# General-relativistic free decay of magnetic fields in a spherically symmetric body

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The decay of a magnetic field penetrating a compact spherical electrically conducting body and continuing in its nonconducting surroundings is systematically studied. The body, considered as a rough model of a compact spherical star, is assumed to be nonrotating and showing no internal motion, and so the metric of the spacetime is static and spherically symmetric. Starting from the absolute space formalism of curved-space electrodynamics the initial value problem for the magnetic field is formulated. The concept of poloidal and toroidal fields is used to reduce the equations describing this problem to equations for the defining scalars of the magnetic field. By expansion of them in a series of spherical harmonics equations are derived for functions of the radial and time coordinates. A solution of these equations for the outer space is given. For the case of time-independent conductivity of the body, the equations for the interior of the body are reduced to ordinary differential equations which pose eigenvalue problems of the Sturm-Liouville type. After these reductions the solution of the initial value problem for the magnetic field is given as a superposition of magnetic field modes decaying exponentially in time. The shape of the modes is determined by the eigenfunctions of the Sturm-Liouville problems mentioned, and the decay rates by the corresponding eigenvalues. Explicit results, mainly gained by solving the relevant equations numerically, are given for the simple extreme case of constant density of the body. Their most striking feature is that all growth rates decrease with the growing compactness of the body. Furthermore, some concentration of the magnetic field in the inner parts occurs for high compactness. The consequences of our findings for the magnetic-field evolution in neutron stars are discussed as well as the implications for dynamo models.

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## I. INTRODUCTION

A magnetic field in an electrically conducting medium at rest is bound to decay. The magnetic energy is converted into heat due to Ohmic dissipation of the electric currents [1]. In flat spacetime this so-called free decay of magnetic fields penetrating a finite conducting body and continuing in its nonconducting surroundings has been extensively studied. For the case of a spherical body with a constant conductivity the general solution of the initial value problem of the relevant equations is available in analytical form [2,3]. It is a superposition of magnetic-field modes each of which decays exponentially in time. Knowledge of the solutions of the free-decay problem is crucial as a background for understanding the behavior of magnetic fields in cosmic bodies even in cases in which they do not decay but are maintained or grow as a consequence of generation processes, for example, by dynamo action of fluid motions [3,4] or thermoelectric instabilities [5–7]. In these cases not only the modes with large spatial scales and small decay rates but also those with small scales and high decay rates are important. In addition study of the free-decay problem provides us with mathematical tools for the investigation of more complex problems. The magnetic-field modes mentioned, taken at a given time, constitute a complete set of vector fields which allows us to represent arbitrary magnetic fields in the region of the conducting body, in particular, solutions of the dynamo equations [8].

In several astrophysical objects showing magnetic phenomena the curvature of the spacetime can no longer be neglected. This applies, for example, for the primordial plasma or for accretion disks around compact objects. We have a particular interest in the neutron stars with their extremely strong magnetic fields. The comparison of observational results with theoretical findings on magnetic-field evolution allows us to draw conclusions concerning the state of matter under extreme conditions.

So it seems very desirable to investigate the free decay of magnetic fields in a curved spacetime. There are already a few results concerning this issue. Geppert *et al.* [9] derived the induction equation on a static spherically symmetric background geometry and on this basis studied the decay of a dipolar magnetic field numerically in a constant-density star model. The most remarkable result is that the decay becomes slower with increasing compactness of the star. Page *et al.* [10] considered more realistic models of neutron stars with compactness ratios between 0.3 and 0.5 but still without rotation, and presented numerical results for the decay of dipolar magnetic fields resulting from electric currents in the crust. If a very soft equation of state applies for the matter of the star the scales of a magnetic field in its interior are much smaller than those for a star with the same mass but a stiff equation of state. Thus, when ignoring relativistic effects, a rapid field decay can be interpreted as a hint of the

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validity of a soft equation of state. With increasing softness, however, the compactness of the star increases and relativistic effects become more pronounced. So the situation is determined by the competition between these two opposite tendencies. For examples with very soft equations of state those effects drastically decelerate the decay of the surface field strengths so that they are larger by a factor of about 100 after $10^{10}$ yr compared to the values obtained by nonrelativistic calculations. That is, the relativistic effects counteract those of the softening of the equation of state. This makes conclusions concerning the state of matter inside neutron stars from observational data on the magnetic-field evolution more difficult.

In this paper we give a systematic treatment of the problem of the free decay of a magnetic field on a static geometry determined by a compact spherical body surrounded by free space. This body is assumed to be nonrotating, that is, the metric of the spacetime is assumed to be spherically symmetric. Although we call this body in the following for brevity a "star" and discuss the results with a view to neutron stars we do not claim to propose a model of such an object. In Sec. II the induction equation in a static spherically symmetric spacetime is derived applying the absolute space formalism, and in Sec. III the free-decay problem is formulated as an initial value problem for a system of vector differential equations. To prepare its reduction to a system of equations for scalar quantities, in Sec. IV the relevant aspects of the concept of poloidal and toroidal vector fields are explained. Then in Sec. V the original equations are reduced to equations for scalar functions of the radial coordinate and the time. After giving a solution for the outer space in Sec. VI, in Sec. VII the assumption of a time-independent conductivity is introduced and the equations for the interior of the body are further reduced to ordinary differential equations. They pose eigenvalue problems of the Sturm-Liouville type, the eigenfunctions and eigenvalues of which define the shape of the decay modes of the magnetic field and their decay rates. These modes are considered in more detail in Sec. VIII. They constitute a complete set of vector fields in the sense that any initial magnetic field and therefore any solution of the initial value problem for the magnetic field can be represented as a superposition of such modes. In Sec. IX we give specific results for the decay modes for a constant-density star gained by numerical calculations. Finally in Sec. X we discuss some implications of our results in view of cosmic objects and of mathematical tools for related problems.

## II. INDUCTION EQUATION ON A STATIC SPACETIME

Let us first deal with the basic equations for the electromagnetic field. We restrict our consideration to an arbitrary nonsingular, globally static, and spherically symmetric spacetime and write the line element in the form

$$ds^2 = -e^{2\Phi}(dx^0)^2 + h_1^2(dx^1)^2 + h_2^2(dx^2)^2 + h_3^2(dx^3)^2; \tag{1}$$

see, e.g., [11]. Here $x^0 = ct$, and $\Phi$ is related to the redshift factor $Z$ via $Z = e^{\Phi}$. The scale factors $h_i$, $i = 1,2,3$, define the spatial metric on the $x^0 = \text{const}$ hypersurfaces. For the present section $Z$ and $h_i$ are arbitrary functions of the spatial coordinates consistent with spherical symmetry.

We start with the covariant form of Maxwell's equations:

$$\nabla^{\alpha} F_{\alpha\beta} = -\frac{4\pi}{c} J_{\beta}, \tag{2}$$

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0. \tag{3}$$

Here $F_{\alpha\beta}$ are the coordinate components of the Maxwell tensor, $J_{\alpha}$ those of the four-current density, and $\nabla$ the covariant derivative operator; greek indices take the values 0, 1, 2, and 3. Clearly, any electric or magnetic polarization of the matter is excluded.

We use in the following the absolute space formalism of curved-space electrodynamics [12]. Once a solution of Eqs. (2) and (3) in the background of (1) has been specified, then a Killing observer with a four-velocity $U^{\alpha}$ defined by $U^{\alpha} = e^{-\Phi}\delta_0^{\alpha}$, $U^{\alpha}U_{\alpha} = -1$, measures an electric field $\mathbf{E}$ and a magnetic one $\mathbf{B}$ defined by the physical components of the corresponding four-vectors with coordinate components

$$E_{\alpha} = F_{\alpha\beta}U^{\beta}, \quad B_{\alpha} = -\frac{1}{2}\epsilon_{\alpha\beta}^{\gamma\delta} F_{\gamma\delta} U^{\beta}, \tag{4}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ stands for the four-dimensional Levi-Civitȧ tensor density. Likewise these observers measure an electric current density $\mathbf{J}$ and charge density $\rho$ which can be derived from the four-vector $J_{\alpha}$. Then the covariant form (2),(3) of the Maxwell equations yields

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{5}$$

$$\boldsymbol{\nabla} \times (Z\mathbf{E}) = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \boldsymbol{\nabla} \times (Z\mathbf{B}) = \frac{4\pi}{c} Z\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{6}$$

Here the symbol $\boldsymbol{\nabla}$ stands for the divergence and curl operations formed out of the scale factors $h_i$ describing the geometry of the $t = \text{const}$ spacelike hypersurfaces with the line element $ds_3^2 = h_1^2(dx^1)^2 + h_2^2(dx^2)^2 + h_3^2(dx^3)^2$. Explicit representations of the vector differential operations used here are given in Appendix A. We note that the time $t$ used here is the universal time, that is, the time measured by an observer at infinity.

We suppose that inside the electrically conducting matter Ohm's law relative to the Killing observers applies in the simple form

$$\mathbf{J} = \sigma \mathbf{E} \tag{7}$$

with $\sigma$ being the electric conductivity.

Let us now assume that the electric conductivity of the matter is sufficiently high so that the displacement current $\partial\mathbf{E}/\partial t$ in Ampère's law, that is, the second equation (6), is negligible in comparison with the conduction current $\mathbf{J}$. Starting then from Faraday's law, that is, the first equation (6), and using Ohm's law (7) as well as Ampère's law without displacement current we readily arrive at the induction equation

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\times\left(\frac{c^2}{4\pi\sigma}\nabla\times(Z\mathbf{B})\right)=\mathbf{0}. \tag{8}$$

As can be easily concluded with the help of Ohm's law the ratio of the magnitudes of displacement and conduction current terms in Ampère's law is $4\pi\sigma Z T_c$ where $T_c$ is a characteristic time scale of the variation of **E**. So the induction equation (8) applies under the condition $4\pi\sigma Z T_c \ll 1$. By the way, if **B** is known and the neglect of the displacement current is justified, **J** and **E** can be readily determined on the basis of Ampère's law and Ohm's law.

## III. FORMULATION OF THE FREE-DECAY PROBLEM FOR A STATIC SPHERICAL STAR

In what follows we deal with the behavior of a magnetic field penetrating a static spherical electrically conducting perfect-fluid body, which we call a "star," and continuing in the surrounding free space. Like all motions of the fluid those due to electromagnetic forces are excluded; furthermore, so is any electric or magnetic polarization of the matter. Finally, the influence of the electromagnetic field on the metric is ignored. We specify the line element (1) to be a corresponding solution of Einstein's equations for a perfect fluid joining smoothly to an exterior Schwarzschild field. Using the familiar Schwarzschild coordinates $r$, $\theta$, and $\phi$ and denoting the corresponding scale factors by $h_r$, $h_\theta$, and $h_\phi$, we have

$$h_r=h(r), \quad h_\theta=r, \quad h_\phi=r\sin\theta \tag{9}$$

with $h$ given by

$$h(r)=\left(1-\frac{2M(r)}{r}\right)^{-1/2}, \tag{10}$$

where $M(r)=Gm(r)/c^2$ with $m(r)$ being the so-called mass function that determines the total mass enclosed within an SO(3) sphere with the radius $r$; see, e.g., [11]. $M(r)$ takes the value $M(R)$ for all $r\geqslant R$, where $r=R$ defines the surface of the star. Instead of $M(R)$ we write simply $M$ in the following. The redshift factor $Z$ depends, of course, on $r$ only. Specific forms of the dependencies of $m$ and $Z$ on $r$ are not needed in this section and will be introduced later. Explicit representations of vector differential operations, etc., in Schwarzschild coordinates used in the following are given in Appendix A.

The magnetic field **B** is assumed to be governed by the equations

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\times[\eta\nabla\times(Z\mathbf{B})]=\mathbf{0}, \quad \nabla\cdot\mathbf{B}=0 \quad \text{in} \quad r<R, \tag{11}$$

where we have introduced the magnetic diffusivity $\eta = c^2/4\pi\sigma$, and

$$\nabla\times(Z\mathbf{B})=\mathbf{0}, \quad \nabla\cdot\mathbf{B}=0 \quad \text{in} \quad r>R, \tag{12}$$

and to satisfy the conditions

$$[\mathbf{B}]=\mathbf{0} \quad \text{across} \quad r=R, \tag{13}$$

where $[\cdots]$ denotes the jump of a quantity across a surface, and

$$\mathbf{B}\to\mathbf{0} \quad \text{as} \quad r\to\infty. \tag{14}$$

In addition we assume that the magnetic energy density is finite everywhere. The requirement for the normal component of **B** contained in Eq. (13) follows from the fact that $\nabla\cdot\mathbf{B}=0$ has to apply everywhere. Those for the tangential components of **B** can be obtained from Eq. (6) if surface currents are excluded, which is natural at least as long as $\eta$ remains finite. The requirement (14) excludes sources for **B** at infinity and ensures together with Eq. (12) that the total magnetic energy remains finite.

The equations and conditions (11)–(14) pose a mixed boundary value problem for **B**. Note that the boundary in that sense is at infinity, and the part of the boundary condition is simply played by Eq. (14). We will deal with this problem for **B** in the following under the additional simplifying assumption that the magnetic diffusivity $\eta$ is spherically symmetric, that is, may depend on $r$ and $t$ but not on $\theta$ or $\phi$. Consequences of deviations from this assumption will be pointed out later.

## IV. POLOIDAL AND TOROIDAL VECTOR FIELDS

For problems such as considered here in flat space it proved to be useful to decompose vector fields like the magnetic field **B** into their poloidal and toroidal parts and to utilize specific properties of these parts. Such a decomposition is rather simple in the case of axisymmetric fields but can also be established in the general case of not necessarily axisymmetric fields; see, e.g., [3,13–15]. Moreover, it can be extended to spherically symmetric metrics in curved spaces; see [16].

We explain this decomposition here with reference to the metric defined by Eq. (9) but consider solenoidal fields only and restrict ourselves to aspects that are important for the reduction of Eqs. (11)–(14). As for the explicit representation of the vector differential operations, etc., we recall again Appendix A. By the way, it might be enlightening to consider our explanations of solenoidal vector fields within the more general concepts applying to not necessarily solenoidal fields that are sketched in Appendix C.

Let us consider an arbitrary solenoidal vector field. Since it will be specified in the next section to be the magnetic field, we denote it already here, without adopting this specification, by **B**. Because $\nabla\cdot\mathbf{B}=0$ we may represent **B** in the form

$$\mathbf{B}=\nabla\times\mathbf{A} \tag{15}$$

with a vector potential **A**. The latter, in turn, can be represented in the form

$$\mathbf{A}=-\mathbf{r}\times\nabla S+\mathbf{r}T+\nabla U \tag{16}$$

with $\mathbf{r}=r\mathbf{e}_r$ and $\mathbf{e}_r$ being the radial basic unit vector in the coordinate system used, with three scalar functions $S$, $T$, and $U$. The representation (16) applies to arbitrary vector fields; it was used, e.g., in [15] and is explained in some detail in Appendix B. When working with this representation it is useful to know vector relations like

$$\nabla\times(\mathbf{r}F)=-\mathbf{r}\times\nabla F, \tag{17}$$

$$\nabla\times[\nabla\times(\mathbf{r}F)]=-\nabla\times(\mathbf{r}\times\nabla F)$$
$$=-\mathcal{D}F\mathbf{r}+\nabla\left(\frac{1}{h}\frac{\partial(rF)}{\partial r}\right), \tag{18}$$

$$\nabla\times\{\nabla\times[\nabla\times(\mathbf{r}F)]\}=-\nabla\times[\nabla\times(\mathbf{r}\times\nabla F)]=\mathbf{r}\times\nabla\mathcal{D}F, \tag{19}$$

$$\mathbf{r}\times(\mathbf{r}\times\nabla F)=\frac{1}{rh}\frac{\partial}{\partial r}(r^2F)\mathbf{r}-\nabla(r^2F), \tag{20}$$

where

$$\mathcal{D}F=\Delta F+\frac{1}{2r}\frac{d}{dr}\left(\frac{1}{h^2}\right)F=\frac{1}{rh}\frac{\partial}{\partial r}\left(\frac{1}{h}\frac{\partial}{\partial r}(rF)\right)+\frac{LF}{r^2}. \tag{21}$$

$F$ is an arbitrary but suitably differentiable scalar field, $\Delta$ the usual three-dimensional Laplacian and $r^{-2}L$ the two-dimensional Laplacian on a surface $r=$const. As a consequence of Eq. (18) we have

$$\mathbf{r}\cdot[\nabla\times(\mathbf{r}\times\nabla F)]=LF. \tag{22}$$

We note that $LF=0$ on a surface $r=$const implies that $F$ is independent of $\theta$ and $\phi$.

Using Eqs. (15) and (16) we now write

$$\mathbf{B}=\mathbf{B}^P+\mathbf{B}^T \tag{23}$$

with two fields $\mathbf{B}^P$ and $\mathbf{B}^T$, which we call "poloidal" and "toroidal," given by

$$\mathbf{B}^P=-\nabla\times(\mathbf{r}\times\nabla S),\quad \mathbf{B}^T=-\mathbf{r}\times\nabla T. \tag{24}$$

With the help of Eq. (22) we conclude from Eqs. (23) and (24) that

$$LS=-\mathbf{r}\cdot\mathbf{B},\quad LT=-\mathbf{r}\cdot(\nabla\times\mathbf{B}). \tag{25}$$

These equations determine $S$ and $T$ for any given $\mathbf{B}$ with the exception of parts independent of $\theta$ and $\phi$. Such terms, however, are without influence on $\mathbf{B}^P$ and $\mathbf{B}^T$; see Eq. (24). Consequently, $\mathbf{B}^P$ and $\mathbf{B}^T$ are uniquely defined by the requirement that they allow representations in the form of Eq. (24). For later use we give them also in the more explicit form

$$\mathbf{B}^P=-\frac{LS}{r}\mathbf{e}_r+\frac{1}{rh}\left[\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial\theta}\right)\mathbf{e}_\theta+\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial\phi}\right)\mathbf{e}_\phi\right], \tag{26}$$

$$\mathbf{B}^T=\frac{1}{\sin\theta}\frac{\partial T}{\partial\phi}\mathbf{e}_\theta-\frac{\partial T}{\partial\theta}\mathbf{e}_\phi$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ are the basic orthonormal vectors associated with the coordinate system.

Our definition of poloidal and toroidal fields implies properties of these fields, a few of which are listed here.

(i) If $\mathbf{B}=\mathbf{0}$ on a surface $r=$const then $\mathbf{B}^P=\mathbf{B}^T=\mathbf{0}$ on this surface,

(ii) both $\mathbf{B}^P$ and $\mathbf{B}^T$ are solenoidal, that is, $\nabla\cdot\mathbf{B}^P=\nabla\cdot\mathbf{B}^T=0$,

(iii) if $f$ is a scalar independent of $\theta$ and $\phi$ then $\nabla\times(f\mathbf{B}^P)$ is toroidal and $\nabla\times(f\mathbf{B}^T)$ is poloidal,

(iv) if $\mathbf{r}\cdot(\nabla\times\mathbf{B}^T)=0$ on a surface $r=$const then $\mathbf{B}^T=\mathbf{0}$ on this surface,

(v) $\mathbf{B}^P$ and $\mathbf{B}^T$ are orthogonal to each other in the sense of $\langle\mathbf{B}^P\cdot\mathbf{B}^T\rangle=0$ where $\langle\cdots\rangle$ means averaging over the solid angle, that is, $\langle\cdots\rangle=(1/4\pi)\int_0^{2\pi}\int_0^{\pi}\cdots\sin\theta\,d\theta\,d\phi$.

To explain statement (i) we refer to the uniqueness of the decomposition of $\mathbf{B}$. Statement (ii) applies since both $\mathbf{B}^P$ and $\mathbf{B}^T$ are defined as curls. Concerning statement (iii) we note that $\nabla\times(f\mathbf{B}^P)$ can be written with the help of Eqs. (18) and (24) in the form $\mathbf{r}\times\nabla\cdots$, and $\nabla\times(f\mathbf{B}^T)$ with the help of Eq. (24) in the form $\nabla\times(\mathbf{r}\times\nabla\cdots)$. As far as statement (iv) is concerned we conclude from Eq. (25) that $LT=0$. Hence $T$ is independent of $\theta$ and $\phi$ and therefore $\mathbf{B}^T=\mathbf{0}$. Finally, statement (v) can easily be verified by expressing the integrand in $\langle\mathbf{B}^P\cdot\mathbf{B}^T\rangle$ with the help of Eq. (26) by derivatives of $S$ and $T$, carrying out integrations by parts, and considering that because of the regularity of $\mathbf{B}^P$, which has to be required, $\partial S/\partial\phi$ has to vanish where $\sin\theta$ vanishes.

It is often useful to remove the ambiguity of $S$ and $T$. This can be done by requiring

$$\langle S\rangle=\langle T\rangle=0. \tag{27}$$

In this context it is of interest that $\langle LF\rangle=0$ for any scalar $F$. As a consequence we have, for example, $\langle\Delta F\rangle=\langle\mathcal{D}F\rangle=0$ as soon as $\langle F\rangle=0$.

## V. REDUCTIONS OF THE BASIC EQUATIONS

We return now to our basic equations (11)–(14) for the magnetic field $\mathbf{B}$ and represent it according to Eq. (23) as a sum of its poloidal and toroidal parts $\mathbf{B}^P$ and $\mathbf{B}^T$ defined by Eq. (24). Using the properties formulated in the statements (i)–(iv) we can easily reduce Eqs. (11) to

$$\frac{\partial\mathbf{B}^P}{\partial t}+\nabla\times[\eta\nabla\times(Z\mathbf{B}^P)]=\mathbf{0}\quad\text{in}\quad r<R, \tag{28a}$$

$$\frac{\partial\mathbf{B}^T}{\partial t}+\nabla\times[\eta\nabla\times(Z\mathbf{B}^T)]=\mathbf{0}\quad\text{in}\quad r<R, \tag{28b}$$

Eqs. (12) to

$$\nabla\times(Z\mathbf{B}^P)=\mathbf{0}, \quad \mathbf{B}^T=\mathbf{0} \quad \text{in} \quad r>R, \tag{29}$$

the conditions (13) to

$$[\mathbf{B}^P]=\mathbf{0} \quad \text{across} \quad r=R, \quad \mathbf{B}^T=\mathbf{0} \quad \text{at} \quad r=R, \tag{30}$$

and the condition (14) to

$$\mathbf{B}^P\to\mathbf{0} \quad \text{as } r\to\infty. \tag{31}$$

The problem for $\mathbf{B}$ formulated with Eqs. (11)–(14) clearly splits into two independent problems for $\mathbf{B}^P$ and $\mathbf{B}^T$. By the way, a dependence of $\eta$ on $\theta$ or $\phi$ would lead to a coupling between $\mathbf{B}^P$ and $\mathbf{B}^T$.

Let us first deal with the problem for $\mathbf{B}^P$. Expressing $\mathbf{B}^P$ according to Eq. (24) by $S$ and using Eq. (18) we can write Eq. (28a) in the form

$$\nabla\times\left\{\mathbf{r}\times\nabla\left[\frac{\partial S}{\partial t}-\eta\left(Z\mathcal{D}S+\frac{1}{rh^2}\frac{dZ}{dr}\frac{\partial}{\partial r}(rS)\right)\right]\right\}=\mathbf{0}. \tag{32}$$

According to Eq. (22) this implies $L[\cdots]=0$, that is, $[\cdots]$ does not depend on $\theta$ and $\phi$. Using Eq. (27) we find $[\cdots]=0$, that is,

$$\frac{\partial S}{\partial t}-\eta\left[\frac{1}{rh}\frac{\partial}{\partial r}\left(\frac{Z}{h}\frac{\partial}{\partial r}(rS)\right)+\frac{Z}{r^2}LS\right]=0 \quad \text{in} \quad r<R. \tag{33}$$

In an analogous way the first equation (29) can be reduced to an equation for $S$ in $r>R$. The Schwarzschild solution of Einstein's equations implies that there $Z=h^{-1}$. So the result can be simplified to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{h^2}\frac{\partial}{\partial r}(rS)\right)+\frac{1}{r^2}LS=0 \quad \text{in} \quad r>R. \tag{34}$$

A look at Eq. (26) shows that the condition (30) is satisfied if

$$[S]=\left[\frac{\partial S}{\partial r}\right]=0 \quad \text{across} \quad r=R. \tag{35}$$

Finally, Eq. (31) requires that

$$\frac{LS}{r},\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial\theta}\right),\ \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial\phi}\right)\ \to 0 \quad \text{as} \quad r\to\infty. \tag{36}$$

It commends itself to expand $S$ in a series of spherical harmonics. More precisely, we consider $S$ as a sum of terms $S_l^m(r,t)Y_l^m(\theta,\phi)$, with the $Y_l^m$ being the familiar spherical harmonics in their real form. The summation runs over all $l$ and $m$ satisfying $l\geqslant 1$ and $|m|\leqslant l$; because of Eq. (27) there is no contribution with $l=0$. We recall that the $Y_l^m$ constitute a complete orthogonal set of functions on any spherical surface and satisfy $LY_l^m=-l(l+1)Y_l^m$. It can easily be seen that the system (33)–(36) implies no couplings between terms differing in $l$ or $m$. So we may put, without any loss of generality,

$$S=S_l^m(r,t)Y_l^m(\theta,\phi). \tag{37}$$

As we will explain later in more detail, $l=1$ corresponds to magnetic-field modes of dipole type, $l=2$ to those of quadrupole type, $l=3$ to those of octupole type, etc.

From Eqs. (33)–(36) together with Eq. (37) we obtain

$$\frac{\partial S_l^m}{\partial t}-\eta\left[\frac{1}{rh}\frac{\partial}{\partial r}\left(\frac{Z}{h}\frac{\partial}{\partial r}(rS_l^m)\right)-\frac{Zl(l+1)}{r^2}S_l^m\right]=0 \quad \text{in} \quad r<R, \tag{38}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{h^2}\frac{\partial}{\partial r}(rS_l^m)\right)-\frac{l(l+1)}{r^2}S_l^m=0 \quad \text{in} \quad r>R, \tag{39}$$

$$[S_l^m]=\left[\frac{\partial S_l^m}{\partial r}\right]=0 \quad \text{across} \quad r=R, \tag{40}$$

$$S_l^m,\ \frac{\partial S_l^m}{\partial r}\to 0 \quad \text{as} \quad r\to\infty. \tag{41}$$

Clearly these equations for $S_l^m$ do not contain $m$. Thus the $S_l^m$ for a given $l$ but different $m$ differ only by factors independent of $r$, that is,

$$S_l^m=C_l^m S_l \tag{42}$$

with $C_l^m$ independent of $r$. Of course, Eqs. (38)–(41) apply also with $S_l^m$ replaced by $S_l$.

As we will explain later in more detail, solutions $S_l^m$ or $S_l$ of Eq. (39) satisfying Eq. (41) are known. Since such a solution is determined only up to a factor independent of $r$ we write $S_l=C_l\tilde{S}_l$ with $C_l$ being such a factor and $\tilde{S}_l$ an arbitrary special solution of Eq. (39) satisfying Eq. (41), for example, that with $\tilde{S}_l=1$ at $r=R$. The remaining problem consists then in finding solutions of Eq. (38) that satisfy the conditions (40). We write these conditions now in the form

$$S_l|_{r=R-0}=C_l\tilde{S}_l|_{r=R+0}, \quad \left.\frac{\partial S_l}{\partial r}\right|_{r=R-0}=C_l\left.\frac{\partial\tilde{S}_l}{\partial r}\right|_{r=R+0}. \tag{43}$$

Eliminating $C_l$ we obtain a connection between $S_l|_{r=R-0}$ and $(\partial S_l/\partial r)|_{r=R-0}$. For reasons that will become clear later we write it in the form

$$\frac{\partial S_l}{\partial r}+(l+1)\frac{f_l}{R}S_l=0 \quad \text{at} \quad r=R \tag{44}$$

with a factor $f_l$ determined by $\tilde{S}_l$,

$$(l+1)\frac{f_l}{R}=-\left.\frac{\partial\tilde{S}_l}{\partial r}\right|_{r=R+0}\Big/\tilde{S}_l|_{r=R+0}. \tag{45}$$

Thus the problem for $\mathbf{B}^P$ defined by Eqs. (28)–(31) is reduced to that of finding solutions of Eq. (38) that satisfy the condition (44) with a factor $f_l$, the value of which we will determine soon. Before doing so, however, we consider the problem for $\mathbf{B}^T$ defined by Eqs. (28)–(31). Proceeding as in the case of $\mathbf{B}^P$ and using, in particular, Eqs. (24) and (27), we find first

$$\frac{\partial T}{\partial t} - \frac{1}{rh}\frac{\partial}{\partial r}\left(\frac{\eta}{h}\frac{\partial}{\partial r}(rZT)\right) - \frac{\eta Z}{r^2}LT = 0 \quad \text{in} \quad r<R \tag{46}$$

and

$$T=0 \quad \text{for} \quad r \geqslant R. \tag{47}$$

With

$$T = T_l^m(r,t)Y_l^m(\theta,\phi) \tag{48}$$

we further obtain

$$\frac{\partial T_l^m}{\partial t} - \frac{1}{rh}\frac{\partial}{\partial r}\left(\frac{\eta}{h}\frac{\partial}{\partial r}(rZT_l^m)\right) + \frac{\eta Z l(l+1)}{r^2}T_l^m = 0 \quad \text{in} \quad r<R, \tag{49}$$

$$T_l^m = 0 \quad \text{for} \quad r \geqslant R. \tag{50}$$

In the sense of Eq. (42) we put

$$T_l^m = D_l^m T_l \tag{51}$$

with $D_l^m$ being independent of $r$. The problem for $\mathbf{B}^T$ consists in finding solutions $T_l^m$ or $T_l$ of Eq. (49) that continuously fit to Eq. (50).

## VI. THE SOLUTIONS FOR THE OUTER SPACE

Let us now return to the problem for $\mathbf{B}^P$ and seek $S_l^m$ or $S_l$ for $r>R$, that is, solutions of Eq. (39) satisfying Eq. (41), from which the value of $f_l$ in Eq. (44) can be calculated. The solution for $l=1$ was first given in a closed form by Ginzburg and Osernoy [17]. For arbitrary $l$ the $S_l$ can be gained in a closed form simply by differentiation from the solutions for the magnetic scalar potential given by Anderson and Cohen [18].

Since it is easier to handle numerically we derive here a solution $S_l$ of Eq. (39) in the form of a series of powers of $1/r$. For this purpose we first put $S_l = s_l/r$ and $1/h^2 = 1 - 2M/r$, where $M$ stands again for $M(R)$. Then Eq. (39) turns into

$$\frac{d}{dr}\left[\left(1-\frac{2M}{r}\right)\frac{ds_l}{dr}\right] - \frac{l(l+1)}{r^2}s_l = 0 \tag{52}$$

or, with $2M/r = \xi$, into

$$\frac{d}{d\xi}\left((1-\xi)\xi^2\frac{ds_l}{d\xi}\right) - l(l+1)s_l = 0. \tag{53}$$

It can easily be checked that

$$s_l = c_l \xi^l \sum_{\nu=0}^{\infty} a_\nu \xi^\nu \tag{54}$$

with an arbitrary factor $c_l$ independent of $\xi$ is a solution of Eq. (53) if

$$a_0 = 1, \quad a_\nu = \frac{(l+\nu)^2-1}{(2l+\nu+1)\nu}a_{\nu-1} \quad \text{for} \quad \nu \geqslant 1. \tag{55}$$

Hence

$$S_l = \frac{C_l}{r^{l+1}}\sum_{\nu=0}^{\infty} a_\nu \left(\frac{2M}{r}\right)^\nu \tag{56}$$

with another arbitrary factor $C_l$ independent of $r$ and $a_\nu$ as above is a solution of Eq. (39), and it satisfies Eq. (41).

With the result (56) we return now to Eq. (45). A straightforward calculation provides us with

$$f_l = \sum_{\nu=0}^{\infty}\frac{l+\nu+1}{l+1}b_\nu \bigg/ \sum_{\nu=0}^{\infty} b_\nu \tag{57}$$

with

$$b_0 = 1, \quad b_\nu = \frac{(l+\nu)^2-1}{(2l+\nu+1)\nu}\epsilon\, b_{\nu-1} \quad \text{for} \quad \nu \geqslant 1 \tag{58}$$

where $\epsilon$ is the compactness ratio,

$$\epsilon = 2M/R. \tag{59}$$

Clearly we have $f_l = 1$ for $\epsilon = 0$, and $f_l$ grows monotonically with $\epsilon$.

## VII. THE EIGENVALUE PROBLEMS FOR THE INTERIOR OF THE BODY

So far we have reduced the problem for $\mathbf{B}$ posed by Eqs. (11)–(14) to problems for the functions $S_l^m$ and $T_l^m$ depending on $r$ and $t$ defined by Eq. (38) with the boundary condition (44), which has to be completed by Eqs. (57)–(59), and by Eq. (49) with the boundary condition (50).

We assume now that $\eta$ does not depend on $t$. Then we can ask for $\mathbf{B}$ modes, that is, for solutions $S_l^m$ and $T_l^m$ of Eqs. (38) and (49), that vary exponentially with $t$. Before formulating the corresponding problems we introduce a dimensionless magnetic diffusivity $\tilde{\eta}$, a dimensionless radial variable $\zeta$, and a dimensionless time variable $\tau$ by

$$\eta = \tilde{\eta}\,\eta_c, \quad r = \zeta R, \quad t = \tau T_\eta, \quad T_\eta = R^2/\eta_c, \tag{60}$$

where $\eta_c$ is a characteristic value of $\eta$; we choose $\eta_c = \eta$ if $\eta$ is constant. $T_\eta$ will prove to be a characteristic time for the decay of a magnetic field with a characteristic length scale $R$. We now put

$$S_l = \hat{S}_l(\zeta)\exp(-\lambda_l^P\tau), \quad T_l = \hat{T}_l(\zeta)\exp(-\lambda_l^T\tau) \tag{61}$$

with dimensionless decay rates $\lambda_l^P$ and $\lambda_l^T$. Like $S_l$ and $T_l$, $\hat{S}_l(\zeta)$ and $\hat{T}_l(\zeta)$ as well as $\lambda_l^P$ and $\lambda_l^T$ are independent of $m$. Then Eq. (38) for $S_l^m$ turns into

$$\tilde{\eta}\left[\frac{1}{\zeta h}\frac{d}{d\zeta}\left(\frac{Z}{h}\frac{d}{d\zeta}(\zeta\hat{S}_l)\right)-\frac{Zl(l+1)}{\zeta^2}\hat{S}_l\right]+\lambda_l^P\hat{S}_l=0 \quad \text{in} \quad \zeta<1, \tag{62}$$

or, which is the same,

$$\frac{d}{d\zeta}\left(\zeta^2\frac{Z}{h}\frac{d\hat{S}_l}{d\zeta}\right)-\left[Zhl(l+1)-\zeta\frac{d}{d\zeta}\left(\frac{Z}{h}\right)\right]\hat{S}_l+\lambda_l^P\zeta^2\frac{h}{\tilde{\eta}}\hat{S}_l=0 \quad \text{in} \quad \zeta<1. \tag{63}$$

Likewise, Eq. (49) for $T_l^m$ turns into

$$\frac{1}{\zeta h}\frac{d}{d\zeta}\left(\frac{\tilde{\eta}}{h}\frac{d}{d\zeta}(\zeta Z\hat{T}_l)\right)-\frac{\tilde{\eta}Zl(l+1)}{\zeta^2}\hat{T}_l+\lambda_l^T\hat{T}_l=0 \quad \text{in} \quad \zeta<1, \tag{64}$$

or

$$\frac{d}{d\zeta}\left(\zeta^2\frac{\tilde{\eta}}{h}\frac{dZ\hat{T}_l}{d\zeta}\right)-\left[\tilde{\eta}hl(l+1)-\zeta\frac{d}{d\zeta}\left(\frac{\tilde{\eta}}{h}\right)\right]Z\hat{T}_l+\lambda_l^T\zeta^2\frac{h}{Z}Z\hat{T}_l=0 \quad \text{in} \quad \zeta<1. \tag{65}$$

If we exclude singularities of $S_l$ and $T_l$ at $\zeta=0$, which would contradict our requirement of finite magnetic energy density, we may conclude from Eqs. (62) and (64) that

$$\hat{S}_l=O(\zeta^l), \quad \hat{T}_l=O(\zeta^l) \quad \text{for} \quad \zeta\to 0. \tag{66}$$

This coincides with the conditions resulting from the requirement that $\mathbf{B}^P$ and $\mathbf{B}^T$, or $S$ and $T$, behave regularly at $r=0$. Equation (63) for $\hat{S}_l$ together with the boundary conditions

$$\hat{S}_l=0 \quad \text{at} \quad \zeta=0, \quad \frac{d\hat{S}_l}{d\zeta}+(l+1)f_l\hat{S}_l=0 \quad \text{at} \quad \zeta=1 \tag{67}$$

that result from Eqs. (66) and (44), pose an eigenvalue problem of Sturm-Liouville type with the eigenvalue parameter $\lambda_l^P$ [19]. Unfortunately, since the coefficient $\zeta^2 Z/h$ occurring with the derivative of $\hat{S}_l$ in Eq. (63) is zero at $\zeta=0$, the problem has to be classified as singular. For the flat case, that is, $Z=h=1$, the solutions of Eq. (63) are well known. They are spherical Bessel functions; see also Sec. IX. Despite the singularity we can then easily conclude from known theorems [23] that there is a countable set of discrete single eigenvalues, which we denote by $\lambda_{ln}^P$, $n=1,2,3,\ldots$, so that $\lambda_{l1}^P<\lambda_{l2}^P<\lambda_{l3}^P<\cdots$, and that the corresponding eigenfunctions $\hat{S}_{ln}$ constitute a complete set of functions in the sense that any function satisfying the boundary conditions (67) can be represented as a series of these functions. In the general case the situation is more complex insofar as we cannot rely on known solutions of Eq. (63). Nevertheless, the above statements on eigenvalues and eigenfunctions can be proved, too [24]. Multiplying both sides of Eq. (62) by $\zeta^2 h\hat{S}_l/\tilde{\eta}$, integrating over all $\zeta$ in $0\leqslant\zeta\leqslant 1$, and considering Eq. (67) we can easily conclude that

$$\lambda_l^P\int_0^1\hat{S}_l^2\frac{h}{\tilde{\eta}}\zeta^2\,d\zeta=\int_0^1\left(\frac{d(\zeta\hat{S}_l)}{d\zeta}\right)^2\frac{Z}{h}d\zeta+l(l+1)\int_0^1\hat{S}_l^2\,Zh\,d\zeta+[(l+1)f_l-1]\left(\hat{S}_l^2\frac{Z}{h}\right)_{\zeta=1}. \tag{68}$$

Since $f_l\geqslant 1$ this means that all $\lambda_{ln}^P$ are positive, $0<\lambda_{l1}^P<\lambda_{l2}^P<\lambda_{l3}^P<\cdots$. The $\hat{S}_{ln}$ are orthogonal in the sense that $\int_0^1\hat{S}_{ln}\hat{S}_{ln'}(h/\tilde{\eta})\,\zeta^2\,d\zeta$ is nonzero only if $n=n'$.

Equation (65) can be understood as an equation for $\overline{T}_l=Z\hat{T}_l$. Together with the boundary conditions

$$\overline{T}_l=0 \quad \text{at} \quad \zeta=0 \quad \text{and} \quad \text{at} \quad \zeta=1 \tag{69}$$

resulting from Eqs. (66) and (50), it poses again an eigenvalue problem of Sturm-Liouville type with the eigenvalue parameter $\lambda_l^T$. The above statements about the $\lambda_{ln}^P$ apply to the $\lambda_{ln}^T$, too, and like the $\hat{S}_{nl}$ the eigenfunctions $\overline{T}_{nl}$ as well as the $\hat{T}_{nl}$ defined by $\overline{T}_{nl}=Z\hat{T}_{nl}$ constitute a complete set of functions which allows us to represent functions satisfying boundary conditions of the type (69). Multiplying both sides of Eq. (64) by $\zeta^2 Zh\hat{T}_l$, integrating over $\zeta$, and considering Eq. (69) we find

$$\lambda_l^T\int_0^1\hat{T}_l^2\,Zh\zeta^2 d\zeta=\int_0^1\left(\frac{d}{d\zeta}(\zeta Z\hat{T}_l)\right)^2\frac{\tilde{\eta}}{h}d\zeta+l(l+1)\int_0^1(Z\hat{T}_l)^2\,\tilde{\eta}h\,d\zeta \tag{70}$$

and conclude that all $\lambda_{ln}^T$ are positive, $0<\lambda_{l1}^T<\lambda_{l2}^T<\lambda_{l3}^T<\cdots$. The $\hat{T}_l$ are orthogonal in the sense that $\int_0^1\hat{T}_{ln}\hat{T}_{ln'}Z h\zeta^2\,d\zeta$ is nonzero only if $n=n'$.

## VIII. THE FREE-DECAY MODES AND THE GENERAL SOLUTION OF THE FREE-DECAY PROBLEM

Let us now return to the original problem for the magnetic field $\mathbf{B}$ defined by Eqs. (11)–(14) and formulate our results on this level. For this purpose we first define magnetic fields $\mathbf{B}_{ln}^{Am}(x^k)$ depending on the space coordinates only, where $A$ stands for $P$ or $T$, by

$$\mathbf{B}_{ln}^{Pm}(x^k)=-\nabla\times\{\mathbf{r}\times\nabla[\hat{S}_l(r)Y_l^m(\theta,\phi)]\}, \tag{71}$$

$$\mathbf{B}_{ln}^{Tm}(x^k)=-\mathbf{r}\times\nabla[\hat{T}_l(r)Y_l^m(\theta,\phi)] \tag{72}$$

where $\hat{S}_l(r)$ and $\hat{T}_l(r)$ mean $\hat{S}_l(\zeta)$ and $\hat{T}_l(\zeta)$ with $\zeta$ replaced by $r/R$. Then the modes

$$\mathbf{B}(x^k,t) = \mathbf{B}^{Am}_{ln}(x^k)\exp(-\lambda^A_{ln} t/T_\eta) \tag{73}$$

are solutions of Eqs. (11)–(14).

For each given $A$, $l$, and $n$ there are $2l+1$ modes differing in $m$. The poloidal modes, $A=P$, with $l=1$ are dipole fields, those with $l=2$ quadrupole fields, those with $l=3$ octupole fields, etc. We rely here on a definition of the multipole nature of poloidal fields on the basis of their variation with the angles $\theta$ and $\phi$ and not that with $r$. As can be seen from Eq. (56) in contrast to the situation in flat space, a dipole field in that sense outside the star does not vary with $r$ simply as $1/r^3$, a quadrupole or octupole field not simply as $1/r^4$ or $1/r^5$, etc.

Among the three dipole modes, $l=1$, belonging to a given $n$ each one can be generated by rotating one of the others. Likewise, each of the three toroidal modes, $A=T$, with $l=1$ and a given $n$ can be obtained by rotating one of the others. This applies no longer, however, for poloidal or toroidal modes with $l>1$. Then among the $2l+1$ modes for given $n$ and $l$ are some that differ in their geometrical structures so that it is no longer possible to bring them to coincidence by rotating one of them. In these cases, of course, the coincidence of the $\lambda^A_{ln}$ for different $m$ is by no means trivial. For given $A$, $l$, and $m$ the radial variation of the magnetic field in the interior of the body, $r<R$, becomes more complex with growing $n$. In outer space, $r>R$, poloidal modes with given $l$ and $m$ but different $n$ have the same structure, that is, they differ there only by factors independent of the space coordinates.

We simplify the notation by writing $\mathbf{B}_i$ instead of $\mathbf{B}^{Am}_{ln}$ and $\lambda_i$ instead of $\lambda^A_{ln}$ where $i$ is a collective index covering $A$, $l$, $m$, and $n$. Of course the $\lambda_i$ are the same for all $i$ that agree in $A$, $l$, and $n$ but differ in $m$. With this notation Eq. (73) reads simply $\mathbf{B}(x^k,t)=\mathbf{B}_i(x^k)\exp(-\lambda_i t/T_\eta)$. Together with Eq. (73), every superposition of such modes

$$\mathbf{B}(x^k,t) = \sum_i c_i \mathbf{B}_i(x^k)\exp(-\lambda_i t/T_\eta) \tag{74}$$

with arbitrary constants $c_i$ is a solution of Eqs. (11)–(14).

According to our construction the $\mathbf{B}_i$ form a complete set of vector functions which allows us to represent vector fields $\mathbf{B}$, which are arbitrarily defined but solenoidal in $r<R$ and continue according to Eqs. (12)–(14) in $r \geqslant R$. We recall here the completeness properties of the $\hat{S}_l$ and $\hat{T}_l$ discussed above and the completeness properties of the $Y^m_l$. Using the orthogonality properties of poloidal and toroidal fields, of the $\hat{S}_{ln}$ and $\hat{T}_{ln}$, and of the $Y^m_l$, it can be shown that the $\mathbf{B}_i$ are orthogonal in the sense of

$$\int_\infty \mathbf{B}_i(x^k)\cdot\mathbf{B}_j(x^k)\, Z\, dv = N_i^2\, \delta_{ij}, \tag{75}$$

where the integral is over all space, $\int_\infty \cdots dv = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\cdots h\, r^2 \sin\theta\, dr\, d\theta\, d\phi$. Hints concerning the proof of Eq. (75) are given in Appendix D. Of course the $\mathbf{B}_i$ can be normalized so that $N_i=1$.

Specifying Eq. (74) to the initial time $t=0$ we have

$$\mathbf{B}(x^k,0) = \sum_i c_i \mathbf{B}_i(x^k). \tag{76}$$

Because of the completeness of the $\mathbf{B}_i$ any $\mathbf{B}(x^k,0)$ satisfying Eqs. (12)–(14) can be represented in the form (76). With Eq. (75) we obtain from Eq. (76) that

$$c_i = \frac{1}{N_i}\int_\infty \mathbf{B}(x^k,0)\cdot\mathbf{B}_i(x^k)\, Z\, dv. \tag{77}$$

Equation (74) together with Eq. (77) is the solution of the problem defined by Eqs. (11)–(14) for an arbitrary initial field $\mathbf{B}(x^k,0)$.

In the limit of large $t$ only the contributions of the modes $\mathbf{B}_i$ to the sum (74) with the smallest values of $\lambda_i$ are of interest. That is why we will pay particular attention to such modes.

Obviously, the validity of Eq. (75) allows us to assign suitably defined energies to the individual modes $c_i\mathbf{B}_i(x^k)\exp(-\lambda_i t/T_\eta)$ the sum of which is equal to the correspondingly defined total energy of the field (74). The energy definition suitable for this to hold is just the one of *redshifted* energy $(\int_\infty \mathbf{B}^2\, Z\, dv)/8\pi$, that is, the redshifted energies of the modes behave additively. According to this we give all energy quantities including local energy densities, in the following, as redshifted quantities.

## IX. SPECIFIC RESULTS FOR A CONSTANT-DENSITY STAR

We restrict our attention now to a star with constant mass density and constant magnetic diffusivity. So we specify the eigenvalue problems formulated by Eqs. (62)–(65) by

$$h=(1-\epsilon\zeta^2)^{-1/2}, \quad Z=\frac{3}{2}(1-\epsilon)^{1/2}-\frac{1}{2}(1-\epsilon\zeta^2)^{1/2},$$

$$\tilde{\eta}=1. \tag{78}$$

The form of the scale factor $h$ is a direct consequence of Eq. (10). As for the redshift factor $Z$ we refer to general representations [11]. Assuming that the geometry is nonsingular we have to require that $Z>0$, which implies that the compactness ratio $\epsilon$ is constrained by $0 \leqslant \epsilon < \frac{8}{9}$.

The special case $\epsilon=0$, that is, $h=Z=1$, corresponds to flat space. In this case the eigenvalue problems formulated here can be solved analytically; see, e.g., [2] or [3]. For the eigenfunctions $\hat{S}_{ln}$ and $\hat{T}_{ln}$ we have then $\hat{S}_{ln} = C_{ln} j_l(z_{l-1,n}\zeta)$ and $\hat{T}_{ln}=D_{ln} j_l(z_{ln}\zeta)$ where $C_{ln}$ and $D_{ln}$ are arbitrary constants, $j_l$ spherical Bessel functions of the first kind, and $z_{ln}$ their zeros, $j_l(z_{ln})=0$, ordered according to $0<z_{l1}<z_{l2}<\cdots$. The eigenvalues are simply given by $\lambda^P_{ln}=z^2_{l-1,n}$ and $\lambda^T_{ln}=z^2_{ln}$. Some values of $\lambda^P_{ln}$ and $\lambda^T_{ln}$ are given in Table I.

TABLE I. Dimensionless decay rates $\lambda_{ln}^P$ and $\lambda_{ln}^T$ of poloidal and toroidal magnetic modes for the flat space, that is, $\epsilon=0$.

| $n$ | $\lambda_{1n}^P$ | $\lambda_{2n}^P=\lambda_{1n}^T$ | $\lambda_{3n}^P=\lambda_{2n}^T$ | $\lambda_{4n}^P=\lambda_{3n}^T$ |
|---|---|---|---|---|
| 1 | $\pi^2$ | 20.1907 | 33.2175 | 48.8312 |
| 2 | $4\pi^2$ | 59.6795 | 82.7192 | 108.5164 |
| 3 | $9\pi^2$ | 118.8999 | 151.8549 | 187.6358 |

A numerical procedure has been developed for the determination of the $\hat{S}_{ln}$ and $\hat{T}_{ln}$ and the $\lambda_{ln}^P$ and $\lambda_{ln}^T$ for arbitrary $\epsilon$ satisfying $0\leq\epsilon<\frac{8}{9}$. A few results for $\lambda_{ln}^P$ and $\lambda_{ln}^T$ are depicted in Figs. 1 and 2. We recall that all statements on variations with the time $t$ and so the values of $\lambda_{ln}^P$ and $\lambda_{ln}^T$ given there refer to an observer at infinity.

Clearly all decay rates decrease with growing compactness ratio $\epsilon$. This can be understood as a consequence of two effects of the spacetime curvature acting in the same direction. First, for an observer at a point inside the body the time runs slower compared to the time $t$ that would be measured by the clock of an observer at infinity. The decay of the magnetic field in the neighborhood of this point inside the body proceeds, of course, according to this local time so that it occurs delayed for the observer at infinity. Secondly, an observer inside the body sees a larger distance between two points in his neighborhood given by their coordinates $r$, $\theta$, and $\phi$ than an observer at infinity. Since the time scale for a change of magnetic structure is proportional to the square of its length scale, this corresponds again to a delay of the magnetic-field decay. Investigating the slowest-decaying dipole mode, Geppert *et al.* [9] found that in this case the first effect is bigger than the second one.

Independently of the compactness ratio $\epsilon$ the smallest decay rate occurs with the poloidal modes with $l=n=1$, that is, with the simplest dipole modes. Sorted according to growing decay rates the next are the toroidal modes with $l=n=1$, that is, the simplest modes with beltlike field structures, and the poloidal modes with $l=2$ and $n=1$, the simplest quadrupole modes. For $\epsilon=0$ the decay rates of these latter modes coincide but for $\epsilon>0$ those of the toroidal modes are smaller than those of the poloidal ones. Table II gives the decay rates for the three groups of modes mentioned for several values of $\epsilon$. As indicated by crossings of lines in Figs. 1 and 2 the sequence of the higher modes depends on $\epsilon$.

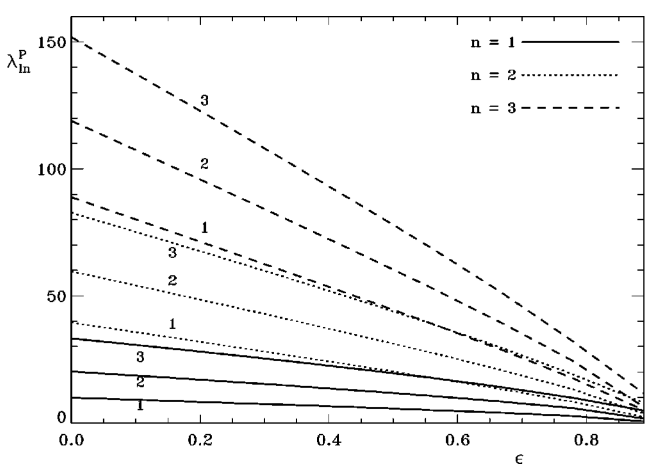


FIG. 1. Dependence of the dimensionless decay rates $\lambda_{ln}^P$ of poloidal magnetic modes on the compactness ratio $\epsilon$. The numbers on the curves give $l$; the different types of line refer to different $n$.

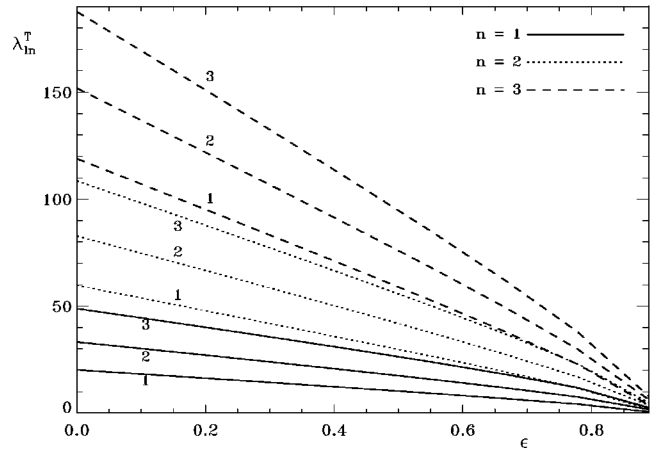


FIG. 2. Same as Fig. 1, but dimensionless decay rates $\lambda_{ln}^T$ of toroidal magnetic modes.

In the evolution of a magnetic field the slowest-decaying modes will dominate after a sufficiently long time. The time needed for reaching a certain dominance is, even if we measure it in units of the decay time of the slowest mode, longer for higher $\epsilon$. More precisely, when starting from a given mixture of modes at some initial time and comparing their magnitudes, for example, after one $e$-folding time of the slowest-decaying mode we find that the dominance of the slowest modes is less pronounced for higher $\epsilon$.

Figure 3 shows magnetic-field lines for some poloidal modes and isolines of the azimuthal field strength in toroidal modes. In Figs. 4–6 the average of the redshifted magnetic energy density $Z\mathbf{B}^2/8\pi$ over surfaces $r=$const is depicted for a few modes. These figures show a tendency toward a concentration of the magnetic field in the central parts of the star for large values of the compactness ratio $\epsilon$. We recall here the two effects responsible for decelerating the decay of the magnetic field discussed above. The concentration of the field in the central region of the star becomes plausible if we consider that the time lapse is large just in this region.

Finally, in Fig. 7 the redshifted magnetic energies stored inside and outside the conducting body are shown for poloidal modes with $l=n=1$ in dependence on $\epsilon$. These energies are defined by integrals $\int(Z\mathbf{B}^2)/8\pi\, dv$, with $dv$ again understood as in Eq. (75), over the region covered by the body or the outer space, respectively. However, they are measured

TABLE II. Three lowest dimensionless decay rates $\lambda_{ln}^A$ for several compactness ratios $\epsilon$.

| Modes | $\epsilon=0.0$ | $\epsilon=0.5$ | $\epsilon=0.8$ | $\epsilon=0.888$ |
|---|---|---|---|---|
| $A=P$, $l=1$, $n=1$ | 9.8696 | 5.5975 | 2.4009 | 0.7545 |
| $A=T$, $l=1$, $n=1$ | 20.1907 | 10.2847 | 3.6592 | 0.8196 |
| $A=P$, $l=2$, $n=1$ | 20.1907 | 11.7121 | 5.2039 | 1.8346 |

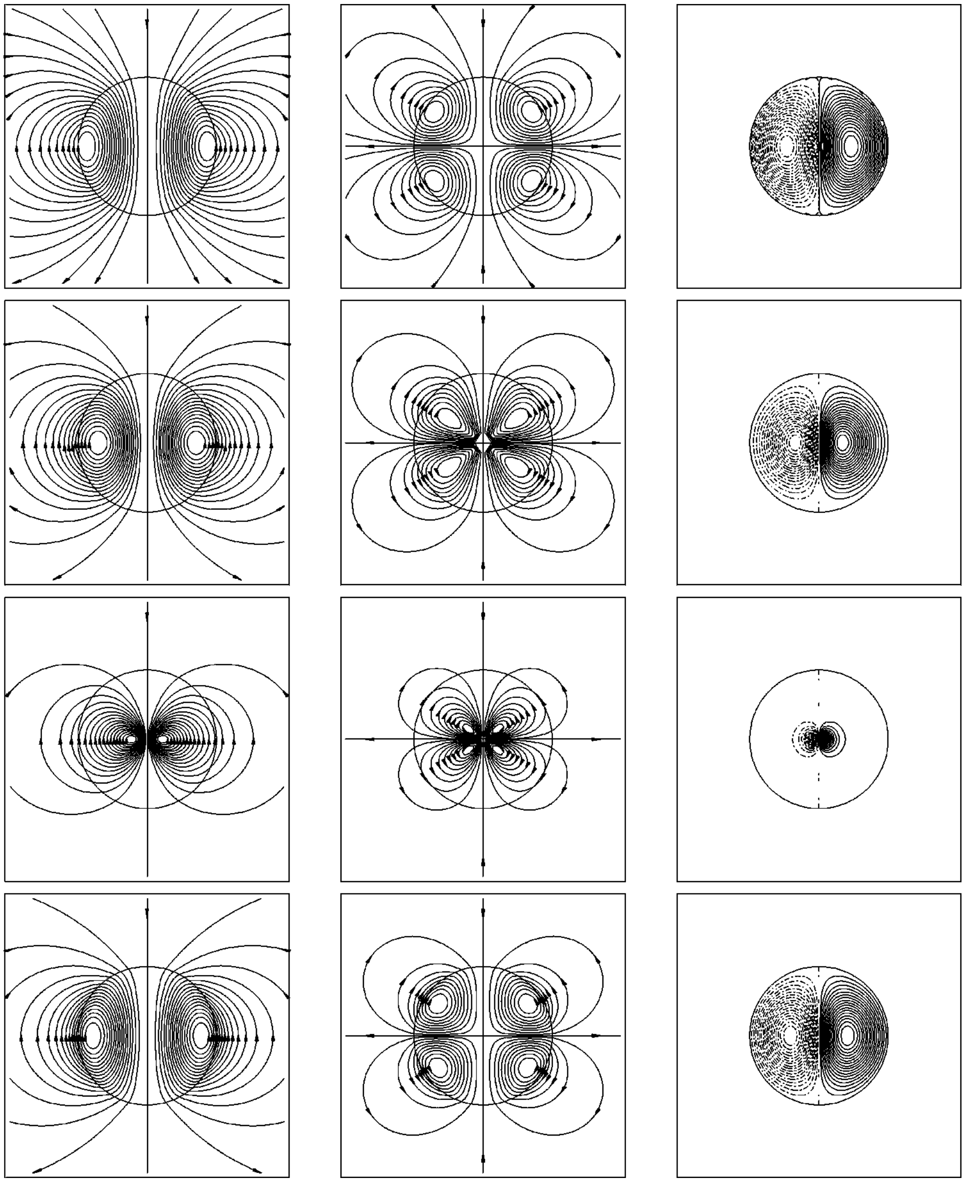

FIG. 3. Magnetic-field lines of the simplest dipole and quadrupole modes (left and middle columns) and isolines of the azimuthal component of the simplest toroidal modes (right column; solid lines, flux out of, and broken lines, flux into the paper plane). First row $\epsilon=0$ (flat space), second row $\epsilon=0.8$, third row $\epsilon=0.888$. In these rows the distance $d(r)=\int_0^r h(r')dr'$ is used as the radial coordinate and the Schwarzschild coordinate $\theta$ as the angle with respect to the vertical axis. Fourth row: same as second row, but with the Schwarzschild coordinate $r$ used as radial coordinate.

here in units of the energy inside the body provided that the energy density is there everywhere equal to its average in the above sense at the surface. The redshifted magnetic energy inside the body is for some range of small $\epsilon$ slightly lower than its value for $\epsilon=0$, but for large $\epsilon$ drastically higher.

## X. DISCUSSION

The results obtained by the above considerations generalize the findings about the effects of general relativity on the decay of a dipolar magnetic field in a nonrotating conducting sphere presented in [9] insofar as the general solution for the free decay as an initial value problem is given. Qualitatively, the compactness of the body affects the spatial structure and the temporal behavior of the smaller-scale field modes in a manner similar to its effect on the fundamental dipolar one.

Let us apply our findings to neutron stars. Of course, this has to be done with care for several reasons. In particular,

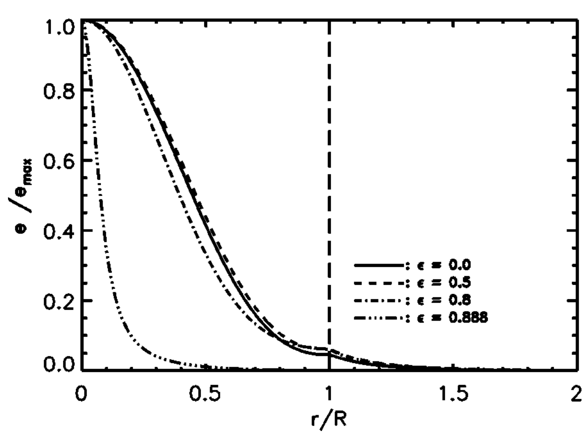


FIG. 4. Redshifted magnetic energy density averaged over surfaces $r=$const, in units of its maximum value, for the dipole modes, $l=1$, with $n=1$ for different $\epsilon$.

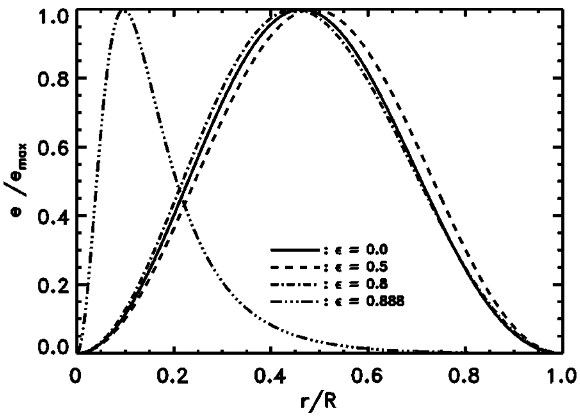


FIG. 5. Same as Fig. 4, but for the toroidal modes with $l = n = 1$.

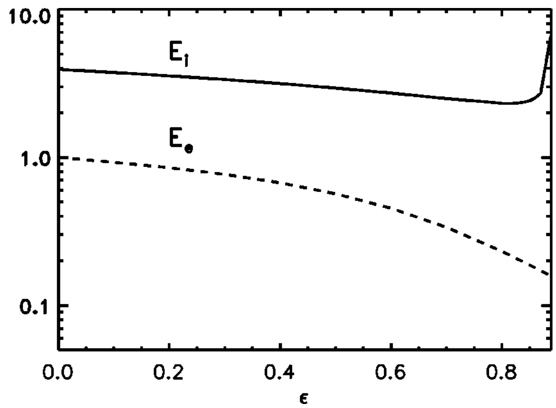


FIG. 7. Redshifted magnetic energies $E_i$ and $E_e$ inside and outside the conducting body in dependence on $\epsilon$. These energies are measured in units of the energy defined as the product of the energy density averaged across the surface and the volume.

any rotation of the body has been ignored in our considerations. In addition the specific results reported in Sec. IX apply only under the problematic assumption of constant density and constant conductivity inside the star.

A reasonable assumption for the electric conductivity of the interior of a neutron star is $\sigma = 10^{25}\ \mathrm{s}^{-1}$, and so for the magnetic diffusivity $\eta = 7.2 \times 10^{-6}\ \mathrm{cm}^2\,\mathrm{s}^{-1}$. Taking for the radius $R = 10$ km we have $T_\eta = 4.4 \times 10^9$ yr. The decay time $T_{\mathrm{decay}}$, understood as $e$-folding time, for the simplest dipole mode is $T_{\mathrm{decay}} = T_\eta / \lambda_{11}^P$. For flat spacetime, $\epsilon = 0$, we have then $T_{\mathrm{decay}} = 4.5 \times 10^8$ yr; for curved spacetime with the compactness ratios $\epsilon = 0.3$ or $\epsilon = 0.5$, however, $T_{\mathrm{decay}} = 5.9 \times 10^8$ yr or $T_{\mathrm{decay}} = 7.9 \times 10^8$ yr, respectively.

When considering stars with equal masses instead of equal radii we get a different picture. If we assume, e.g., $M_* = 1.4 M_\odot$ and a constant conductivity equal to that above, it turns out that with growing compactness the decay times decrease. For $\epsilon = 0.3$ we get $T_{\mathrm{decay}} = 1.2 \times 10^9$ yr and for $\epsilon = 0.5$ only $T_{\mathrm{decay}} = 5.6 \times 10^8$ yr. That is, the acceleration of the decay due to the decrease in the length scales of the field now dominates the decelerating effect of the curvature.

For comparison, decay times have been calculated for the flat space case, too, using radii that result from the total mass and the (constant) density. They read $T_{\mathrm{decay}} = 8.7 \times 10^8$ yr for the density corresponding to $\epsilon = 0.3$ and $T_{\mathrm{decay}} = 3.5 \times 10^8$ yr for that corresponding to $\epsilon = 0.5$. The difference from the values above elucidates the amount of deceleration due to curvature.

When taking into account the growth of the conductivity with growing density, however, it is no longer clear *ab initio* whether the accelerating or decelerating effect is the overwhelming one.

For the generation of magnetic fields self-excited dynamos are of great interest. The dynamo action of fluid motions has been extensively studied in flat spacetime; see, e.g., [21,22]. Our findings on the decelerating effect of the curvature on the decay of magnetic fields suggest that the requirements concerning the intensity of the motions in kinematic dynamo models may be lower compared to the flat case.

An interesting side product of our investigation is the construction of a complete orthogonal set of vector functions $\mathbf{B}_i$ allowing the representation of magnetic fields $\mathbf{B}$ that are arbitrary in $r < R$ and continue in $r \geq R$ according to Eqs. (12)–(14). This might be of importance also beyond the free-decay problem. In the flat-space case, in which this set can be given analytically, it has been used, for example, for the reduction of the induction equation with convection terms to a system of ordinary differential equations for functions of $t$ only, which can be solved numerically. In this way spherical dynamo models have been studied; see, e.g., [8]. Analogous applications in curved-space cases seem possible.

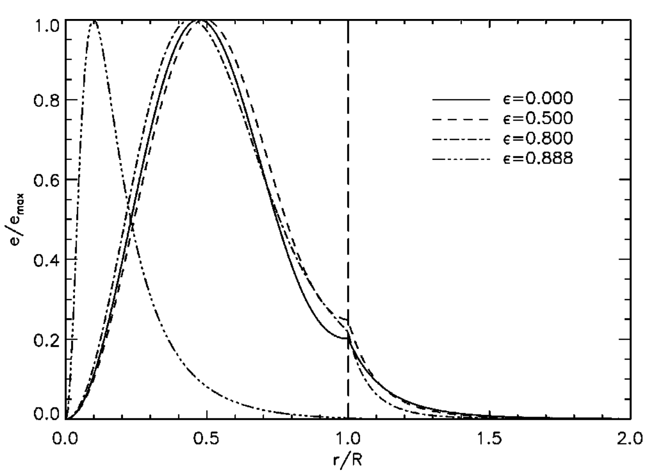


FIG. 6. Same as Fig. 4, but for the quadrupole modes, $l = 2$, with $n = 1$.

## ACKNOWLEDGMENTS

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## APPENDIX A

We give here representations of the three-dimensional vector differential operators with respect to a coordinate system defined by the line element (1) specified to a $t=$const spacelike hypersurface, that is, $ds_3^2=h_1^2(dx^1)^2+h_2^2(dx^2)^2+h_3^2(dx^3)^2$. For arbitrary but suitably differentiable scalar fields $F$ or vector fields $\mathbf{F}$ on a three-dimensional space, we have

$$\nabla F=\frac{1}{h_1}\frac{\partial F}{\partial x^1}\mathbf{e}_1+\frac{1}{h_2}\frac{\partial F}{\partial x^2}\mathbf{e}_2+\frac{1}{h_3}\frac{\partial F}{\partial x^3}\mathbf{e}_3, \tag{A1}$$

$$\nabla\cdot\mathbf{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(h_2h_3F_1)+\frac{\partial}{\partial x^2}(h_1h_3F_2)+\frac{\partial}{\partial x^3}(h_1h_2F_3)\right], \tag{A2}$$

$$\nabla\times\mathbf{F}=\frac{1}{h_2h_3}\left(\frac{\partial(h_3F_3)}{\partial x^2}-\frac{\partial(h_2F_2)}{\partial x^3}\right)\mathbf{e}_1+\frac{1}{h_1h_3}\left(\frac{\partial(h_1F_1)}{\partial x^3}-\frac{\partial(h_3F_3)}{\partial x^1}\right)\mathbf{e}_2+\frac{1}{h_1h_2}\left(\frac{\partial(h_2F_2)}{\partial x^1}-\frac{\partial(h_1F_1)}{\partial x^2}\right)\mathbf{e}_3, \tag{A3}$$

where the $\mathbf{e}_i$ are the basic orthonormal vectors associated with the coordinate system and $F_i$ the physical components of $\mathbf{F}$.

In spherical Schwarzschild coordinates $r$, $\theta$, and $\phi$ specified by Eq. (9) we have, in particular,

$$\nabla F=\frac{1}{h}\frac{\partial F}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial F}{\partial\theta}\mathbf{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial F}{\partial\phi}\mathbf{e}_\phi, \tag{A4}$$

$$\nabla\cdot\mathbf{F}=\frac{1}{hr^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta F_\theta)+\frac{\partial F_\phi}{\partial\phi}\right), \tag{A5}$$

$$\nabla\times\mathbf{F}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\,F_\phi)-\frac{\partial F_\theta}{\partial\phi}\right)\mathbf{e}_r+\left(\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi}-\frac{1}{hr}\frac{\partial}{\partial r}(rF_\phi)\right)\mathbf{e}_\theta+\left(\frac{1}{hr}\frac{\partial}{\partial r}(rF_\theta)-\frac{1}{r}\frac{\partial F_r}{\partial\theta}\right)\mathbf{e}_\phi. \tag{A6}$$

We add that

$$\Delta F=\nabla\cdot\nabla F=\frac{1}{hr^2}\frac{\partial}{\partial r}\left(\frac{r^2}{h}\frac{\partial F}{\partial r}\right)+\frac{1}{r^2}LF, \tag{A7}$$

$$LF=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2F}{\partial\phi^2} \tag{A8}$$

where $\Delta$ is the usual three-dimensional Laplacian and $r^{-2}L$ the two-dimensional Laplacian on a surface $r=$const.

## APPENDIX B

We want to show here that the representation (16) applies to an arbitrary vector field $\mathbf{A}$. Consider first the two-dimensional field on a surface $r=$const that coincides with the field $\mathbf{A}^*=\mathbf{A}-(\mathbf{A}\cdot\mathbf{e}_r)\mathbf{e}_r$ on this surface. It is known that any two-dimensional field can be represented as a sum $-\mathbf{r}\times\nabla' S+\nabla' U$ with a stream function $S$ and a potential $U$, where $\nabla'$ means the two-dimensional version of $\nabla$. Suppose now that $S$ and $U$ are known for arbitrary values of $r$ and replace $\nabla'$ in this sum by $\nabla$. Then the sum represents a three-dimensional field which in general differs from $\mathbf{A}$ in the radial component, and it agrees with $\mathbf{A}$ after adding a term $\mathbf{r}T$ with a proper $T$.

## APPENDIX C

Our explanations given above on the decomposition of a solenoidal vector field into poloidal and toroidal parts are sufficient for the reduction of our basic equations but show some strange aspects otherwise. In particular, for any solenoidal field $\mathbf{B}$ such parts are well defined but not for $f\mathbf{B}$ with a scalar $f$ depending on the space coordinates, since such a field is in general no longer solenoidal. Therefore it is useful to look at the decomposition of a solenoidal field from a level on which a decomposition is defined for arbitrary, not necessarily solenoidal fields.

With this in mind let us consider an arbitrary vector field $\mathbf{F}$ with the component representation

$$\mathbf{F}=F_r\,\mathbf{e}_r+F_\theta\,\mathbf{e}_\theta+F_\phi\,\mathbf{e}_\phi \tag{C1}$$

with respect to a spherical coordinate system $(r,\theta,\phi)$. We write it as a sum of a poloidal field $\mathbf{F}^P$ and a toroidal field $\mathbf{F}^T$,

$$\mathbf{F}=\mathbf{F}^P+\mathbf{F}^T, \tag{C2}$$

which we will define in the following.

In the case of an axisymmetric vector field $\mathbf{F}$ we may choose the coordinate system so that the components $F_r$, $F_\theta$, and $F_\phi$ do not depend on $\phi$. Then we define $\mathbf{F}^P$ and $\mathbf{F}^T$ simply by

$$\mathbf{F}^P=F_r\,\mathbf{e}_r+F_\theta\,\mathbf{e}_\theta,\quad \mathbf{F}^T=F_\phi\,\mathbf{e}_\phi. \tag{C3}$$

Clearly $\mathbf{F}=\mathbf{0}$ implies $\mathbf{F}^P=\mathbf{F}^T=\mathbf{0}$ and vice versa, and $f\mathbf{F}^P$ and $f\mathbf{F}^T$ with any axisymmetric scalar $f$ are again poloidal and toroidal, respectively. Moreover, our definition implies, for example, that $\nabla\cdot\mathbf{F}^T=0$ and that $\nabla\times\mathbf{F}^P$ and $\nabla\times\mathbf{F}^T$ are toroidal and poloidal, respectively. These properties of $\mathbf{F}^P$ and $\mathbf{F}^T$ can easily be concluded on the basis of Eqs. (A5) and (A6).

Turning now to the general, that is, not necessarily axisymmetric, case we first note that an arbitrary vector field $\mathbf{F}$ can always be represented in the form

$$\mathbf{F}=\mathbf{r}\times\nabla U+\mathbf{r}V+\nabla W, \tag{C4}$$

with $\mathbf{r}=r\mathbf{e}_r$, by three scalars $U$, $V$, and $W$ depending on $r$, $\theta$, and $\phi$.

This is the same statement as mentioned with Eq. (16) and explained in Appendix B. The component representation of $\mathbf{F}$ that corresponds to Eq. (C4) reads

$$\mathbf{F}=\left(rV+\frac{1}{h}\frac{\partial W}{\partial r}\right)\mathbf{e}_r+\left(-\frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}+\frac{1}{r}\frac{\partial W}{\partial\theta}\right)\mathbf{e}_\theta+\left(\frac{\partial U}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial W}{\partial\phi}\right)\mathbf{e}_\phi. \tag{C5}$$

We recall here the vector relations (17)–(20).

The determination of the scalars $U$, $V$, and $W$ that occur in Eq. (C4) for a given vector field $\mathbf{F}$ requires in general the integration of partial differential equations. Starting from Eq. (C4) and using Eqs. (17)–(19) we find

$$LU=\mathbf{r}\cdot(\nabla\times\mathbf{F}),\quad LW=\mathbf{r}\cdot[\nabla\times(\mathbf{r}\times\mathbf{F})], \tag{C6}$$

$$rV+\frac{1}{h}\frac{\partial W}{\partial r}=\mathbf{r}\cdot\mathbf{F}. \tag{C7}$$

If then the components of $\mathbf{F}$ lying in a spherical surface $r=\text{const}$ are given, the integration of the first two independent partial differential equations with respect to $\theta$ and $\phi$ provides us $U$ and $W$ in this surface. If $W$ is known for some $r$ interval, $V$ can be calculated in this interval from the radial component of $\mathbf{F}$ using the last equation.

Obviously $\mathbf{F}$ in Eq. (C4) is invariant under certain gauge transformations $U\to U+u$, $V\to V+v$, and $W\to W+w$. We conclude from Eqs. (C6) and (C7) that all possible transformations are given by $Lu=Lw=0$, which means that $u$ and $w$ cannot depend on $\theta$ and $\phi$, and by $rv+(1/h)(\partial w/\partial r)=0$. Remarkably enough, these transformations leave not only $\mathbf{F}$ unchanged but also its parts $\mathbf{r}\times\nabla U$ and $\mathbf{r}V+\nabla W$.

With the last finding in mind we define now the fields $\mathbf{F}^P$ and $\mathbf{F}^T$ simply by requiring that they allow representations of the form

$$\mathbf{F}^P=\mathbf{r}V+\nabla W,\quad \mathbf{F}^T=\mathbf{r}\times\nabla U \tag{C8}$$

or, which is the same,

$$\mathbf{F}^P=\left(rV+\frac{1}{h}\frac{\partial W}{\partial r}\right)\mathbf{e}_r+\frac{1}{r}\frac{\partial W}{\partial\theta}\mathbf{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial W}{\partial\phi}\mathbf{e}_\phi, \tag{C9}$$

$$\mathbf{F}^T=-\frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}\mathbf{e}_\theta+\frac{\partial U}{\partial\theta}\mathbf{e}_\phi \tag{C10}$$

with scalar functions $U$, $V$, and $W$. This definition essentially includes that given for the axisymmetric case. However, the specific definition given for the axisymmetric case is local in the sense that, if $\mathbf{F}$ is given only at a point or, which is the same, on a circle defined by one value of $r$ and one of $\theta$, then it immediately determines $\mathbf{F}^P$ and $\mathbf{F}^T$ at this point irrespective of the situation at other points. In contrast to this, our general definition is nonlocal. It works only if we know $\mathbf{F}$ at least on a surface $r=\text{const}$, it requires the solution of at least one partial differential equation on this surface, and it gives us then $\mathbf{F}^P$ and $\mathbf{F}^T$ on the whole surface.

Our definition implies several interesting properties of $\mathbf{F}^P$ and $\mathbf{F}^T$ a few of which are listed here.

(i) If, on a surface $r=\text{const}$, $\mathbf{F}=\mathbf{0}$ then $\mathbf{F}^P=\mathbf{F}^T=\mathbf{0}$ on this surface and vice versa.

(ii) If $f$ is a scalar independent of $\theta$ and $\phi$ then $f\mathbf{F}^P$ is poloidal and $f\mathbf{F}^T$ is toroidal.

(iii) $\mathbf{r}\times\mathbf{F}^P$ is toroidal and $\mathbf{r}\times\mathbf{F}^T$ poloidal.

(iv) $\mathbf{F}^T$ is solenoidal, that is, $\nabla\cdot\mathbf{F}^T=0$.

(v) $\nabla\times\mathbf{F}^P$ is toroidal and $\nabla\times\mathbf{F}^T$ poloidal.

(vi) If, on a surface $r=\text{const}$, $\mathbf{r}\cdot(\nabla\times\mathbf{F}^T)=0$, then $\mathbf{F}^T=\mathbf{0}$ on this surface.

(vii) $\mathbf{F}^P$ and $\mathbf{F}^T$ are orthogonal in the sense of $\langle\mathbf{F}^P\cdot\mathbf{F}^T\rangle=0$ where $\langle\cdots\rangle$ means the average over the solid angle, that is, $\langle\cdots\rangle=(1/4\pi)\int_0^{2\pi}\int_0^{\pi}\cdots\sin\theta\, d\theta\, d\phi$.

Statement (i) can easily be proved on the basis of Eqs. (C6) and (C7). The validity of (ii) to (v) becomes clear with a look at Eqs. (17)–(20). As for (vi) we note that $\mathbf{r}\cdot(\nabla\times\mathbf{F}^T)=0$ is according to Eq. (22) equivalent to $LU=0$. This implies that $U$ is independent of $\theta$ and $\phi$, which leads immediately to $\mathbf{F}^T=\mathbf{0}$. The proof of (vii) can easily be given by expressing $\langle\mathbf{F}^P\cdot\mathbf{F}^T\rangle$ according to Eqs. (C9) and (C10) by an integral over some combination of derivatives of $U$ and $W$, carrying out integrations by parts, and considering that the regularity of $\mathbf{F}$ requires that $\partial U/\partial\phi$ and $\partial W/\partial\phi$ vanish at points where $\sin\theta$ does.

It is often useful to remove the ambiguity of the scalars $U$, $V$, and $W$. We may use the mentioned possibilities of gauge transformations and fix $u$ and $w$ so that

$$\langle U\rangle=\langle W\rangle=0. \tag{C11}$$

With the condition (C11) all three scalars $U$, $V$, and $W$ are uniquely fixed. In this context it is of interest that $\langle LF\rangle=0$ for any scalar $F$. As a consequence we have, for example, $\langle\Delta F\rangle=\langle\mathcal{D}F\rangle=0$ as soon as $\langle F\rangle=0$.

Let us now assume that $\mathbf{F}$ is solenoidal, that is, $\nabla\cdot\mathbf{F}=0$. Then in addition to $\mathbf{F}^T$, which according to (iv) is solenoidal anyway, $\mathbf{F}^P$ has to be solenoidal, too. Thus we may put $\mathbf{F}^P=\nabla\times\mathbf{G}$ with some vector potential $\mathbf{G}$. According to (v) the poloidal part of $\mathbf{G}$ cannot contribute to $\mathbf{F}^P$, and so we may put without loss of generality $\mathbf{G}=\mathbf{r}\times\nabla\hat{U}$ with a scalar potential $\hat{U}$. In this way we arrive at a justification of Eq. (24).

## APPENDIX D

Let us sketch here the essential ideas for a proof of the orthogonality relation (75). Recall first that the index $i$ is a collective index covering $A$, $l$, $m$, and $n$. According to statement (v) of Sec. IV the integral on the right hand side can be nonzero only if $A=A'$. Consider then first $A=A'=P$ and express the integrand according to Eq. (71) by $\hat{S}_{ln}$, $\hat{S}_{l'n'}$, $Y_l^m$, and $Y_{l'}^{m'}$. Carrying out proper integrations by parts and using $L\,Y_l^m=-l(l+1)\,Y_l^m$, the integrand can be brought into a form in which the $Y_l^m$ and $Y_{l'}^{m'}$ occur only as their product $Y_l^m Y_{l'}^{m'}$. From the orthogonality of the $Y_l^m$ it follows then that the integral is nonzero only if $l=l'$ and $m=m'$. In this case it reads, apart from a factor,

$$\int_0^\infty \left( \frac{1}{h^2} \frac{d}{d\,\zeta}(\zeta \hat{S}_{ln}) \frac{d}{d\,\zeta}(\zeta \hat{S}_{ln'}) + l(l+1) \hat{S}_{ln} \hat{S}_{ln'} \right) Zh\, d\zeta \tag{D1}$$

where the variable $r$ is again replaced by $\zeta = r/R$. Split this integral into two, $\int_0^1 \cdots d\zeta$ and $\int_1^\infty \cdots d\zeta$. Carrying out proper integrations by parts and using Eq. (62) we find for the first one

$$\lambda_{ln}^P \int_0^1 \hat{S}_{ln} \hat{S}_{ln'} \frac{h}{\tilde{\eta}} \zeta^2\, d\zeta + \left( \frac{Z}{h} \frac{d}{d\,\zeta}(\zeta \hat{S}_{ln})\, \hat{S}_{ln'} \right)_{\zeta=1-0}. \tag{D2}$$

Analogously with Eq. (39), rewritten by replacing $S_l^m$ by $\hat{S}_{ln}$, and considering that $Zh=1$ for $\zeta \geqslant 1$ we find for the second one

$$-\left( \frac{Z}{h} \frac{d}{d\zeta}(\zeta \hat{S}_{ln})\, \hat{S}_{ln'} \right)_{\zeta=1+0}. \tag{D3}$$

Since $\hat{S}_{ln}$ and $d\hat{S}_{ln}/d\zeta$ are continuous at $\zeta=1$ the integral (D1) is equal to

$$\lambda_{ln}^P \int_0^1 \hat{S}_{ln} \hat{S}_{ln'} \frac{h}{\tilde{\eta}} \zeta^2\, d\zeta. \tag{D4}$$

Because of the orthogonality relation for the $\hat{S}_{ln}$ mentioned in Sec. VII this integral is nonzero only if $n=n'$. For $A=A'=T$ we can proceed in the same way, but it is simpler since the integral that occurs instead of Eq. (D1) is from the very beginning of the type $\int_0^1 \cdots d\zeta$. As a result the integral on the right hand side of Eq. (75) is indeed nonzero only if $A=A'$, $l=l'$, $m=m'$, and $n=n'$.

Another possibility for proving the orthogonality relation (75) consists in generalizing a method used by Rheinhardt for the flat-space case [20]. Consider in addition to the magnetic-field modes $\mathbf{B}_i$ also the corresponding electric-field modes $\mathbf{E}_i$. Then Faraday's law contained in Eq. (6) gives

$$\lambda_i \mathbf{B}_i = -cT_\eta \boldsymbol{\nabla} \times (Z\mathbf{E}_i) \tag{D5}$$

and, hence,

$$\lambda_i \int_\infty \mathbf{B}_i \cdot \mathbf{B}_j\, Z\, dv = cT_\eta \int_\infty [\boldsymbol{\nabla} \times (Z\mathbf{E}_i)] \cdot B_j\, Z\, dv. \tag{D6}$$

With standard manipulations, considering that the tangential components of $\mathbf{B}_i$ and $\mathbf{E}_i$ have to be continuous across the surface $r=R$, we find

$$\lambda_i \int_\infty \mathbf{B}_i \cdot \mathbf{B}_j\, Z\, dv = cT_\eta \int_V \mathbf{E}_i \cdot [\boldsymbol{\nabla} \times (Z\mathbf{B}_j)]\, Z\, dv, \tag{D7}$$

where $V$ means the sphere $r \leqslant R$. Using Ohm's law (7) and Ampere's law contained in (6) with the displacement current neglected, we find further

$$\lambda_i \int_\infty \mathbf{B}_i \cdot \mathbf{B}_j\, Z\, dv = R^2 \int_V [\boldsymbol{\nabla} \times (Z\mathbf{B}_i)] \cdot [\boldsymbol{\nabla} \times (Z\mathbf{B}_j)]\, \tilde{\eta}\, dv. \tag{D8}$$

Like Eqs. (D5)–(D7) this relation applies also with $i$ and $j$ interchanged. Hence we have

$$(\lambda_i - \lambda_j) \int_\infty \mathbf{B}_i \cdot \mathbf{B}_j\, Z\, dv = 0. \tag{D9}$$

This proves the orthogonality of the $\mathbf{B}_i$ belonging to different $\lambda_i$. The remaining orthogonalities can be concluded as above from the orthogonality of poloidal and toroidal fields and the orthogonality of the spherical harmonics.

An interesting side product of this proof is the orthogonality of the $\boldsymbol{\nabla} \times (Z\mathbf{B}_i)$ in the sense of

$$\int_V [\boldsymbol{\nabla} \times (Z\mathbf{B}_i)] \cdot [\boldsymbol{\nabla} \times (Z\mathbf{B}_j)]\, \tilde{\eta}\, dv = \frac{\lambda_i}{R^2} N_i^2 \delta_{ij}, \tag{D10}$$

which can be immediately concluded from Eqs. (75) and (D8).

The flat-space versions of Eqs. (75) and (D10) play an important part in the theory of a numerical code for solving the equations governing spherical dynamo models [8].


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