Noncommutative quantum mechanics

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A general noncommutative quantum mechanical system in a central potential $V=V(r)$ in two dimensions is considered. The spectrum is bounded from below and, for large values of the anticommutative parameter θ, we find an explicit expression for the eigenvalues. In fact, any quantum mechanical system with these characteristics is equivalent to a commutative one in such a way that the interaction $V(r)$ is replaced by $V=V(\hat{H}_{HO},\hat{L}_z)$, where $\hat{H}_{HO}$ is the Hamiltonian of the two-dimensional harmonic oscillator and $\hat{L}_z$ is the z component of the angular momentum. For other finite values of θ the model can be solved by using perturbation theory.

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Recent results in string theory [1] suggest that spacetime could be nonconmutative [2]. This intriguing possibility implies new and deep changes in our concept of spacetime that could be visualized at the quantum mechanical level. For example, unitarity in quantum mechanics is assured if time is commutative, but spatial noncommutativity, although it is completely consistent with the standard rules of quantum mechanics, implies the new Heisenberg relation

$$\Delta x\Delta y\sim\theta, \tag{1}$$

where θ is the strength of the noncommutative effects and plays an analogous role to $\hbar$ in the usual quantum mechanics.

In this Brief Report we would like to discuss a general noncommutative quantum mechanical system, stressing the differences from the equivalent commutative case. More precisely, we show that any two-dimensional noncommutative system in a central potential $V=V(r)$ where $r=\sqrt{|\mathbf{x}|^2}$ is equivalent to a commutative system decribed by the potential

$$V=V(\hat{H}_{HO},\hat{L}_z), \tag{2}$$

where $\hat{H}_{HO}$ is the Hamiltonian of the usual (commutative) two-dimensional harmonic oscillator and $\hat{L}_z$ is the z component of the angular momentum.

In noncommutative space one replaces the ordinary product with the Moyal or star product:

$$\mathbf{A}\star\mathbf{B}(\mathbf{x})=e^{(i/2)\theta^{ij}\partial_i^{(1)}\partial_j^{(2)}}\mathbf{A}(\mathbf{x}_1)\mathbf{B}(\mathbf{x}_2)|_{\mathbf{x}_1=\mathbf{x}_2=\mathbf{x}}. \tag{3}$$

The only modification to the Schrödinger equation

$$i\frac{\partial\Psi(\mathbf{x},t)}{\partial t}=\left[\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})\right]\Psi(\mathbf{x},t) \tag{4}$$

is to replace

$$V(\mathbf{x})\star\Psi(\mathbf{x},t)\to V\left(\mathbf{x}-\frac{\tilde{\mathbf{p}}}{2}\right)\Psi, \tag{5}$$

where $\tilde{p}_{i_k}=\theta^{i_kj_k}p_{j_k}$, and $\theta_{ij}=\theta\epsilon_{ij}$ with ϵ_{ij} the antisymmetric tensor. This formula, which appeared recently in connection with string theory, was written in [3] although there is an older version also known as Bopp's shift [4].

The next step is to consider a central potential in two dimensions. The right hand side of Eq. (5) becomes

$$V\left(\left|\mathbf{x}-\frac{\tilde{\mathbf{p}}}{2}\right|^2\right)\Psi=V\left(\frac{\theta^2}{4}p_x^2+x^2+\frac{\theta^2}{4}p_y^2+y^2-\theta L_z\right)$$
$$=V(\hat{\aleph}), \tag{6}$$

where the $\hat{\aleph}$ operator is defined as

$$\hat{\aleph}=\hat{H}_{HO}-\theta\hat{L}_z, \tag{7}$$

and corresponds to a two-dimensional harmonic oscillator with effective mass $m=2/\theta^2$, frequency $\omega=\theta$, and angular momentum $L_z=xp_y-yp_x$. The symmetry group for this system is SU(2) and the spectrum of $\hat{\aleph}$ can be computed noting that

$$L_x\frac{1}{2}(a_x^\dagger a_x-a_y^\dagger a_y),$$

$$L_y=\frac{1}{2}(a_x^\dagger a_y+a_y^\dagger a_x), \tag{8}$$

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$$L_z=\frac{1}{2i}(a_x^\dagger a_y - a_y^\dagger a_x)$$

are symmetry generators satisfying the Lie algebra $[L_i, L_j] = i\epsilon_{ijk}L_k$ and, therefore, $\{\hat{\aleph}, \hat{\mathbf{L}}^2, J_z = \frac{1}{2}\hat{L}_z\}$ is a complete set of conmuting observables. If we denote the eigenvalues and eigenvectors by λ_{jm} and $|j,m\rangle$, respectively, then we have the selection rules

$$j=0,\frac{1}{2},1,\frac{3}{2}, \ldots,$$

$$m=j, j-1, j-2, \ldots, -j. \tag{9}$$

The eigenfunctions $|j,m\rangle$ are well known [5] and the eigenvalues of $\hat{\aleph}$ are given by

$$\lambda_{jm}=\theta[2j+1-2m]. \tag{10}$$

Using these results, the calculation of the eigenvalues of $V(\hat{\aleph})$ is straightforward. Indeed, if the eigenvalues of the operator $\hat{A}$ are a_n, then the function $f(\hat{A})$, after expanding for small values of ϵ, is

$$\begin{aligned} f(\hat{A}+\epsilon)\psi_n &= \left(f(\hat{A}) + f'(\hat{A})\epsilon + \frac{1}{2!}f''(\hat{A})\epsilon^2 \cdots \right)\psi_n \\ &= \left(f(a_n) + f'(a_n)\epsilon + \frac{1}{2!}f''(a_n)\epsilon^2 \cdots \right)\psi_n \\ &= f(a_n+\epsilon)\psi_n \to f(a_n)\psi_n, \end{aligned} \tag{11}$$

and, as a consequence, the eigenvalue equation of $V(\hat{\aleph})$ is

$$V(\hat{\aleph})|j,m\rangle = V[\theta(2j+1-2m)]|j,m\rangle. \tag{12}$$

Once Eq. (12) is found, one must compute the spectrum of the full Hamiltonian given by

$$\begin{aligned} H &= \frac{\mathbf{p}^2}{2M} + V(\hat{\aleph}) \\ &= \frac{2}{M\theta^2}\left(\frac{\theta^2}{4}\mathbf{p}^2 + \mathbf{r}^2 - \theta L_z\right) - \frac{2}{M\theta^2}\mathbf{r}^2 + V(\hat{\aleph}) + \frac{2}{M\theta}L_z \\ &= \frac{2}{M\theta^2}\hat{\aleph} + V(\hat{\aleph}) - \frac{2}{M\theta^2}\mathbf{r}^2 + \frac{2}{M\theta}L_z \\ &\equiv H_0 - \frac{2}{M\theta^2}\mathbf{r}^2 + \frac{2}{M\theta}L_z. \end{aligned} \tag{13}$$

Using Eqs. (10) and (12) one find that the eigenvalues of $\hat{H}_0$ are

$$\Lambda_{j,m} = \frac{2}{M\theta}[2j+1-2m] + V[\theta(2j+1-2m)]. \tag{14}$$

The second term of the Hamiltonian can be treated as a perturbation for large (but finite) values of θ. For regular, polynomial-like potentials, the situation is similar to the solitonic case as in [6].

We will concentrate on the expectation values of the full Hamiltonian, $E_{jm} = \langle j,m|\hat{H}|j,m\rangle$, i.e.,

$$\begin{aligned} E_{jm} &= \langle j,m|\hat{H}_0|j,m\rangle - | \frac{2}{M\theta^2}\langle j,m|\mathbf{r}^2|j,m\rangle \\ &\quad + \frac{4}{M\theta}\langle j,m|L_z|j,m\rangle, \\ &= \frac{2}{M\theta}[2j+1] + V[\theta(2j+1-2m)] \\ &\quad - \frac{2}{M\theta^2}\langle j,m|\mathbf{r}^2|j,m\rangle. \end{aligned} \tag{15}$$

The next to last term in the right hand side in Eq. (15) can be calculated using perturbation theory for large values of θ. Indeed, in this case $|j,m\rangle$ corresponds to the two-dimensional harmonic oscillator eigenvectors

$$|j,m\rangle = \frac{a_+^{\dagger\, j+m} a_-^{\dagger\, j-m}}{\sqrt{(j+m)!(j-m)!}}|0,0\rangle, \tag{16}$$

where in Eq. (16) we have used the Schwinger representation for the two-dimensional harmonic oscillator [5] and $\langle j,m|\mathbf{r}^2|j,m\rangle$ becomes

$$\langle j,m|\mathbf{r}^2|j,m\rangle = \frac{\theta}{2}[2j+1]. \tag{17}$$

Let us consider now two kinds of singular potential.

(a) If $V(r) = -\gamma/r^\alpha$, then $V(\theta) = -\gamma[\theta(2j+1-2m)]^{-\alpha/2}$. Note that this term varies as $\theta^{-\alpha/2}$, and therefore the relevant contribution to the spectrum for $\alpha > 2$ is given by the first term in Eq. (15). In fact, the difference between the levels $2j$ and $2j+1$ is $2/M\theta$.

(b) From the physical point of view, probably the most interesting case is the Coulomb potential $V(r) = \gamma \ln(r)$ which corresponds to $V(\theta) = \gamma/2 \ln[\theta(2j+1-2m)]$. This could have a relation to the quantum Hall effect where electrons are confined in a plane. We would like to remark that spectroscopy in two-dimensional systems could be a sensible mechanism for detecting noncommutative corrections to quantum mechanics [7].

Finally, we would like to point out that our results seem to indicate that the connection between the commutative and noncommutative regimes is abrupt, i.e., the limit $\theta \to 0$ cannot be taken directly [8].

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