle in the Gott-Hiscock space-time and in an infinitely thin cosmic string space-time $\mathcal{U}_{\text{inf. thin}}$ defined below,

$$U_{\text{inf. thin}} = \frac{q^2}{2r_{\text{o}}} \frac{\nu^2 - 1}{\nu} \mathcal{U}_{\text{inf. thin}},$$

(77)

$$\mathcal{U}_{\text{inf. thin}} = \frac{\nu}{\nu^2 - 1} \frac{1}{R} L_0(\nu),$$

as a function of distance from the string's origin $D = d/r_{\text{o}}$.

For an infinitely thin cosmic string space-time $R = D$ and for the Gott-Hiscock space-time we have

$$R = \begin{cases} D + 1 - \dfrac{\epsilon}{\tan \epsilon}, & R \geqslant 1 \\[2mm] \dfrac{\sin(D \tan \epsilon)}{\sin \epsilon}, & R \leqslant 1. \end{cases}$$

(78)

The self-force, according to Eq. (20), is minus the derivative of the self-energy given by Eq. (54) with respect to the particle position. It is zero in the string's origin and it tends to zero such as in the infinitely thin cosmic string space-time far away from the string. In the neighborhood of the string's surface, for $|R - 1| \ll 1$, it tends to infinity logarithmically according to (see the Appendix)

$$F_r \approx -\frac{q^2}{2r_{\text{o}}^2} \frac{\nu^2 - 1}{8} \ln|R - 1|.$$
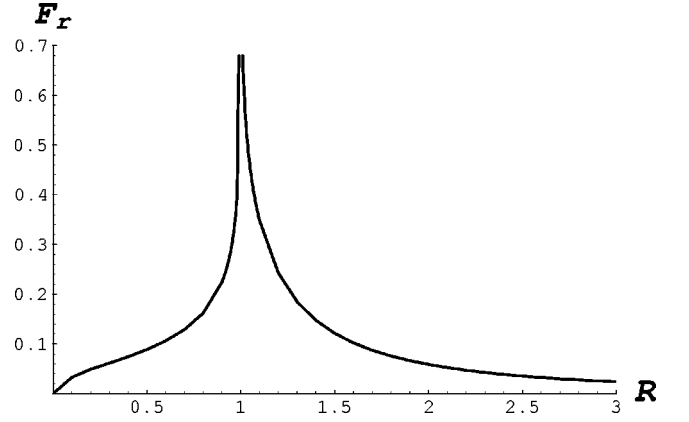
(79)

FIG. 4. The self-force $F_r(\nu, R) = (q^2/2r_{\text{o}})(\nu^2 - 1)/\nu \mathcal{F}_r(\nu, R)$. Shown here is the plot of the dimensionless self-force for a particle in a thick cosmic string space-time $\mathcal{F}_r$ as a function of $R = r_p/r_{\text{o}}$ for $\epsilon = 0.1$.

This divergence is rather formal. The function $\ln x$ is an integrable function at $x = 0$. For this reason the work against this self-force is finite and equal to the height of the barrier U_{max} given by Eq. (72).

Because the self-potential U has the structure given in Eq. (71) we represent the self-force in the same way as

$$F_r = \frac{q^2}{2r_{\text{o}}^2} \frac{\nu^2 - 1}{\nu} \mathcal{F}_r(\nu, R).$$

(80)

The numerical simulation of the function $\mathcal{F}_r$ is displayed in Fig. 4.

V. CONCLUSIONS

The aim of the paper is to calculate the self-energy and self-force for a charged particle at rest in the space-time of an infinitely long, straight cosmic string with a nontrivial internal structure. The relevance of this calculation is to clear up the role of the nonzero thickness of the string. It is well-known that for a particle at rest in the infinitely thin cosmic string space-time, the self-energy and self-force fall down far from the string and tend to infinity at the string's core [8,9]. Obviously the origin of these singularities is associated with the delta-like model of the string's interior.

In the proceeding sections we considered the self-energy and the self-force of charged particles at rest in the cosmic string space-time with simplest nontrivial interior, suggested by Gott [5] and Hiscock [6]. This model of string is an exact solution of the Einstein equations and it corresponds to the cylindrical distribution of matter of constant energy density inside it. The exterior of the string is the flat conical space-time and the interior is the constant curvature space-time. The model is usually named the ''ballpoint pen'' model [18]. It is suitable for our goal because this model contains a dimensional parameter, the radius of string r_{o}, with respect to which we may analyze our problem. Let us summarize the main results in what follows.

To calculate the self-energy and self-force we used the approach of Refs. [8,9] in which the quantities under consid-

eration are expressed in terms of the renormalized Green's function of the three-dimensional Laplace operator.

We analyzed the self-energy and self-force for different positions r_p of the particle. The self-energy falls down outside the string and tends to the same result obtained in the case of an infinitely thin cosmic string space-time far from the string's surface. Namely, the self-energy has the following structure:

$$U(r_p) = \frac{q^2}{2r_p}\left[L_0(\nu) + L_1\left(\nu, \frac{r_p}{r_o}\right)\right], \qquad (81)$$

where the function L_1 exponentially tends to zero far away from the string and the function L_0 is the same one that corresponds to the case of an infinitely thin cosmic string space-time. The additional contribution to the self-energy is expressed as momentum expansion in terms of the S matrix of the scattering problem in the imaginary axis.

Inside the string the self-energy grows up and tends to a constant in the string's origin, which is, in fact, the height of the potential barrier. For a cosmic string which is considered in grand unified theory with $\nu - 1 \approx 10^{-6}$ and $r_o \approx 10^{-29}$ cm, the height of the energy barrier is 2.8×10^5 GeV.

In the limit of zero radius of the string ($r_o \rightarrow 0$), the self-potential tends to the same value corresponding to a particle in the infinitely thin cosmic string space-time and the maximum of the self-energy tends to infinity as $1/r_o$.

The self-force, which is the minus gradient of the self-energy has only a radial component and it is repulsive for any position of the particle. It is zero in the string's origin and tends to the self-force in the infinitely thin cosmic string space-time. In the surface of the string it has a maximum value. In the framework of Gott-Hiscock thick cosmic string space-time the self-force tends to infinity logarithmically. This is an integrable divergence and the total work against the self-force is finite and equal to the maximum of the self-energy at the center of the string.

Therefore the nonzero radius of the string drastically changes the behavior of self-energy and self-force close to the string's core. Outside the string's surface the additional contributions due to the string's radius exponentially fall down. The self-energy and self-force are equal, in fact, to that in an infinitely thin cosmic string space-time starting from the distance of two radii of the string. We expect that this behavior of self-energy and self-force will be, in general, the same for a cosmic string in the Abelian-Higgs model, considered in [7].

ACKNOWLEDGMENTS

APPENDIX

In this appendix we discuss the numerical analysis of the self-potential given by Eq. (54). Using Eq. (53) we can represent the self-potential in the following form:

$$U(r_p) = \frac{q^2}{2r_o}\begin{cases} \dfrac{1}{R}L(\nu, R), & R \geq 1 \\[2mm] H(\nu, R), & R \leq 1, \end{cases} \qquad (A1)$$

where

$$L(\nu, R) = \frac{2}{\pi}\int_0^\infty dx \sum_{n=-\infty}^{+\infty}\left\{\nu K_{|n|\nu}[x]I_{|n|\nu}[x] - K_{|n|}\left[\frac{x}{\nu}\right]I_{|n|}\left[\frac{x}{\nu}\right]\right. $$
$$\left. + \nu S_{\text{out}}\left[\frac{x}{R}, |n|, \nu\right]K_{|n|\nu}^2[x]\right\}, \qquad (A2)$$

$$H(\nu, R) = \frac{2}{\pi}\int_0^\infty dx \sum_{n=-\infty}^{\infty}\left\{p_{\mu_x}^{|n|}[x_R](q_{\mu_x}^{|n|}[x_R]\right. $$
$$\left. + S_{\text{in}}[x, |n|, \nu]p_{\mu_x}^{|n|}[x_R]) - K_{|n|}\left[\frac{xR}{\nu}\right]I_{|n|}\left[\frac{xR}{\nu}\right]\right\}, $$
$$\qquad (A3)$$

$$S_{\text{out}}[x, |n|, \nu]$$
$$= -\frac{\nu\left(1 - \dfrac{1}{\nu^2}\right)p_{\mu_x}^{|n|\,\prime}\left[\dfrac{1}{\nu}\right]I_{|n|\nu}[x] + xp_{\mu_x}^{|n|}\left[\dfrac{1}{\nu}\right]I'_{|n|\nu}[x]}{\nu\left(1 - \dfrac{1}{\nu^2}\right)p_{\mu_x}^{|n|\,\prime}\left[\dfrac{1}{\nu}\right]K_{|n|\nu}[x] + xp_{\mu_x}^{|n|}\left[\dfrac{1}{\nu}\right]K'_{|n|\nu}[x]}, $$
$$\qquad (A4)$$

$$S_{\text{in}}[x, |n|, \nu]$$
$$= -\frac{\nu\left(1 - \dfrac{1}{\nu^2}\right)q_{\mu_x}^{|n|\,\prime}\left[\dfrac{1}{\nu}\right]K_{|n|\nu}[x] + xq_{\mu_x}^{|n|}\left[\dfrac{1}{\nu}\right]K'_{|n|\nu}[x]}{\nu\left(1 - \dfrac{1}{\nu^2}\right)p_{\mu_x}^{|n|\,\prime}\left[\dfrac{1}{\nu}\right]K_{|n|\nu}[x] + xp_{\mu_x}^{|n|}\left[\dfrac{1}{\nu}\right]K'_{|n|\nu}[x]}. $$
$$\qquad (A5)$$

In the last term in Eq. (A3) we used Eq. (53) and changed the argument of the Bessel functions from xR to xR/ν. By construction, the self-energy is a C^1 regular function at the surface of the string because Green's function is expressed in terms of the functions given in Eq. (30). The renormalization is done by subtracting the same function in the regions outside and inside the string. Therefore the expression given previously which corresponds to the self-potential is a C^1 regular function at the string's surface mode by mode, which is more suitable for numerical simulations.

First of all let us consider the self-energy for a particle situated outside the string with $R \geq 1$. In the neighborhood of the string's surface the series converges very slowly. To simplify numerical calculations we represent it in the following form:

$$\frac{1}{R}L(\nu,R) = \frac{2}{\pi R}\int_0^\infty dx\left\{\nu K_0[x]I_0[x] - K_0\left[\frac{x}{\nu}\right]I_0\left[\frac{x}{\nu}\right]\right.$$

$$\left. + \nu S_{\text{out}}\left[\frac{x}{R},0,\nu\right]K_0^2[x]\right\}$$

$$+ \frac{4}{\pi R}\sum_{n=1}^N n\int_0^\infty dx\left\{\nu K_{n\nu}[nx]I_{n\nu}[nx]\right.$$

$$\left. - K_n\left[\frac{nx}{\nu}\right]I_n\left[\frac{nx}{\nu}\right] + \nu S_{\text{out}}\left[\frac{nx}{R},n,\nu\right]K_{n\nu}^2[nx]\right\}$$

$$+ \frac{4}{\pi R}\sum_{n=N+1}^{+\infty} n\int_0^\infty dx\left\{\nu K_{n\nu}[nx]I_{n\nu}[nx]\right.$$

$$\left. - K_n\left[\frac{nx}{\nu}\right]I_n\left[\frac{nx}{\nu}\right] + \nu S_{\text{out}}\left[\frac{nx}{R},n,\nu\right]K_{n\nu}^2[nx]\right\}.$$

$$(A6)$$

In the last term of the previous equation, let us use (for sufficiently great N) the uniform expansion for Bessel functions [21]

$$K_p[pz] \approx \sqrt{\frac{\pi t}{2p}}e^{-p\eta}\sum_{k=0}^\infty (-p)^{-k}u_k[t],$$

$$I_p[pz] \approx \sqrt{\frac{t}{2\pi p}}e^{p\eta}\sum_{k=0}^\infty p^{-k}u_k[t],$$

$$(A7)$$

$$K_p'[pz] \approx -\sqrt{\frac{\pi}{2ptz^2}}e^{-p\eta}\sum_{k=0}^\infty (-p)^{-k}\bar{u}_k[t],$$

$$I_p'[pz] \approx \sqrt{\frac{1}{2\pi ptz^2}}e^{p\eta}\sum_{k=0}^\infty p^{-k}\bar{u}_k[t],$$

where

$$t[z] = \frac{1}{\sqrt{1+z^2}}, \quad \eta[z] = \sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}},$$

$$u_k[t] = \frac{1}{2}t^2(1-t^2)u_{k-1}'[t]$$

$$+ \frac{1}{8}\int_0^t (1-5t^2)u_{k-1}[t]dt, \quad u_0[t]=1,$$

$$\bar{u}_k[t] = u_k[t] + t(t^2-1)\left\{\frac{1}{2}u_{k-1}[t] + tu_{k-1}'[t]\right\},$$

$$(A8)$$

$$\bar{u}_0[t] = 1,$$

and the uniform expansion of Legendre's functions found in Ref. [23] which have the form

$$p_\mu^n[z] = \frac{1}{n!}\left[\frac{1+\gamma^2v^2}{1+\gamma^2}\right]^{1/4}e^{nS}\sum_{k=0}^\infty n^{-k}\Pi_k[v],$$

$$q_\mu^n[z] = \frac{(n-1)!}{2}\left[\frac{1+\gamma^2v^2}{1+\gamma^2}\right]^{1/4}e^{-nS}\sum_{k=0}^\infty (-n)^{-k}\Pi_k[v],$$

$$\frac{1}{n}\frac{d}{dz}p_\mu^n = -\frac{1}{n!}\left[\frac{1+\gamma^2v^2}{1+\gamma^2}\right]^{3/4}\frac{1+\gamma^2}{1-v^2}e^{nS}\sum_{k=0}^\infty n^{-k}\bar{\Pi}_k[v],$$

$$(A9)$$

$$\frac{1}{n}\frac{d}{dz}q_\mu^n = \frac{(n-1)!}{2}\left[\frac{1+\gamma^2v^2}{1+\gamma^2}\right]^{3/4}\frac{1+\gamma^2}{1-v^2}e^{-nS}$$

$$\times\sum_{k=0}^\infty (-n)^{-k}\bar{\Pi}_k[v],$$

where

$$\mu = -\frac{1}{2} + \frac{1}{2}\sqrt{1-\left(\frac{2nx}{\tan\epsilon}\right)^2}, \quad v = \frac{z}{\sqrt{1+\gamma^2(1-z^2)}},$$

$$\gamma = \frac{x}{\tan\epsilon},$$

$$S = \frac{1}{2}\ln\left[\frac{1-v}{1+v}\frac{1}{1+\gamma^2}\right] - \gamma(\arctan[\gamma v] - \arctan[\gamma]),$$

$$\Pi_{k+1}[v]$$

$$= \frac{1-v^2}{2}\frac{1+\gamma^2v^2}{1+\gamma^2}\Pi_k'[v] - \frac{\gamma^2}{8(1+\gamma^2)}$$

$$\times\int_1^v dv'\left\{5v'^2 + \frac{1}{\gamma^2} - 1 - \frac{1+\gamma^2}{\gamma^2(1+\gamma^2v'^2)}\right\} \quad (A10)$$

$$\times\Pi_k[v'], \quad \Pi_0[v]=1,$$

$$\bar{\Pi}_k[v] = \Pi_k[v] - \frac{\gamma^2v(1-v^2)}{2(1+\gamma^2)}\Pi_{k-1}[v]$$

$$- \frac{(1-v^2)(1+\gamma^2v^2)}{1+\gamma^2}\Pi_{k-1}'[v], \quad \bar{\Pi}_0[v]=1.$$

Then, taking into account these previous formulas we have the following expression for the last term in Eq. (A6):

$$\frac{4}{\pi R}\sum_{n=N+1}^{+\infty} n\int_0^\infty dx\left\{\nu K_{n\nu}[nx]I_{n\nu}[nx] - K_n\left[\frac{nx}{\nu}\right]I_n\left[\frac{nx}{\nu}\right]\right.$$

$$\left. + \nu S_{\text{out}}\left[\frac{nx}{R},n,\nu\right]K_{n\nu}^2[nx]\right\}$$

$$\approx \frac{\nu^2-1}{4\pi\nu R}\sum_{n=N+1}^\infty \frac{1}{n^2}\int_0^\infty dz\left\{-z^2t[z]^5(1-5t[z]^2)\right.$$

$$\left. + \frac{z^2}{R^2}t[z]t\left[\frac{z}{R}\right]^4 e^{-2n\nu(\eta[z]-\eta[z/R])}\right\}, \quad (A11)$$

which may be expressed in terms of the function

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+z)^{-s}z^{n}. \qquad (A12)$$

Using integral representation for this function [24] we obtain, finally, the following expression for this term in Eq. (A6):

$$\frac{4}{\pi R}\sum_{n=N+1}^{+\infty}n\int_{0}^{\infty}dx\left\{vK_{n\nu}[nx]I_{n\nu}[nx]-K_{n}\left[\frac{nx}{\nu}\right]I_{n}\left[\frac{nx}{\nu}\right]\right.$$
$$\left.+\nu S_{\text{out}}\left[\frac{nx}{R},n,\nu\right]K_{n\nu}^{2}[nx]\right\}$$
$$\approx\frac{\nu^{2}-1}{4\pi\nu R}\left\{\frac{1}{3}\zeta_{H}[2,N+1]+\frac{1}{R^{2}}\int_{0}^{\infty}dz\,z^{2}t[z]t\left[\frac{z}{R}\right]^{4}\right.$$
$$\left.\times\int_{0}^{\infty}dy\,y\frac{e^{-N[y+2\nu(\eta[z]-\eta[z/R])]}}{e^{y+2\nu(\eta[z]-\eta[z/R])}-1}\right\}, \qquad (A13)$$

which is suitable for numerical calculations.

We use the same approach for the function $H(\nu,R)$ and obtain the following result for the series corresponding to the function $H(\nu,R)$:

$$\frac{4}{\pi}\sum_{n=N+1}^{+\infty}n\int_{0}^{\infty}dx\left\{p_{\mu_{x}}^{n}[x_{R}](q_{\mu_{x}}^{n}[x_{R}]+S_{\text{in}}[x,n,\nu]p_{\mu_{x}}^{n}[x_{R}])\right.$$
$$\left.-K_{n}\left[\frac{xR}{\nu}\right]I_{n}\left[\frac{xR}{\nu}\right]\right\}$$
$$\approx\frac{\nu^{2}-1}{4\pi\nu}\left\{R\zeta_{H}[2,N+1]-\int_{0}^{\infty}dz\,z^{2}t[zR]t[z]^{4}\right.$$
$$\left.\times\int_{0}^{\infty}dy\,y\frac{e^{-N[y+2(\widetilde{\eta}[z,1]-\widetilde{\eta}[z,R])]}}{e^{y+2(\widetilde{\eta}[z,1]-\widetilde{\eta}[z,R])}-1}\right\}, \qquad (A14)$$

where

$$\widetilde{\eta}[z,R]=\ln\frac{zR}{\sqrt{1+z^{2}R^{2}}+\sqrt{1-R^{2}\sin^{2}\epsilon}}$$
$$-\frac{z}{\sin\epsilon}\left[\arctan\frac{z\sqrt{1-R^{2}\sin^{2}\epsilon}}{\sin\epsilon\sqrt{1+z^{2}R^{2}}}-\arctan\frac{z}{\sin\epsilon}\right], \qquad (A15)$$

$$\eta[z]=\sqrt{1+z^{2}}+\ln\frac{z}{1+\sqrt{1+z^{2}}}, \qquad (A16)$$

and $\zeta_{H}(s,x)$ is the Hurwitz zeta function. At the surface of the string $R=1$, and expressions given by Eqs. (A13) and (A14) coincide and are equal to

$$\frac{\nu^{2}-1}{\nu}\frac{\zeta_{H}[2,N+1]}{6\pi}. \qquad (A17)$$

Therefore the self-energy is continuous at the string's surface. For numerical simulations we used previous formulas for $N=0$.

In order to analyze the self-force near the string's surface let us consider more carefully the self-energy for a particle outside the string. For $N=0$ we have

$$\frac{1}{R}L(\nu,R)=\frac{2}{\pi R}\int_{0}^{\infty}dx\left\{\nu K_{0}[x]I_{0}[x]-K_{0}\left[\frac{x}{\nu}\right]I_{0}\left[\frac{x}{\nu}\right]\right.$$
$$\left.+\nu S_{\text{out}}\left[\frac{x}{R},0,\nu\right]K_{0}^{2}[x]\right\}+\frac{\pi}{72}\frac{\nu^{2}-1}{\nu R}$$
$$+\frac{\nu^{2}-1}{4\pi\nu}\int_{0}^{\infty}dz\,z^{2}t[zR]t[z]^{4}$$
$$\times\int_{0}^{\infty}dy\frac{y}{e^{y+2\nu(\eta[zR]-\eta[z])}-1}. \qquad (A18)$$

The last term will give a logarithmic divergence at the string's surface. In order to see this let us represent it in the form

$$\int_{0}^{\infty}\frac{y\,dy}{e^{y+2\nu p}-1}=\int_{2\nu p}^{\infty}\frac{y\,dy}{e^{y}-1}+2\nu p\ln(1-e^{-2\nu p}), \qquad (A19)$$

where

$$p=\eta[zR]-\eta[z]=\ln R$$
$$+\ln\frac{\sqrt{1+z^{2}}+1}{\sqrt{1+z^{2}R^{2}}+1}+\sqrt{1+z^{2}R^{2}}-\sqrt{1+z^{2}}. \qquad (A20)$$

The derivative of the integral in the right-hand side with respect to R is finite at the point $R=1$, but the derivative of the second term in the right-hand side gives a logarithmic divergence at $R=1$ because

$$\frac{dp}{dR}=\frac{\sqrt{1+z^{2}R^{2}}}{R} \qquad (A21)$$

is finite at the string's surface.

Taking into account previous formulas, we obtain the following results which are divergent at $R=1$:

$$\frac{1}{R}L(\nu,R)=\frac{\nu^{2}-1}{8}(R-1)\ln(R-1)+l(\nu,R), \quad R>1, \qquad (A22)$$

$$H(\nu,R)=-\frac{\nu^{2}-1}{8}(1-R)\ln(1-R)+h(\nu,R), \quad R<1. \qquad (A23)$$

The functions $l(\nu,R)$ and $h(\nu,R)$ and their first derivatives with respect to R are finite at $R=1$.

Taking the derivative of the above expressions inside and outside the string we find that the divergence at the string's surface is given by Eq. (79). This expression does not depend on the number N because this divergence appears associated

with small y, but for small y the integrand does not depend on N. Therefore each term in the expression for the self-force is finite and continuous at the string surface but the sum of the series is logarithmically divergent.

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