Derivative expansion and the parity violating effective action for thermal (2+1)-dimensional QED at higher orders

F. T. Brandt,[1] Ashok Das,[2] J. Frenkel,[1] S. Pereira,[1] and J. C. Taylor[3]
[1]Instituto de Física, Universidade de São Paulo, São Paulo, SP 05315-970, Brazil
[2]Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627-0171
[3]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, United Kingdom

(Received 28 March 2001; published 24 August 2001)

We systematically study induced higher order corrections to the parity violating part in thermal QED_{2+1}, in the static limit, using the method of derivative expansion. We explicitly calculate the parity violating parts of the quadratic, cubic and the quartic terms (in fields) of the effective action which is linear in the magnetic field. We show that each of these actions can be summed, in principle, to all orders in the derivatives. However, such a structure is complicated and not very useful. On the other hand, at every order in the powers of the derivatives, we show that the effective action can also be summed to all orders in the A_0 fields. The resulting thermal parity violating actions can be expressed in terms of the leading order effective action in the static limit. We prove gauge invariance, both *large* and *small* of the resulting effective actions, within the framework of derivative expansion. Various other features of the theory are also brought out.

DOI: 10.1103/PhysRevD.64.065018 PACS number(s): 11.15.Tk, 11.10.Wx, 11.30.Er

I. INTRODUCTION

Chern-Simons (CS) theories, in $2+1$ dimensions, have been of interest in recent years [1,2]. Non-Abelian CS theories are invariant under *large gauge* transformations provided the coefficient of the CS term is quantized. At finite temperature, however, the induced CS term has a continuous coefficient (temperature dependent), which is incompatible with the discreteness of the CS coefficient necessary for *large gauge* invariance to hold [3–5]. This puzzle of violation of *large gauge* invariance at finite temperature is well understood now, at least in Abelian theory [6–9].

As is well known, at finite temperature, amplitudes [10] as well as the effective action become nonanalytic [11,12], unlike at zero temperature. As a result, it becomes essential to talk of the effective action only in certain limits—the conventional ones being the long wave and static limits. It has already been shown within the context of Abelian CS theory in $2+1$ dimensions that *large gauge* invariance is not an issue in the long wave limit [9]. On the other hand, a truncation of the effective action at any finite order in the static limit leads to a violation of *large gauge* invariance, although the complete action has *large gauge* invariance. This has been checked for leading order terms in the effective action in the static limit.

The leading order parity violating effective action in the static limit can be determined exactly, either through functional methods [7,8] or through the use of a *large gauge* Ward identity [9,13]. This, of course, raises the question of higher order corrections to this action and the issue of *large gauge* invariance for such terms. In this paper, we address this question.

Higher order corrections can naturally be obtained through a derivative expansion (powers of momentum) [4,14,15]. However, as is known in simple models, in a model with *large gauge* invariance, a derivative expansion does lead to new subtleties [16,18]. In fact, some such subtlety was already noted earlier in the Abelian $(2+1)$-dimensional model [16], even at the leading order in the static limit. In Sec. II, we extend and improve on the analysis of Ref. [16] to rederive, within the context of derivative expansion, the leading order parity violating effective action in the static limit which is linear in B, the magnetic field. We also show, within the framework of a derivative expansion, that the parity violating part of the effective action does not contain any higher order terms in $\vec{A}$ in that limit, so that this action is a complete parity violating effective action in that limit, consistent with the results obtained in Refs. [7–9]. In Sec. III, we tackle the question of going beyond the leading order, in the static limit, and evaluate the parity violating effective action using a derivative expansion in the coordinate space (see Appendix A for the corresponding momentum space calculation). Beyond the leading order in the static limit, the parity violating part of the effective action will contain nonlinear terms in $\vec{A}$. However, our calculations in this paper are restricted only to terms linear in $\vec{A}$, for simplicity, although the method of derivative expansion can be extended to more general cases as well. We calculate a closed form expression for the quadratic part of the parity violating effective action. In fact, the effective action, at any order, can be obtained in a closed form, but the closed form expressions are not necessarily simple. Rather, a power series expansion in the number of derivatives gives a simpler expression to the quadratic, cubic and quartic (in fields) terms of the effective action. In Sec. IV, we analyze the general features of our results. In particular, we show that from our low order (in fields) calculations, we can, in fact, predict the behavior of the effective action with one, three and five derivatives to all orders in the fields. In fact, to all orders, we find that these effective actions are completely determined from the form of the leading order parity violating effective action in the static limit. They are manifestly invariant under *small* as well as *large* gauge transformations. We present a brief conclusion in Sec. V. One of the features that arises, within the derivative expansion, is the fact that expressions for these actions are not manifestly invariant under *small* gauge transformations, although they can be brought to a gauge invariant form (at least for lower orders) through the use of various algebraic identities. A general proof of gauge invariance and locality of the effective action linear in $\vec{A}$ under *small* gauge transformations is presented in Appendix B, where we recast the derivative expansion also in a gauge invariant form.

II. LEADING ORDER DERIVATIVE EXPANSION IN THE STATIC LIMIT

Let us consider a fermion interacting with an Abelian gauge background described by the Lagrangian density (in $2+1$ dimensions)

$$\mathcal{L}=\bar{\psi}[\gamma^{\mu}(i\partial_{\mu}+eA_{\mu})-M]\psi. \tag{1}$$

Here, for simplicity, we will assume that $M>0$. The effective action following from this is formally given by

$$\begin{aligned}\Gamma_{eff}[A,M]&=-i\ln\det[\gamma^{\mu}(i\partial_{\mu}+eA_{\mu})-M]\\&=-i\,\mathrm{Tr}\ln[\gamma^{\mu}(i\partial_{\mu}+eA_{\mu})-M],\end{aligned} \tag{2}$$

where ''Tr'' stands for a trace over Dirac indices as well as over a complete basis of states. As is well known, at finite temperature, the effective action is not well defined everywhere [11,12], as a result of which it can be expanded in powers of derivatives only in some limit. This is a simple reflection of the fact that thermal amplitudes are nonanalytic at the origin in the energy-momentum plane [10]. This was explicitly shown in thermal QED_{2+1} [9], where we calculated the leading term in the parity violating part of the box diagram at finite temperature, and where we also showed that *large gauge* invariance is an issue in the static limit but not in the long wave limit. In this paper, we systematically calculate the higher order corrections to the earlier result, in the static limit, by using the method of derivative expansion [14,15]. Although we earlier summed the leading order terms in the static limit using a large gauge Ward identity [9], in this section we will rederive this result from the derivative expansion.

The leading order behavior in the static limit, as shown in Ref. [9], is consistent with assuming a specific form of the background gauge fields, namely [7,8],

$$A_0=A_0(t),\quad \vec{A}=\vec{A}(\vec{x}). \tag{3}$$

[In other words, even though background (3) is not what one would call a static background, explicit perturbative calculations show [9] that the leading order behavior of the parity violating amplitudes, in the static limit, corresponds to this choice of a gauge background.] The parity violating part of the effective action, in such a background, was already calculated in Refs. [7–9,16,17], and here, as a warmup, we rederive the result from a derivative expansion which will also help settle some subtlety of this method. It is well known that, by a suitable gauge transformation [8,9,19]

$$A_{\mu}\to A_{\mu}+\partial_{\mu}\Omega,\quad \Omega(t)=\left(-\int_0^t+\frac{t}{\beta}\int_0^{\beta}\right)dt'A_0(t'), \tag{4}$$

a gauge background of the form of Eq. (3) can always be brought to the form

$$A_0(t)\to\frac{a}{\beta}=\frac{1}{\beta}\int_0^{\beta}dt\,A_0(t),\quad \vec{A}=\vec{A}(\vec{x}), \tag{5}$$

so that under a *large gauge* transformation,

$$a\to a+\frac{2\pi n}{e}. \tag{6}$$

We will use the imaginary time formalism [11,20,21] in evaluating the finite temperature determinant, where energy takes discrete values. Rotating to Euclidean space, the effective action takes the form

$$\Gamma_{eff}[A,M]=-\sum_n\mathrm{Tr}\ln(\not{p}+\gamma_0\tilde{\omega}_n+M+e\not{A}), \tag{7}$$

where we have defined $\not{p}=\vec{\gamma}\cdot\vec{p}$ and similarly $\not{A}=\vec{\gamma}\cdot\vec{A}$. Furthermore,

$$\tilde{\omega}_n=\omega_n+\frac{ea}{\beta}=\frac{(2n+1)\pi}{\beta}+\frac{ea}{\beta}, \tag{8}$$

where β represents the inverse temperature in units where the Boltzmann constant is unity. The momentum in the above expression is to be understood as an operator which does not commute with coordinate dependent quantities. Let us also note that we are working with the following representations of gamma matrices in Euclidean space:

$$\gamma_0=i\sigma_3,\quad \gamma_1=i\sigma_1,\quad \gamma_2=i\sigma_2. \tag{9}$$

Let us next define

$$K_n=\frac{1}{\not{p}+\gamma_0\tilde{\omega}_n+M}. \tag{10}$$

Then, taking out a factor (which does not contribute to the parity violating effective action) we can write the effective action as

$$\begin{aligned}\Gamma_{eff}[A,M]&=-\sum_n\mathrm{Tr}\ln(1+eK_n\not{A})\\&=-\sum_n\mathrm{Tr}\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}(eK_n\not{A})^{j+1}.\end{aligned} \tag{11}$$

This expression shows that the effective action contains all powers of $\vec{A}$. However, let us next show that quadratic and higher powers of $\vec{A}$ do not occur in the parity violating part of the effective action. To this end, let us note that if we define [8]

$$\gamma_0\tilde{\omega}_n+M=\rho_n e^{\gamma_0\phi_n}, \tag{12}$$

where

$$\rho_n=\sqrt{\tilde{\omega}_n^2+M^2}, \quad \phi_n=\tan^{-1}\frac{\tilde{\omega}_n}{M}, \tag{13}$$

we can write

$$K_n=\frac{1}{\not{p}+\gamma_0\tilde{\omega}_n+M}$$
$$=e^{-\gamma_0\phi_n/2}\frac{1}{\not{p}+\rho_n}e^{-\gamma_0\phi_n/2}=e^{-\gamma_0\phi_n/2}K_n^{(0)}e^{-\gamma_0\phi_n/2}. \tag{14}$$

Using this, the terms in Eq. (11) containing higher powers of $\vec{A}$ can be written as

$$\Gamma_{eff}^{(higher)}[A,M]=-\sum_n \mathrm{Tr}\sum_{j=1}\frac{(-1)^j}{j+1}(eK_n\not{A})^{j+1}$$
$$=-\sum_n \mathrm{Tr}\sum_{j=1}\frac{(-1)^j}{j+1}$$
$$\times(e^{-\gamma_0\phi_n/2}eK_n^{(0)}e^{-\gamma_0\phi_n/2}\not{A})^{j+1}$$
$$=-\sum_n \mathrm{Tr}\sum_{j=1}\frac{(-1)^j}{j+1}(eK_n^{(0)}\not{A})^{j+1}, \tag{15}$$

where the intermediate phase factors cancel because of the gamma matrix algebra, whereas the initial and final phase factors cancel because of the cyclicity of the trace. It now follows that the parity violating part of this action is

$$\Gamma_{eff}^{\mathrm{PV}(higher)}[A,M]=\frac{1}{2}(\Gamma_{eff}^{(higher)}[A,M]-\Gamma_{eff}^{(higher)}[A,-M])$$
$$=0, \tag{16}$$

which follows because expression (15) is an even function of the fermion mass. This shows that the parity violating part of the effective action is at best linear in $\vec{A}$. However, as is clear from this derivation, it says nothing about the parity conserving part of the effective action, which, in general will contain higher powers of $\vec{A}$. In fact, as we can see from Eq. (15), the parity conserving part will have a quadratic term of the form

$$\Gamma_{eff}^{PC(2)}=\frac{e^2}{2}\sum_n \mathrm{Tr}\,K_n^{(0)}\not{A}K_n^{(0)}\not{A}$$
$$=\frac{e^2}{2}\sum_n \mathrm{Tr}\frac{1}{\not{p}+\rho_n}\not{A}\frac{1}{\not{p}+\rho_n}\not{A}$$
$$=-e^2\sum_n\int \mathrm{d}^2x\frac{\mathrm{d}^2p}{(2\pi)^2}\left(\frac{2p_ip_j+i(p_i\partial_j+p_j\partial_i)-\delta_{ij}[\rho_n^2+p_k(p_k+i\partial_k)]}{(\vec{p}^2+\rho_n^2)[(\vec{p}+i\vec{\nabla})^2+\rho_n^2]}A_j\right)A_i$$
$$=-\frac{e^2}{2\pi}\sum_n\int \mathrm{d}\alpha\,\mathrm{d}^2x\;\alpha(1-\alpha)\,A_i\frac{(\partial_i\partial_j-\nabla^2\delta_{ij})}{[-\alpha(1-\alpha)\nabla^2+\rho_n^2]}A_j$$
$$=\frac{e^2}{2\pi}\sum_n\int \mathrm{d}\alpha\,\mathrm{d}^2x\;\alpha(1-\alpha)B\frac{1}{\rho_n^2-\alpha(1-\alpha)\nabla^2}B, \tag{17}$$

where we have defined the magnetic field as

$$B\equiv\epsilon_{ij}\partial_iA_j, \quad i,j=1,2. \tag{18}$$

and combined the denominators using the Feynman combination formula in the intermediate steps. The sum over the discrete frequencies can be done in a simple manner using the formula

$$\sum_{n=-\infty}^{\infty}\frac{1}{\left(\frac{(2n+1)\pi}{\beta}+\frac{\theta}{\beta}\right)^2+\mu^2}$$
$$=\frac{\beta}{4\mu}\left[\tanh\frac{1}{2}(\beta\mu-i\theta)+\tanh\frac{1}{2}(\beta\mu+i\theta)\right]$$
$$=\frac{\beta}{\mu}\frac{\partial}{\partial\theta}\tan^{-1}\left(\tanh\frac{\beta\mu}{2}\tan\frac{\theta}{2}\right), \tag{19}$$

and leads to

$$\Gamma_{eff}^{PC(2)}=\frac{e^2\beta}{8\pi}\int \mathrm{d}\alpha\,\mathrm{d}^2x\,\alpha(1-\alpha)B\frac{1}{\sqrt{M^2-\alpha(1-\alpha)\nabla^2}}\times\left[\tanh\frac{1}{2}[\beta\sqrt{M^2-\alpha(1-\alpha)\nabla^2}-iea]+\tanh\frac{1}{2}[\beta\sqrt{M^2-\alpha(1-\alpha)\nabla^2}+iea]\right]B$$
$$=\frac{e\beta}{2\pi}\int \mathrm{d}\alpha\,\mathrm{d}^2x\,\alpha(1-\alpha)B\frac{1}{\sqrt{M^2-\alpha(1-\alpha)\nabla^2}}\frac{\partial}{\partial a}\times\tan^{-1}\left(\tanh\frac{\beta\sqrt{M^2-\alpha(1-\alpha)\nabla^2}}{2}\tan\frac{ea}{2}\right)B. \tag{20}$$

This is completely in agreement with the results of Ref. [16], and it is clear that this action is manifestly invariant under *large gauge* transformations [see Eq. (6)]. (That is, the arctan changes by a constant under a *large gauge* transformation. However, the quadratic effective action involves a derivative and, therefore, this action is invariant under *large gauge* transformations.)

The term in the effective action, linear in $\vec{A}$, has to be evaluated more carefully since this term, as it stands [see Eq. (11)], needs to be regularized. It was suggested in Ref. [16] to look alternately at the linear term in the derivative of the effective action:

$$\frac{\partial\Gamma_{eff}^{(1)}}{\partial a}=\frac{e^2}{\beta}\sum_n \mathrm{Tr}\,K_n\not{A}K_n\gamma_0. \tag{21}$$

This would correspond to making one subtraction. However, this expression is still not fully regularized (it does not satisfy cyclicity as can be easily checked) so that the effective action linear in $\vec{A}$ was derived in Ref. [16] in a limiting manner from this (where cyclicity was still an issue). Let us note, however, that we are interested in the parity violating part of the effective action. Thus, from Eq. (7), we obtain

$$\frac{\partial\Gamma_{eff}^{\mathrm{PV}}}{\partial a}=\frac{1}{2}\left(\frac{\partial\Gamma_{eff}[A,M]}{\partial a}-\frac{\partial\Gamma_{eff}[A,-M]}{\partial a}\right)$$
$$=-\frac{e}{2\beta}\sum_n \mathrm{Tr}\left(\frac{1}{\not{p}+\gamma_0\tilde{\omega}_n+M+e\not{A}}-\frac{1}{\not{p}+\gamma_0\tilde{\omega}_n-M+e\not{A}}\right)\gamma_0$$
$$=-\frac{eM}{\beta}\sum_n \mathrm{Tr}\frac{1}{p^2+\tilde{\omega}_n^2+M^2+e(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}-i\gamma_0B)+e^2\vec{A}^2}\gamma_0. \tag{22}$$

The linear term (in $\vec{A}$) of this expression gives

$$\frac{\partial\Gamma_{eff}^{\mathrm{PV}(1)}}{\partial a}=\frac{e^2M}{\beta}\sum_n \mathrm{Tr}\frac{1}{p^2+\tilde{\omega}_n^2+M^2}(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}-i\gamma_0B)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\gamma_0. \tag{23}$$

This expression is well defined and satisfies the cyclicity condition. Evaluating the Dirac trace gives (“tr” simply denotes trace over a complete basis)

$$\frac{\partial\Gamma_{eff}^{\mathrm{PV}(1)}}{\partial a}=\frac{2ie^2M}{\beta}\sum_n \mathrm{tr}\frac{1}{(p^2+\tilde{\omega}_n^2+M^2)^2}B$$
$$=\frac{ie^2M}{2\pi\beta}\sum_n\int \mathrm{d}^2x\frac{1}{\tilde{\omega}_n^2+M^2}B$$
$$=\frac{ie^2}{8\pi}\int \mathrm{d}^2x\left[\tanh\frac{1}{2}(\beta M-iea)+\tanh\frac{1}{2}(\beta M+iea)\right]B$$
$$=\frac{ie}{2\pi}\frac{\partial}{\partial a}\int \mathrm{d}^2x\,\tan^{-1}\left(\tanh\frac{\beta M}{2}\tan\frac{ea}{2}\right)B. \tag{24}$$

This determines the parity violating effective action linear in B which precisely coincides with the action derived earlier [8,9,16]; for future use, let us define

$$\Gamma(a,M)=\frac{e}{2\pi}\tan^{-1}\left(\tanh\frac{\beta M}{2}\tan\frac{ea}{2}\right), \tag{25}$$

so that we can write

$$\Gamma_{eff}^{\mathrm{PV}(1)}=i\int \mathrm{d}^2x\,B\,\Gamma(a,M). \tag{26}$$

[In general, of course, Eq. (24) determines the effective action up to an additive constant. However, if we assume that the effective action is normalized such that it vanishes when the external fields vanish, then, the additive constant vanishes and Eq. (26) gives the parity violating part of the effective action linear in $\vec{A}$.] Furthermore, as we have already shown, the parity violating part of the effective action does not contain higher order terms in $\vec{A}$. Consequently, this is the complete parity violating part of the effective action in the particular background we have chosen, consistent with the results of Refs. [7,8].

The particular gauge background, as we argued earlier, gives the leading terms in the static limit; consequently, this action would correspond to the leading order parity violating effective action in that limit. We will next try to extend these calculations to higher orders in derivatives (but still to linear order in $\vec{A}$).

III. DERIVATIVE EXPANSION AT HIGHER ORDERS

In trying to determine the higher order terms (in derivatives) in the static limit, we let the A_0 field depend on space as well [in contrast to the discussion in Sec. II, Eq. (3)] and make the decomposition

$$A_0(t,\vec{x})=\overline{A_0}(t)+\hat{A}_0(\vec{x}), \qquad \int d^2x\,\hat{A}_0(\vec{x})=0. \tag{27}$$

That is, we separate out the zero mode of the space dependent part into the first term, which can always be done using a box normalization. Once again, by a suitable gauge transformation [see Eq. (4)], the gauge fields can be brought to the form

$$A_0(t,\vec{x})\rightarrow\frac{a}{\beta}+\hat{A}_0(\vec{x}), \quad \vec{A}=\vec{A}(\vec{x}). \tag{28}$$

With such a separation, we have also separated the behavior of the fields under a *small* and a *large* gauge transformation. That is, under a *large gauge* transformation only a transforms as

$$a\rightarrow a+\frac{2\pi n}{e}, \tag{29}$$

while under a *small gauge* transformation only $\vec{A}$ transforms as [$\hat{A}_0$ does not transform under a *small gauge* transformation in the static limit, since we have already used this freedom to bring A_0 to the form in Eq. (28)]:

$$\vec{A}\rightarrow\vec{A}+\vec{\nabla}\epsilon. \tag{30}$$

In this case, the effective action [see Eq. (11)] takes the form

$$\Gamma_{eff}[A,M] = -\sum_n \mathrm{Tr}\ln\{1+[\not{p}+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M]^{-1}(e\not{A})\}. \tag{31}$$

The linear term in $\vec{A}$ has the simple form [there will now be higher order terms in $\vec{A}$ in the parity violating (PV) action, but we restrict to linear terms for simplicity]

$$\Gamma^{(1)}_{eff}[\hat{A}_0,M]=-e\sum_n \mathrm{Tr}\,\frac{1}{\not{p}+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M}\not{A}. \tag{32}$$

It is clear now that if we expand the denominator in powers of $\hat{A}_0$ and carry out the trace, we will obtain all the higher derivative corrections to the effective action in the static limit. However, it is also clear that the expansion would bring out more and more factors of Dirac matrices in the numerator, so that calculations will become increasingly difficult as we go to higher orders. Thus we look for an alternate method for obtaining the result.

Let us note that we are really interested in the parity violating part of the effective action, which is obtained as

$$\Gamma^{\mathrm{PV}(1)}_{eff}=\frac{1}{2}(\Gamma^{(1)}_{eff}[\hat{A}_0,M]-\Gamma^{(1)}_{eff}[\hat{A}_0,-M]). \tag{33}$$

Furthermore, let us also note the identity

$$\frac{1}{2}\left(\frac{1}{\not{p}+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M}-\frac{1}{\not{p}+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)-M}\right) =\frac{M}{p^2+\tilde{\omega}_n^2+M^2+L}, \tag{34}$$

where

$$L=2e\tilde{\omega}_n\hat{A}_0-ie\gamma_0(\not{\partial}\hat{A}_0)+e^2\hat{A}_0^2 \tag{35}$$

contains all the field dependent terms and has a much simpler Dirac matrix structure. Using this, we can write

$$\Gamma^{\mathrm{PV}(1)}_{eff}=-eM\sum_n \mathrm{Tr}\,\frac{1}{p^2+\tilde{\omega}_n^2+M^2+L}\not{A}. \tag{36}$$

The denominator can now be expanded, and the effective action can be calculated for any number of $\hat{A}_0$ fields in a simple and systematic manner.

As an example, let us note that the part of the parity violating action containing one $\hat{A}_0$ field in addition to a B field arises as (in the first two lines $\vec{p}$ represents the momentum operator, while in the last line it corresponds to the eigenvalues of the operator [14,15])

$$\begin{aligned}(\Gamma^{\mathrm{PV}(1)}_{eff})^{(1)}&=eM\sum_n \mathrm{Tr}\,\frac{1}{p^2+\tilde{\omega}_n^2+M^2}[-ie\gamma_0(\not{\partial}\hat{A}_0)] \times\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\not{A}\\ &=-2ie^2M\sum_n \mathrm{Tr}\frac{1}{p^2+\tilde{\omega}_n^2+M^2} \times\frac{1}{(\vec{p}+i\vec{\nabla})^2+\tilde{\omega}_n^2+M^2}(\partial_i\hat{A}_0)\epsilon_{ij}A_j\end{aligned}$$

$$=2ie^2M\sum_n \int d^2x \frac{d^2p}{(2\pi)^2}\frac{1}{p^2+\tilde{\omega}_n^2+M^2} \times\frac{1}{(\vec{p}+i\vec{\nabla})^2+\tilde{\omega}_n^2+M^2}\hat{A}_0 B. \tag{37}$$

Here the derivatives act only on $\hat{A}_0$ and not on B. The momentum integral can be evaluated by combining the denominators using the Feynman combination formula. Even the sum over the discrete frequency modes can also be exactly evaluated [see Eq. (19)] and the parity violating effective action containing one $\hat{A}_0$ field in addition to the B field has the form

$$(\Gamma_{eff}^{\mathrm{PV}(1)})^{(1)}=\frac{ie^2M\beta}{8\pi}\int d^2x\, d\alpha\, B\frac{1}{\sqrt{M^2-\alpha(1-\alpha)\nabla^2}} \times\Bigg[\tanh\frac{1}{2}[\beta\sqrt{M^2-\alpha(1-\alpha)\nabla^2}-iea] +\tanh\frac{1}{2}[\beta\sqrt{M^2-\alpha(1-\alpha)\nabla^2}+iea]\Bigg]\hat{A}_0. \tag{38}$$

This is an exact, closed form expression which can also be expanded in powers of derivatives, and takes the form

$$(\Gamma_{eff}^{\mathrm{PV}(1)})^{(1)}=\frac{ie^2M\beta}{8\pi}\sum_{s=0}^{\infty}\int d^2x\frac{B(s+1,s+1)}{s!}B[(-\nabla^2)^s\hat{A}_0] \times\frac{\partial^s}{(\partial M^2)^s}\left\{\frac{1}{M}\left[\tanh\frac{1}{2}(\beta M-iea) +\tanh\frac{1}{2}(\beta M+iea)\right]\right\}$$
$$=iM\beta\sum_{s=0}^{\infty}\int d^2x\frac{B(s+1,s+1)}{s!}B[(-\nabla^2)^s\hat{A}_0] \times\frac{\partial^s}{(\partial M^2)^s}\left(\frac{1}{M}\frac{\partial}{\partial a}\Gamma(a,M)\right), \tag{39}$$

which can be compared with the result from the momentum space calculation [given in Appendix A, Eq. (A18), recalling that the coefficients of the momentum space amplitudes are related to those of the real space amplitudes by a factor of i/β]. Let us explicitly write out the first few terms, which have the forms

$$(\Gamma_{eff}^{\mathrm{PV}(1)})^{(1)}=i\beta\int d^2x\, B\left[\hat{A}_0\frac{\partial\Gamma}{\partial a}-\frac{M}{6}(\nabla^2\hat{A}_0)\frac{\partial}{\partial M^2}\left(\frac{1}{M}\frac{\partial\Gamma}{\partial a}\right) +\frac{M}{60}(\nabla^4\hat{A}_0)\frac{\partial^2}{(\partial M^2)^2}\left(\frac{1}{M}\frac{\partial\Gamma}{\partial a}\right)+\cdots\right]. \tag{40}$$

We note here that this effective action will give an amplitude of type $\hat{A}_0$-B with any number of a insertions which can be thought of as zero momentum A_0 fields.

Without going into detail, let us simply note here that the parity violating effective action containing two $\hat{A}_0$ fields, in addition to the B field (and, of course, any number of a fields), can also be evaluated in a similar manner and has the form

$$(\Gamma_{eff}^{\mathrm{PV}(1)})^{(2)} =-4ie^3M\sum_n \mathrm{tr}\,\frac{\tilde{\omega}_n}{(p^2+\tilde{\omega}_n^2+M^2)[(\vec{p}+i\vec{\nabla}_1)^2+\tilde{\omega}_n^2+M^2]} \times\frac{1}{[(\vec{p}+i\vec{\nabla}_1+i\vec{\nabla}_2)^2+\tilde{\omega}_n^2+M^2]}\hat{A}_0^{(1)}\hat{A}_0^{(2)}B. \tag{41}$$

Here we have put indices on the derivatives as well as the $\hat{A}_0$ fields to indicate the action of these operators. The momentum integral as well as the sum over the discrete frequencies can also be carried out in this case, and the final form can be obtained in a closed form. However, let us make a power series expansion in the derivatives, and explicitly write the first few terms:

$$(\Gamma_{eff}^{\mathrm{PV}(1)})^{(2)}=i\beta^2\int d^2x\, B\Bigg[\frac{1}{2!}\hat{A}_0^2\frac{\partial^2\Gamma}{\partial a^2}-\frac{M}{12}[2(\nabla^2\hat{A}_0)\hat{A}_0 +(\vec{\nabla}\hat{A}_0)\cdot(\vec{\nabla}\hat{A}_0)]\frac{\partial}{\partial M^2}\left(\frac{1}{M}\frac{\partial^2\Gamma}{\partial a^2}\right) +\frac{M}{60}\bigg(2\hat{A}_0(\nabla^4\hat{A}_0)+4(\nabla^2\partial_i\hat{A}_0)(\partial_i\hat{A}_0) +\frac{4}{3}(\partial_i\partial_j\hat{A}_0)^2+\frac{5}{3}(\nabla^2\hat{A}_0)^2\bigg) \times\frac{\partial^2}{(\partial M^2)^2}\left(\frac{1}{M}\frac{\partial^2\Gamma}{\partial a^2}\right)+\cdots\Bigg]. \tag{42}$$

Calculations become algebraically more tedious as we go to higher orders. For example, the parity violating part of the effective action containing three $\hat{A}_0$ fields in addition to the B field (and any number of a fields) can also be evaluated, and has the form (before simplification)

$$(\Gamma_{eff}^{\mathrm{PV}(1)})^{(3)}=2ie^4M\sum_n \mathrm{tr}\Bigg[\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\hat{A}_0^2\frac{1}{p^2+\tilde{\omega}_n^2+M^2}(\partial_i\hat{A}_0)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\epsilon_{ij}A_j$$

$$+\frac{1}{p^2+\tilde{\omega}_n^2+M^2}(\partial_i\hat{A}_0)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\hat{A}_0^2\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\epsilon_{ij}A_j$$

$$+\frac{4\tilde{\omega}_n^2}{p^2+\tilde{\omega}_n^2+M^2}\hat{A}_0\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\hat{A}_0\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\hat{A}_0\frac{1}{p^2+\tilde{\omega}_n^2+M^2}B$$

$$-\frac{1}{p^2+\tilde{\omega}_n^2+M^2}(\partial_k\hat{A}_0)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}(\partial_k\hat{A}_0)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}(\partial_i\hat{A}_0)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}\epsilon_{ij}A_j$$

$$-\frac{\epsilon_{ij}}{p^2+\tilde{\omega}_n^2+M^2}(\partial_i\hat{A}_0)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}(\partial_j\hat{A}_0)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}(\partial_k\hat{A}_0)\frac{1}{p^2+\tilde{\omega}_n^2+M^2}A_k\Bigg]. \tag{43}$$

It is interesting to note that the expression above does not look manifestly invariant under *small* gauge transformations in Appendix B. We will give a proof of gauge invariance within the framework of a derivative expansion. For the present, let us simply note that if we were to evaluate this expression in powers of derivatives, the leading order term, which is linear in the derivatives, has the form

$$=-\frac{ie^4M}{3\pi^2}\sum_n\int \mathrm{d}^2x\,\mathrm{d}^2p\left[\frac{1}{(p^2+\tilde{\omega}_n^2+M^2)^3}-\frac{6\tilde{\omega}_n^2}{(p^2+\tilde{\omega}_n^2+M^2)^4}\right]B\hat{A}_0^3$$

$$=-\frac{ie^4M}{6\pi}\sum_n\int \mathrm{d}^2x\left[\frac{1}{(\tilde{\omega}_n^2+M^2)^2}-\frac{4\tilde{\omega}_n^2}{(\tilde{\omega}_n^2+M^2)^3}\right]B\hat{A}_0^3$$

$$=\frac{ie^2M\beta^2}{12\pi}\sum_n\int \mathrm{d}^2x\frac{\partial^2}{\partial a^2}\left(\frac{1}{\tilde{\omega}_n^2+M^2}\right)B\hat{A}_0^3$$

$$=i\beta^3\int \mathrm{d}^2x\frac{1}{3!}B\hat{A}_0^3\frac{\partial^3\Gamma}{\partial a^3}. \tag{44}$$

[This can again be compared with Eq. (A24).]

The term cubic (only odd powers of derivatives arise) in the derivatives has the form

$$=\frac{ie^4M}{2\pi^2}\sum_n\int \mathrm{d}^2x\,\mathrm{d}^2p\left(\frac{8\tilde{\omega}_n^2}{(p^2+\tilde{\omega}_n^2+M^2)^5}-\frac{1}{(p^2+\tilde{\omega}_n^2+M^2)^4}\right)[(\nabla^2\hat{A}_0)\hat{A}_0+(\vec{\nabla}\hat{A}_0)\cdot(\vec{\nabla}\hat{A}_0)]\hat{A}_0B$$

$$=-\frac{iM\beta^3}{12}\int \mathrm{d}^2x\,B\hat{A}_0[(\nabla^2\hat{A}_0)\hat{A}_0+(\vec{\nabla}\hat{A}_0)\cdot(\vec{\nabla}\hat{A}_0)]\frac{\partial}{\partial M^2}\left(\frac{1}{M}\frac{\partial^3\Gamma}{\partial a^3}\right) \tag{45}$$

[which agrees with Eq. (A32)]. Finally, the term fifth order in the derivatives has the form

$$=-\sum_n\frac{ie^4M}{20\pi(\tilde{\omega}_n^2+M^2)^5}\int \mathrm{d}^2x\Bigg[(7\tilde{\omega}_n^2-M^2)\Bigg(\hat{A}_0^2[(\nabla^2)^2\hat{A}_0]B+4\hat{A}_0(\partial_k\hat{A}_0)(\partial_k\nabla^2\hat{A}_0)B+\frac{5}{3}\hat{A}_0(\nabla^2\hat{A}_0)^2B+\frac{4}{3}\hat{A}_0(\partial_k\partial_l\hat{A}_0)^2B\Bigg)$$

$$+\left(11\tilde{\omega}_n^2-\frac{7M^2}{3}\right)(\partial_k\hat{A}_0)^2(\nabla^2\hat{A}_0)B+\left(\frac{40\tilde{\omega}_n^2}{3}-\frac{8M^2}{3}\right)(\partial_k\hat{A}_0)(\partial_l\hat{A}_0)(\partial_k\partial_l\hat{A}_0)B\Bigg]$$

$$=-\frac{iM}{120}\int \mathrm{d}^2x\,B\Bigg[\beta^3\Bigg[\hat{A}_0^2(\nabla^4\hat{A}_0)+4\hat{A}_0(\partial_i\hat{A}_0)(\nabla^2\partial_i\hat{A}_0)+\frac{5}{3}\hat{A}_0(\nabla^2\hat{A}_0)^2+\frac{4}{3}\hat{A}_0(\partial_i\partial_j\hat{A}_0)^2+\frac{5}{3}(\partial_i\hat{A}_0)^2(\nabla^2\hat{A}_0)$$

$$+2(\partial_i\hat{A}_0)(\partial_j\hat{A}_0)(\partial_i\partial_j\hat{A}_0)\Bigg]\frac{\partial^2}{(\partial M^2)^2}\left(\frac{1}{M}\frac{\partial^3\Gamma}{\partial a^3}\right)+\frac{4e^2\beta}{3}[(\partial_i\hat{A}_0)^2(\nabla^2\hat{A}_0)+(\partial_i\hat{A}_0)(\partial_j\hat{A}_0)(\partial_i\partial_j\hat{A}_0)]\frac{\partial^3}{(\partial M^2)^3}\left(\frac{1}{M}\frac{\partial\Gamma}{\partial a}\right)\Bigg]. \tag{46}$$

There are several things to note from these results. First, the results obtained up to fifth order in the derivatives above agree completely with the momentum space calculations (given in Appendix A). Second, even though the expression in Eq. (43) is not manifestly gauge invariant, terms up to fifth order in derivatives are explicitly invariant under *small* gauge transformations. A gauge invariant form of the effective action, derived above, needs the use of various algebraic identities, and it is not *a priori* clear that, at higher orders, gauge invariant expressions will be obtained. In Appendix B, we will show, within the framework of derivative expansion, that the PV effective action, linear in $\vec{A}$, is both gauge invariant and local.

IV. GENERAL FEATURES OF THE EFFECTIVE ACTION

It is clear that, while at every order the effective action can be determined in a closed form, its structure may not be that simple. On the other hand, from the analysis of the effective action up to fourth order (in fields) brings out some nice features that are worth discussing.

First, the structures in Eqs. (40), (42), and (44) suggest that, to all orders (in the $\hat{A}_0$ fields) the leading order term in the parity violating part of the effective action has the form (terms linear in the derivative)

$$(\Gamma_{eff}^{PV(1)})_1 = i\sum_{n=0}\int d^2x \frac{1}{n!}B(\beta\hat{A}_0)^n \frac{\partial^n\Gamma(a,M)}{\partial a^n} = i\int d^2x\, B\,\Gamma(a+\beta\hat{A}_0,M). \tag{47}$$

Here the subscript refers to the number of derivatives contained in the effective action. This gives the simple result that the leading order correction to the static result can be obtained completely from the static result itself. Furthermore, this action is invariant under *large* gauge transformations whenever the action with $\hat{A}_0=0$ is. Finally, we note that, at very high temperatures, $\beta\rightarrow 0$, so that the action reduces to Eq. (26), which is consistent with the fact that the action in Eq. (26) gives the leading terms of the parity violating action at high temperatures.

Even the next order terms in the expansion (namely, third order in derivatives) in Eqs. (40), (42), and (45) seem to have a nice structure and, with a little bit of analysis, suggest that they can be summed to a simple form. This can be done in the following way. Let us recall that we are interested in evaluating the effective action (up to normalization)

$$\Gamma_{eff} = -\sum_n \mathrm{Tr}\ln\left\{\not p + e\not A + \gamma_0\left[\omega_n + e\left(\frac{a}{\beta}+\hat{A}_0\right)+M\right] = -\sum_n \mathrm{Tr}\ln(\not p + e\not A + \gamma_0\overline{\omega}_n + M)\right\}, \tag{48}$$

where we have defined

$$\overline{\omega}_n = \omega_n + \frac{e}{\beta}(a+\beta\hat{A}_0) = \frac{(2n+1)\pi}{\beta} + \frac{e}{\beta}(a+\beta\hat{A}_0). \tag{49}$$

Following Eqs. (12) and (13), we can now define [8]

$$\overline{\rho}_n = \sqrt{\overline{\omega}_n^2 + M^2}, \qquad \overline{\phi}_n = \tan^{-1}\left(\frac{\overline{\omega}_n}{M}\right), \tag{50}$$

where $\overline{\rho}_n$ and $\overline{\phi}_n$ are now coordinate dependent because of the presence of $\hat{A}_0$. The effective action, in these variables, takes the form

$$\begin{aligned}\Gamma_{eff} &= -\sum_n \mathrm{Tr}\ln(\not p + e\not A + \overline{\rho}_n e^{\gamma_0\overline{\phi}_n})\\ &= -\sum_n \mathrm{Tr}\ln e^{\gamma_0\overline{\phi}_n/2}\left(\not p + e\not A + \overline{\rho}_n - \frac{i}{2}\gamma_0(\not\partial\overline{\phi}_n)\right)e^{\gamma_0\overline{\phi}_n/2}\\ &= -\sum_n\left\{\mathrm{Tr}\ln\left(\not p + e\not A + \overline{\rho}_n - \frac{i}{2}\gamma_0(\not\partial\overline{\phi}_n)\right) + \frac{ie}{2\pi}\int d^2x\left[\overline{\phi}_n\epsilon_{jk}\partial_j A_k + \frac{1}{4}\overline{\phi}_n\nabla^2\overline{\phi}_n\right]\right\},\end{aligned} \tag{51}$$

where the terms in the square brackets arise from the Jacobian of the $(1+1)$-dimensional chiral rotation [7,8]. Note that the contribution which is quadratic in $\overline{\phi}_n$ is irrelevant to the parity-breaking part, because it is invariant under the change $M\rightarrow -M$.

The parity violating part of the effective action can be obtained from Eq. (51) through a derivative expansion and would have an odd number of $\overline{\phi}_n$ terms. The action is gauge invariant [7] (also see Appendix B) and, if we are interested in terms linear in $\vec{A}$, would depend linearly on B as well as terms with derivatives acting on $\overline{\rho}_n$ and $\overline{\phi}_n$. From the definition of these variables, we see that $\overline{\rho}_n$ has the canonical dimension of energy while $\overline{\phi}_n$ is dimensionless. This allows us to organize the successive terms in the expansion.

At the order of terms cubic in the derivatives, let us note that the most general local term we can write for the parity violating effective action will have the form

$$\begin{aligned}(\Gamma_{eff}^{PV(1)})_3 &= \sum_n\int d^2x\, B\left[b_1\frac{(\nabla^2\overline{\phi}_n)}{\overline{\rho}_n^2} + b_2\frac{(\vec{\nabla}\overline{\rho}_n)\cdot(\vec{\nabla}\overline{\phi}_n)}{\overline{\rho}_n^3}\right]\\ &= \sum_n\int d^2x\, B\left[eMb_1\left(\frac{(\nabla^2\hat{A}_0)}{\overline{\rho}_n^4} - \frac{2e\overline{\omega}_n}{\overline{\rho}_n^6}(\vec{\nabla}\hat{A}_0)\cdot(\vec{\nabla}\hat{A}_0)\right) + e^2Mb_2\frac{\overline{\omega}_n}{\overline{\rho}_n^6}(\vec{\nabla}\hat{A}_0)\cdot(\vec{\nabla}\hat{A}_0)\right].\end{aligned} \tag{52}$$

It is clear that the contribution of the first term starts with terms of the type $B\hat{A}_0$, while the second structure has contribution starting with $B\hat{A}_0^3$. Consequently, the coefficients b_1 and b_2 can be identified from our earlier calculations [see Eqs. (40) and (45)] and take the values $b_1 = ie/12\pi$ and $b_2 = 0$. Thus the parity violating part of the effective action which is cubic in the derivatives can be written as

$$\begin{aligned}(\Gamma_{eff}^{PV(1)})_3 &= \frac{ie^2M}{12\pi}\sum_n \int d^2x\, B\left[\frac{(\nabla^2\hat{A}_0)}{(\overline{\omega}_n^2+M^2)^2} - \frac{2e\overline{\omega}_n}{(\overline{\omega}_n^2+M^2)^3}(\vec{\nabla}\hat{A}_0)\cdot(\vec{\nabla}\hat{A}_0)\right]\\ &= -\frac{ie^2M}{24\pi}\sum_n \int d^2x\, B\frac{\partial}{\partial M^2}\left[\frac{(\nabla^2\hat{A}_0)}{(\overline{\omega}_n^2+M^2)} + \frac{1}{eM}\left(\nabla^2\tan^{-1}\frac{\overline{\omega}_n}{M}\right)\right]\\ &= -\frac{iM}{12}\int d^2x\, B\left[\beta(\nabla^2\hat{A}_0)\frac{\partial}{\partial a}+\nabla^2\right]\times\frac{\partial}{\partial M^2}\left(\frac{\Gamma(a+\beta\hat{A}_0,M)}{M}\right). \end{aligned} \tag{53}$$

If we go to the next order, namely terms containing five derivatives, there are 12 possible structures that arise. However, let us note from the terms with three derivatives, that terms with $\overline{\phi}_n$ appear only in combination with ∇^2 acting on them. [This corresponds to finding $b_2 = 0$ in Eq. (52).] If a similar pattern continues to hold at orders higher than the box amplitude, the parity violating effective action with five derivatives can be uniquely determined from our results for the two and four point amplitudes [see Eqs. (40) and (46)], and takes the simple form

$$\begin{aligned}(\Gamma_{eff}^{PV(1)})_5 = -\frac{ie}{60\pi}\sum_n \int d^2x\, B &\times\left[\frac{\nabla^4\overline{\phi}_n}{\overline{\rho}_n^4} - \frac{3(\nabla^2\overline{\phi}_n)(\nabla^2\overline{\rho}_n)+4(\partial_i\overline{\rho}_n)(\partial_i\nabla^2\overline{\phi}_n)}{\overline{\rho}_n^5}\right.\\ &\left. + \frac{7(\nabla^2\overline{\phi}_n)(\partial_i\overline{\rho}_n)^2}{\overline{\rho}_n^6}\right]. \end{aligned} \tag{54}$$

Upon doing the sum over the discrete frequencies, this determines the following form for the corresponding all orders (in fields) effective action

$$\begin{aligned}(\Gamma_{eff}^{PV(1)})_5 = -\frac{iM}{30}\int \mathrm{d}^2x\, B\Bigg\{ & e^2\beta^2[(\partial_i\hat{A}_0)^2]^2\left(\frac{13}{8}+\frac{5}{12}M^2\frac{\partial}{\partial M^2}\right)\frac{\partial}{\partial a}\frac{\partial}{\partial M^2}\\ & + e^2\beta\left[(\partial_i\hat{A}_0)^2(\nabla^2\hat{A}_0)\left(\frac{11}{2}+\frac{5}{3}M^2\frac{\partial}{\partial M^2}\right)+(\partial_i\hat{A}_0)(\partial_j\hat{A}_0)(\partial_i\partial_j\hat{A}_0)\left(\frac{20}{3}+2M^2\frac{\partial}{\partial M^2}\right)\right]\frac{\partial}{\partial M^2}\\ & -\beta^2\left[(\partial_i\hat{A}_0)(\partial_i\nabla^2\hat{A}_0)+\frac{5}{12}(\nabla^2\hat{A}_0)^2+\frac{1}{3}(\partial_i\partial_j\hat{A}_0)^2\right]\frac{\partial}{\partial a}-\frac{\beta}{2}(\nabla^4\hat{A}_0)\Bigg\}\frac{\partial}{\partial a}\left(\frac{\partial}{\partial M^2}\right)^2\left(\frac{\Gamma(a+\beta\hat{A}_0,M)}{M}\right). \end{aligned} \tag{55}$$

This discussion makes it clear that such an analysis can be carried out systematically to any order in the derivatives (of course, one needs to calculate higher point functions), which, in turn, would determine the corresponding all order effective action. Interestingly, all such effective actions can be determined completely from a knowledge of the leading order parity violating action in the static limit.

V. CONCLUSION

In this paper, we have tried to go beyond the leading order term in the static limit of the induced parity violating effective action for thermal QED_{2+1} using the derivative expansion. We have discussed the various subtleties that arise in using derivative expansion in such a theory, and have improved and extended the earlier proposed method [16] for calculating the leading order term in this approach. We have shown, in this approach, that the leading order term in the static limit of the parity violating thermal effective action, is linear in the $\vec{A}$ field. In going beyond the leading order we have used the derivative expansion in the coordinate space to determine the parity violating effective action up to fourth order in fields (linear in $\vec{A}$) All these actions can be obtained in closed form (namely, powers of derivatives can be summed) in principle. However, their forms are neither very illuminating nor useful. In contrast, at any given order of the derivatives, we can sum the effective action containing all possible A_0 fields. The resulting effective actions are determined completely by the leading order action in the static limit. We have also shown, within the framework of the derivative expansion, that all the higher order terms, which are linear in $\vec{A}$, are *large* gauge invariant and local. We have tried to discuss the possible origin of the interesting structure of the higher order terms that arise in the derivative expansion.

ACKNOWLEDGMENTS

This work was supported in part by U.S. DOE Grant No. DE-FG 02-91ER40685, and by CNPq and FAPSEP, Brazil.

APPENDIX A: MOMENTUM SPACE CALCULATIONS

In this appendix we will present the main results of the perturbative momentum space one-loop amplitudes which can be derived from the Lagrangian density given by Eq. (1), in the framework of finite temperature field theory [11,20,21]. The main results are given in Eqs. (A18), (A24), (A32), and (A33).

In order to simplify the presentation of the results, we express the N-point amplitudes in terms of Bose symmetric combinations of the basic quantities

$$\mathcal{A}_{\mu_1\cdots\mu_N}(\{k\};M)=-\frac{e^N}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int \mathrm{d}^2\vec{p}\times\frac{\mathcal{N}_{\mu_1\cdots\mu_N}(p,\{k\};M)}{(p^2-M^2)\cdots[(p+k_{1(N-1)})^2-M^2]}, \tag{A1}$$

where $\{k\}\equiv k_1,\ldots,k_{N-1}$ represents the set of $N-1$ independent external 3-momenta, $k_{1i}\equiv k_1+k_2+\cdots+k_i$, and

$$\mathcal{N}_{\mu_1\cdots\mu_N}=\mathrm{Tr}[\gamma_{\mu_1}(\not{p}+\not{k}_1+M)\times\gamma_{\mu_2}(\not{p}+\not{k}_{12}+M)\cdots\gamma_{\mu_N}(\not{p}+M)]. \tag{A2}$$

The external bosonic lines in Eq. (A1) are such that the zero component of its 3-momenta is quantized and purely imaginary (for instance $k_{10}=2i\pi l/\beta$, with $l=0,\pm1,\pm2,\ldots$). Similarly, the zero component of the 3-momenta associated with a fermion loop is given by

$$p_0=\frac{i\pi(2n+1)}{\beta}\equiv i\omega_n,\quad n=0,\pm1,\pm2,\ldots. \tag{A3}$$

Every thermal N-point amplitude is the sum of a parity violating and a parity conserving part. In what follows, we will concentrate only on the former, which can be written as [also see Eq. (16)]

$$\mathcal{A}^{\mathrm{PV}}_{\mu_1\cdots\mu_N}(\{k\};M)=\frac{1}{2}[\mathcal{A}_{\mu_1\cdots\mu_N}(\{k\};M)-\mathcal{A}_{\mu_1\cdots\mu_N}(\{k\};-M)]. \tag{A4}$$

Since the denominator in Eq. (A1) is an even function of M, only the odd powers of M from the numerator $\mathcal{N}_{\mu_1\cdots\mu_N}$ in Eq. (A4) will contribute to $\mathcal{A}^{\mathrm{PV}}_{\mu_1\cdots\mu_N}$. Consequently, the parity violating parts of thermal amplitudes come only from those terms in Eq. (A2) which involve the trace of an odd number of Dirac gamma matrices.

Expressing the two terms on the right hand side of Eq. (A4) in terms of the integral in Eq. (A1) and performing the change of variable $p\to -p$, we can easily verify that

$$\mathcal{A}^{\mathrm{PV}}_{\mu_1\cdots\mu_N}(\{-k\};M)=(-1)^{N+1}\mathcal{A}^{\mathrm{PV}}_{\mu_1\cdots\mu_N}(\{k\};M). \tag{A5}$$

This result confirms that the procedure of antisymmetrization in the mass gives a result which is in agreement with the usual concept of parity violation, according to which the N-point amplitude is odd under the concomitant interchange of the sign of all external gauge fields as well as their respective momenta.

Of course, we do not expect to be able to compute the amplitudes $\mathcal{A}^{\mathrm{PV}}_{\mu_1\cdots\mu_N}$ for general arbitrary momenta at finite temperature. This is because, at finite temperature, amplitudes are nonanalytic and, therefore, one can at best describe them in some limit. In what follows, we will calculate the thermal amplitudes in the static limit $k_{i0}=0$, where *large gauge* invariance is known to be an issue. In this limit, the parity violating part of the basic amplitudes can be written as

$$\mathcal{A}^{\mathrm{static,PV}}_{\mu_1\cdots\mu_N}(\{k\};M)=(-1)^{N+1}\frac{e^N}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int \mathrm{d}^2\vec{p}\times\frac{\mathcal{N}^{\mathrm{static,PV}}_{\mu_1\cdots\mu_N}(p,\{k\};M)}{(\vec{p}^2+M_\omega^2)\cdots[(\vec{p}+\vec{k}_{1(N-1)})^2+M_\omega^2]}, \tag{A6}$$

where $M_\omega^2\equiv\omega_n^2+M^2$, with ω_n given by Eq. (A3), and

$$\mathcal{N}^{\mathrm{static,PV}}_{\mu_1\cdots\mu_N}=\frac{1}{2}[\mathcal{N}_{\mu_1\cdots\mu_N}(p,\{k\};M)-\mathcal{N}_{\mu_1\cdots\mu_N}(p,\{k\};-M)]|_{k_{10},\ldots,k_{(N-1)0}=0}. \tag{A7}$$

To evaluate the two dimensional integral in Eq. (A6), we can use the standard Feynman parametrization to combine the N denominators. After performing appropriate shifts, the integration over $\vec{p}$ can be easily performed. A closed form expression for the Feynman parameter integrals can be obtained in the limit $|\vec{k}_i|\ll M_\omega$, in which case we can employ a derivative expansion (the term "derivative" is reminiscent of the configuration space transformation $k_i\to -i\partial_{x_i}$).

In the Abelian theory all the odd point amplitudes vanish simply because of charge conjugation invariance. Let us consider the even functions. From Eq. (A6), with $N=2$, the self-energy is given by

$$\Pi^{\text{static,PV}}_{\mu_1\mu_2}(k) \equiv \mathcal{A}^{\text{static,PV}}_{\mu_1\mu_2}(k;M)$$

$$= -\frac{e^2}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int d^2\vec{p}$$

$$\times \frac{\mathcal{N}^{\text{static,PV}}_{\mu_1\mu_2}(p,k;M)}{(\vec{p}^2+M_\omega^2)[(\vec{p}+\vec{k})^2+M_\omega^2]}. \quad \text{(A8)}$$

Using the Feynman combination formula, we can write

$$\Pi^{\text{static,PV}}_{\mu_1\mu_2}(k) = -\frac{e^2}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int_0^1 d\alpha \int d^2\vec{p}$$

$$\times \frac{\mathcal{N}^{\text{static,PV}}_{\mu_1\mu_2}(p,k;M)}{[(\vec{p}+\alpha\vec{k})^2+\alpha(1-\alpha)\vec{k}^2+M_\omega^2]^2}. \quad \text{(A9)}$$

Performing the simple trace of three gamma matrices one easily obtains

$$\mathcal{N}^{\text{static,PV}}_{\mu_1\mu_2} = (2M\, k_\alpha \epsilon_{\alpha,\mu_1,\mu_2})^{\text{static}} = 2M\, k_j \epsilon_{j\mu_1\mu_2}. \quad \text{(A10)}$$

Since we are in the static limit, namely $k_0=0$, either the index μ_1 or μ_2 has to be in the time direction. Choosing $\mu_1=0$ and $\mu_2=i$, and noting that

$$\epsilon_{0ij} \equiv \epsilon_{ij}, \quad \text{(A11)}$$

we obtain

$$\Pi^{\text{static,PV}}_{0i}(k) = -\epsilon_{ij}k_j \frac{2e^2M}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int_0^1 d\alpha \int d^2\vec{p}$$

$$\times \frac{1}{[\vec{p}^2+\alpha(1-\alpha)\vec{k}^2+M_\omega^2]^2}, \quad \text{(A12)}$$

where we have performed the shift $\vec{p}\to\vec{p}-\alpha\vec{k}$. The integration in $\vec{p}$ is now elementary, giving the result

$$\Pi^{\text{static,PV}}_{0i}(k) = -\epsilon_{ij}k_j \frac{e^2M}{2\pi\beta}\sum_{n=-\infty}^{\infty}\int_0^1 d\alpha$$

$$\times \frac{1}{\alpha(1-\alpha)\vec{k}^2+M^2+\omega_n^2}. \quad \text{(A13)}$$

We can now proceed in one of two ways, namely, either perform a derivative expansion, as described earlier, or perform the sum over n, using the formula

$$\mathcal{S}(\mu) \equiv \sum_{n=-\infty}^{\infty}\frac{1}{\mu^2+\omega_n^2} = \frac{1}{2\mu T}\tanh\left(\frac{\mu}{2T}\right), \quad \text{(A14)}$$

with $\mu = \sqrt{\alpha(1-\alpha)\vec{k}^2+M^2}$. Using the latter approach, we obtain

$$\Pi^{\text{static,PV}}_{0i}(k) = -\epsilon_{ij}k_j \frac{e^2M}{4\pi}\int_0^1 d\alpha$$

$$\times \frac{\tanh\left(\frac{\beta\sqrt{\alpha(1-\alpha)\vec{k}^2+M^2}}{2}\right)}{\sqrt{\alpha(1-\alpha)\vec{k}^2+M^2}}. \quad \text{(A15)}$$

This expression shows that even in the simplest case of the one-loop self-energy in the static limit, one cannot obtain a simple closed form expression. Of course, the integration over the Feynman parameter can be performed order by order using a derivative expansion of Eqs. (A13) or (A15). It is clear from Eq. (A15) that, at any $2s+1$ order, the polynomial in the Feynman parameter can be systematically expressed in terms of Euler's beta function B which is defined as

$$\mathrm{B}(s+1,s+1) = \int_0^1 d\alpha\, \alpha^s(1-\alpha)^s, \quad \text{(A16)}$$

so that the expansion of the integrand in Eq. (A13) in powers of $(\vec{k})^2$ yields

$$\Pi^{\text{static,PV}}_{0i}(k) = -\epsilon_{ij}k_j \frac{e^2M}{2\pi}\sum_{s=0}^{\infty}(-1)^s \mathrm{B}(s+1,s+1)$$

$$\times (\vec{k}^2)^s \frac{(-1)^s}{s!}\frac{\partial^s}{(\partial M^2)^s}\left[\frac{1}{2M}\tanh\left(\frac{M}{2T}\right)\right]. \quad \text{(A17)}$$

Using Eqs. (19) and (25), we finally obtain

$$\Pi^{\text{static,PV}}_{0i}(k) = -\epsilon_{ij}k_j M \sum_{s=0}^{\infty}\frac{\mathrm{B}(s+1,s+1)}{s!}(\vec{k}^2)^s \frac{\partial^s}{(\partial M^2)^s}$$

$$\times \left(\frac{1}{M}\frac{\partial}{\partial a}\Gamma(a,M)\bigg|_{a=0}\right). \quad \text{(A18)}$$

Equation (A18) gives the momentum space two point amplitude which is obtained from the parity violating, quadratic effective action by taking functional derivative with respect to $A_0(\vec{k})$ and $A_i(-\vec{k})$.

Let us next consider the box diagram which is obtained from Eq. (A6) with $N=4$:

$$\mathcal{A}^{\text{static,PV}}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3;M)=-\frac{e^4}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int d^2\vec{p}\,\frac{\mathcal{N}^{\text{static,PV}}_{\mu_1\mu_2\mu_3\mu_4}(p,k_1,k_2,k_3;M)}{(\vec{p}^2+M_\omega^2)[(\vec{p}+\vec{k}_1)^2+M_\omega^2][(\vec{p}+\vec{k}_{12})^2+M_\omega^2][(\vec{p}+\vec{k}_{123})^2+M_\omega^2]}. \tag{A19}$$

From our experience with the previous example the self-energy, we do not expect to obtain a closed form for Eq. (A19) for arbitrary values of $\vec{k}$. Therefore, right from the beginning we will adopt the derivative approximation $\vec{k}_i \ll M_\omega$. Furthermore, instead of trying to obtain the general term of the series, we will separately analyze each individual order up to the fifth order in the external momenta and consider the specific components $\mu_1=\mu_2=\mu_3=0$ and $\mu_4=i$, which correspond to the part of the effective action containing three A_0 fields and one magnetic field.

The parity violating numerator in Eq. (A19) is an odd function of the external momenta which can have degree one or three [this can easily be verified from Eq. (A2) and the definition of parity violating numerator as an antisymmetric function of M]. Making the external momenta equal to zero inside the denominators, and keeping only the linear contribution from the numerator in Eq. (A19), we obtain the leading linear contribution

$$\mathcal{A}^{\text{static,PV}}_{000i}\Big|^{(1)}=-\frac{e^4}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int d^2\vec{p}\times\frac{\mathcal{N}^{\text{static,PV}}_{000i}(p,k_1,k_2,k_3;M)|^{(1)}}{(\vec{p}^2+M^2+\omega_n^2)^4}, \tag{A20}$$

where

$$\mathcal{N}^{\text{static,PV}}_{000i}\Big|^{(1)}=2M\epsilon_{ij}[(3k_1+4k_2+3k_3)_j\omega_n^2-(k_1+k_3)_j(\vec{p}^2+M^2)] \tag{A21}$$

comes from the trace computation. Substituting Eq. (A21) into Eq. (A20), and performing the six permutations of the external momenta and indices yields the following Bose symmetric expression for the box diagram ($\Pi^{\text{static,PV}}_{000i}$ is the sum of six permutations of $\mathcal{A}^{\text{static,PV}}_{000i}$)

$$\Pi^{\text{static,PV}}_{000i}\Big|^{(1)}=-8\epsilon_{ij}k_{4j}\frac{e^4M}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int d^2\vec{p}\times\frac{\vec{p}^2+M^2-5\omega_n^2}{(\vec{p}^2+M^2+\omega_n^2)^4}, \tag{A22}$$

where we have used the momentum conservation $k_1+k_2+k_3=-k_4$. Performing the integration over $\vec{p}$ in Eq. (A22), we obtain

$$\Pi^{\text{static,PV}}_{000i}\Big|^{(1)}=-2\,\epsilon_{ij}k_{4j}\frac{e^4M}{2\pi\beta}\times\sum_{n=-\infty}^{\infty}\left[\frac{4M^2}{(M^2+\omega_n^2)^3}-\frac{3}{(M^2+\omega_n^2)^2}\right]. \tag{A23}$$

Using Eq. (A14) we can perform the sum and express the result in terms of derivatives of Eq. (25) in the following way:

$$\Pi^{\text{static,PV}}_{000i}\Big|^{(1)}=\epsilon_{ij}k_{4j}\,\beta^2\frac{\partial^3}{\partial a^3}\Gamma(a,M)\Bigg|_{a=0}. \tag{A24}$$

In order to obtain the higher order derivative contributions, we will have to take into account the external momenta dependence inside the denominators of Eq. (A19). Using the Feynman combination formula we can write

$$\mathcal{A}^{\text{static,PV}}_{000i}=-\frac{6e^4}{(2\pi)^2\beta}\sum_{n=-\infty}^{\infty}\int_0^1 d\alpha_1\int_0^{1-\alpha_1}d\alpha_2\int_0^{1-\alpha_2}d\alpha_3\int d^2\vec{p}\,\frac{\mathcal{N}^{\text{static,PV}}_{000i}(p_0,\vec{p}-\alpha_1\vec{k}_1-\alpha_2\vec{k}_{12}-\alpha_3\vec{k}_{13},\vec{k}_1,\vec{k}_2,\vec{k}_3;M)}{(\vec{p}^2+M^2+\omega_n^2+K^2)^4}, \tag{A25}$$

where $\vec{k}_{12}\equiv\vec{k}_1+\vec{k}_2$, $\vec{k}_{13}\equiv\vec{k}_1+\vec{k}_2+\vec{k}_3$ and

$$K^2\equiv\vec{k}_1^2\alpha_1(1-\alpha_1)+\vec{k}_{12}^2\alpha_2(1-\alpha_2)+\vec{k}_{13}^2\alpha_3(1-\alpha_3)-2(\vec{k}_1\cdot\vec{k}_{12}\alpha_1\alpha_2+\vec{k}_2\cdot\vec{k}_{13}\alpha_2\alpha_3+\vec{k}_1\cdot\vec{k}_{13}\alpha_1\alpha_3). \tag{A26}$$

Except for structures like

$$Mp_ip_jk_l\epsilon_{jl}\quad\text{or}\quad Mp_lp_lk_j\epsilon_{ij}$$

which appear in the numerator $\mathcal{N}^{\text{static,PV}}_{000i}$, the $d^2\vec{p}$ integration in Eq. (A25) is as straightforward as the ones that arose in the self-energy calculation. In order to obtain a simple scalar integral we first perform the elementary angular integrations with the help of

$$\int_0^{2\pi} \mathrm{d}\theta\, p_i\, p_j = \pi \vec{p}^2 \delta_{ij}\,. \tag{A27}$$

In this way, Eq. (A25) leads to

$$\mathcal{A}_{000i}^{\text{static,PV}} = -\frac{6e^4}{2\pi\beta}\sum_{n=-\infty}^{\infty}\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1}\mathrm{d}\alpha_2 \int_0^{1-\alpha_2}\mathrm{d}\alpha_3 \times \int_0^\infty p\,\mathrm{d}p \frac{n_i^{(1)}\vec{p}^2 + N_i^{(1)} + N_i^{(3)}}{(\vec{p}^2+M^2+\omega_n^2+K^2)^4}. \tag{A28}$$

The compact notation in the numerator of Eq. (A28) means that $n_i^{(1)}$ and $N_i^{(1)}$ are of first order in the external momenta, while $N_i^{(3)}$ is of third order in the external momenta. [Of course, the algebra has become very much involved by now. Just to give an idea of how involved it is, the numerator in Eq. (A28) contains 242 terms.] Performing the integration in dp and expanding the result up to fifth order in the external momenta yields the following third and fifth order expressions:

$$\mathcal{A}_{000i}^{\text{static,PV}}\big|^{(3)} = -\frac{e^4}{2\pi\beta}\sum_{n=-\infty}^{\infty}\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1}\mathrm{d}\alpha_2 \int_0^{1-\alpha_2}\mathrm{d}\alpha_3 \times\left[\frac{N_i^{(3)}-K^2 n_i^{(1)}}{(M^2+\omega_n^2)^3} - 3\frac{K^2 N_i^{(1)}}{(M^2+\omega_n^2)^4}\right] \tag{A29}$$

and

$$\mathcal{A}_{000i}^{\text{static,PV}}\big|^{(5)} = -\frac{e^4}{2\pi\beta}\sum_{n=-\infty}^{\infty}\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1}\mathrm{d}\alpha_2 \int_0^{1-\alpha_2}\mathrm{d}\alpha_3 \times\left[\frac{6K^4 N_i^{(1)}}{(M^2+\omega_n^2)^5} + \frac{3}{2}\,\frac{K^4 n_i^{(1)} - 2K^2 N_i^{(3)}}{(M^2+\omega_n^2)^4}\right]. \tag{A30}$$

The parametric integrals in the above expressions are very involved, but straightforward, since there are only powers of the Feynman parameters. As in the previous cases, the sum over discrete energy can also be performed using Eq. (A14) and the result can be expressed in terms of derivatives of $\Gamma(a,M)$ defined in Eq. (25). The complete four photon amplitude is then obtained adding the six permutations of external momenta and indices.

Of course, the final result must preserve the small gauge invariance, being proportional to $\epsilon_{ij}k_{4j}$, like the leading order result given by Eq. (A24), so that the contraction with k_{4i} gives zero [this is a consequence of the invariance under a small gauge transformation $\vec{A}(k_4)\rightarrow\vec{A}(k_4)+\vec{k}_4$ in the momentum space]. However, at this higher order, our explicit calculation shows that the small gauge invariance will only be explicitly manifest, when we make use of some identities involving the two-dimensional vectors. A simple example is the Jacobi identity

$$(k_{1l}k_{2m}k_{3i}+k_{2l}k_{3m}k_{1i}+k_{3l}k_{1m}k_{2i})\epsilon_{lm}=0. \tag{A31}$$

The emergence of these identities is, in fact, expected, because the very nature of the sub-leading contributions (higher powers of the external momenta) leaves room to write the two-dimensional structures involving ϵ_{ij} and the vectors $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ in many equivalent ways. Our strategy to single out the unique gauge invariant form, was to decompose each vector in a two-dimensional basis and verify (by brute force, using the computer) that the unique function of the components is indeed gauge invariant. Then, from the expressions in terms of components, we were able to identify the two-dimensional scalar functions which multiplies $\epsilon_{ij}k_{4j}$. This leads to the following results:

$$\begin{aligned}\Pi_{000i}^{\text{static,PV}}\big|^{(3)} &= \frac{e^4}{3\pi}\epsilon_{ij}k_{4j}(\vec{k}_1^2+\vec{k}_2^2+\vec{k}_3^2+\vec{k}_1\cdot\vec{k}_2+\vec{k}_2\cdot\vec{k}_3 + \vec{k}_2\cdot\vec{k}_3)\frac{M}{\beta}\sum_{n=-\infty}^{\infty}\frac{M^2-5\omega_n^2}{(M^2+\omega_n^2)^4}\\ &= \frac{\epsilon_{ij}k_{4j}}{6}(\vec{k}_1^2+\vec{k}_2^2+\vec{k}_3^2+\vec{k}_1\cdot\vec{k}_2+\vec{k}_2\cdot\vec{k}_3+\vec{k}_2\cdot\vec{k}_3)\\ &\quad\times M\beta^2\frac{\partial}{\partial M^2}\left[\frac{1}{M}\,\frac{\partial^3}{\partial a^3}\Gamma(a,M)\right]_{a=0}\end{aligned} \tag{A32}$$

and

$$\Pi_{000i}^{\text{static,PV}}\big|^{(5)} = \frac{e^4 M}{30\pi\beta}\epsilon_{ij}k_{4j}\Bigg\{(\vec{k}_1^2\vec{k}_2\cdot\vec{k}_3+\vec{k}_1\cdot\vec{k}_2\vec{k}_2\cdot\vec{k}_3)\sum_{n=-\infty}^{\infty}\frac{2}{(M^2+\omega_n^2)^4} - [\vec{k}_1^2(3\vec{k}_1^2+6\vec{k}_1\cdot\vec{k}_2+6\vec{k}_1\cdot\vec{k}_3+5\vec{k}_2^2+5\vec{k}_2\cdot\vec{k}_3) + 4(\vec{k}_1\cdot\vec{k}_2)^2+6\vec{k}_1\cdot\vec{k}_2\vec{k}_2\cdot\vec{k}_3]\sum_{n=-\infty}^{\infty}\frac{7\omega_n^2-M^2}{(M^2+\omega_n^2)^5}\Bigg\} + (\text{two cyclic permutations of})\ \ \vec{k}_1,\ \ \vec{k}_2,\ \text{and}\ \ \vec{k}_3$$

$$=-\frac{M}{60}\epsilon_{ij}k_{4j}\Bigg\{(\vec{k}_1^2\vec{k}_2\cdot\vec{k}_3+\vec{k}_1\cdot\vec{k}_2\vec{k}_2\cdot\vec{k}_3)\frac{4e^2}{3}\frac{\partial^3}{(\partial M^2)^3}\left(\frac{1}{M}\frac{\partial\Gamma}{\partial a}\right)_{a=0}+[\vec{k}_1^2(3\vec{k}_1^2+6\vec{k}_1\cdot\vec{k}_2+6\vec{k}_1\cdot\vec{k}_3+5\vec{k}_2^2+5\vec{k}_2\cdot\vec{k}_3)$$
$$+4(\vec{k}_1\cdot\vec{k}_2)^2+6\vec{k}_1\cdot\vec{k}_2\vec{k}_2\cdot\vec{k}_3]\frac{\beta^2}{3}\frac{\partial^2}{(\partial M^2)^2}\left(\frac{1}{M}\frac{\partial^3\Gamma}{\partial a^3}\right)\Bigg\}_{a=0}+(\text{two cyclic permutations of})\ \ \vec{k}_1,\ \vec{k}_2,\ \text{and}\ \vec{k}_3. \tag{A33}$$

APPENDIX B: *SMALL* GAUGE INVARIANCE AND LOCALITY

It is known [7] that the effective action resulting from a fermion interacting with a gauge background is small gauge invariant. However, as we saw in Sec. III, within the framework of derivative expansion, gauge invariance is not manifest. There is also an issue of locality of the resulting effective action in this approach. In this appendix we show, within the derivative expansion, that the PV action linear in $\vec{A}$ is both small gauge invariant and local.

Let us consider the effective action in Eq. (32) which is linear in $\vec{A}$. If we now make a gauge transformation, $A\!\!\!/\rightarrow A\!\!\!/+\partial\!\!\!/\alpha$, where α is the parameter of transformation, then the change in the effective action is given by

$$\delta\Gamma_{eff}^{(1)}=-e\sum_n \mathrm{Tr}\frac{1}{p\!\!\!/+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M}(\partial\!\!\!/\alpha). \tag{B1}$$

Let us now use the standard canonical commutation relation

$$[p_i,\alpha]=-i(\partial_i\alpha)$$

as well as the cyclicity of the trace (we note here that the zeroth order term in this expression is the only term that needs regularization and we have already seen that it is manifestly gauge invariant. The higher order terms are well defined and satisfy cyclicity of trace.) to write

$$\delta\Gamma_{eff}^{(1)}=ie\sum_n \mathrm{Tr}\left[p\!\!\!/,\frac{1}{p\!\!\!/+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M}\right]\alpha. \tag{B2}$$

Let us next write

$$p\!\!\!/=p\!\!\!/+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M-[\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M]. \tag{B3}$$

Using this leads to

$$\delta\Gamma_{eff}^{(1)}=ie\sum_n \mathrm{Tr}\Bigg[-[\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M]\times\frac{1}{p\!\!\!/+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M}+\frac{1}{p\!\!\!/+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M}\times[\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M]\Bigg]\alpha$$
$$=ie\sum_n \mathrm{Tr}\Bigg[-[\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M]\times\frac{1}{p\!\!\!/+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M}[\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M]\times\frac{1}{p\!\!\!/+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M}\Bigg]\alpha$$
$$=0. \tag{B4}$$

Here we have used the cyclicity of the trace in the second term and the fact that the factor in the numerator is a multiplicative operator which commutes with α. This proves that the expression that we are interested in is invariant under *small* gauge transformations, even though it may not be manifest.

This, therefore, raises the question as to whether we can have a derivative expansion which will give a manifestly (*small*) gauge invariant expression for the effective action. The answer, not surprisingly, is in the affirmative. Let us recall that

$$\Gamma_{eff}[A,M]=-\sum_n \mathrm{Tr}\ln[p\!\!\!/+\gamma_0(\tilde{\omega}_n+e\hat{A}_0)+M+eA\!\!\!/]. \tag{B5}$$

In two dimensions, the vector field has the simple decomposition

$$A_i=\partial_i\sigma+\epsilon_{ij}\partial_j\rho,$$

from which it can be determined that

$$(\partial^2\rho)=-\epsilon_{ij}\partial_iA_j=-B. \tag{B6}$$

Using this decomposition and the familiar properties of gamma matrices in two dimensions, we can write

$$\Gamma_{eff}[A,M] = -\sum_n \mathrm{Tr}\ln e^{-ie\sigma}[\not{p} + e\gamma_0(\not{\partial}\rho) + \gamma_0(\tilde{\omega}_n + e\hat{A}_0) + M]e^{ie\sigma}$$
$$= -\sum_n \mathrm{Tr}\ln[\not{p} + e\gamma_0(\not{\partial}\rho) + \gamma_0(\tilde{\omega}_n + e\hat{A}_0) + M]. \qquad \text{(B7)}$$

From the definition of the parity violating effective action in Eq. (33), it now follows that

$$\frac{\partial\Gamma^{\mathrm{PV}}_{eff}}{\partial a} = -\frac{e}{2\beta}\sum_n \mathrm{Tr}\left[\frac{1}{\not{p} + e\gamma_0(\not{\partial}\rho) + \gamma_0(\tilde{\omega}_n + e\hat{A}_0) + M} - \frac{1}{\not{p} + e\gamma_0(\not{\partial}\rho) + \gamma_0(\tilde{\omega}_n + e\hat{A}_0) - M}\right]\gamma_0$$
$$= -\frac{eM}{\beta}\sum_n \mathrm{Tr}\frac{1}{p^2 + \tilde{\omega}_n^2 + M^2 + N}\gamma_0, \qquad \text{(B8)}$$

where we have defined

$$N = [-ie\gamma_0(\not{\partial}\hat{A}_0) + 2e\tilde{\omega}_n\hat{A}_0 + e^2\hat{A}_0^2] + ie\gamma_0(\partial^2\rho) - e^2(\partial_i\rho)(\partial_i\rho) \qquad \text{(B9)}$$

Using Eq. (B6), the last two terms in N can be expressed in terms of B and while the last one has a nonlocal form in terms of B, the penultimate term is local. Expression (B8) can now be expanded to linear order in the B field to give

$$\frac{\partial\Gamma^{\mathrm{PV}(1)}_{eff}}{\partial a} = -\frac{e^2M}{\beta}\sum_n \mathrm{Tr}\left[\frac{1}{p^2 + \tilde{\omega}_n^2 + M^2 + [-ie\gamma_0(\not{\partial}\hat{A}_0) + 2e\tilde{\omega}_n\hat{A}_0 + e^2\hat{A}_0^2]}(i\gamma_0 B) \times \frac{1}{p^2 + \tilde{\omega}_n^2 + M^2 + [-ie\gamma_0(\not{\partial}\hat{A}_0) + 2e\tilde{\omega}_n\hat{A}_0 + e^2\hat{A}_0^2]}\right]\gamma_0$$
$$= \frac{2ie^2M}{\beta}\sum_n \mathrm{tr}\frac{1}{(p^2 + \tilde{\omega}_n^2 + M^2 + 2e\tilde{\omega}_n\hat{A}_0 + e^2\hat{A}_0^2)^2 - e^2(\partial_i\hat{A}_0)^2}B. \qquad \text{(B10)}$$

To any order in the $\hat{A}_0$ fields, the denominator can be expanded in a systematic manner, as discussed earlier. However, this form has the advantage that it is manifestly gauge invariant to begin with. Furthermore, there are no Dirac matrices or momentum operators in the numerator to complicate the calculation. The only complication may be that integrating over a to obtain the action may be nontrivial.

[1] S. S. Chern and J. Simons, Ann. Math. **99**, 48 (1974).
[2] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).
[3] A. J. Niemi and G. W. Semenoff, Phys. Rep. **135**, 99 (1986).
[4] K. S. Babu, A. Das, and P. Panigrahi, Phys. Rev. D **36**, 3725 (1987).
[5] R. Pisarski, Phys. Rev. D **35**, 664 (1987).
[6] G. Dunne, K. Lee, and C. Lu, Phys. Rev. Lett. **78**, 3434 (1997).
[7] S. Deser, L. Griguolo, and D. Seminara, Phys. Rev. Lett. **79**, 1976 (1997); Phys. Rev. D **57**, 7444 (1998).
[8] C. Fosco, G. L. Rossini, and F. A. Schaposnik, Phys. Rev. Lett. **79**, 1980 (1997); Phys. Rev. D **56**, 6547 (1997).
[9] F. T. Brandt, A. Das, and J. Frenkel, Phys. Rev. D **62**, 085012 (2000); hep-ph/0004195.
[10] H. A. Weldon, Phys. Rev. D **47**, 594 (1993); P. F. Bedaque and A. Das, *ibid.* **47**, 601 (1993).
[11] For a review, see A. Das, *Finite Temperature Field Theory* (World Scientific, Singapore, 1997).
[12] A. Das and M. Hott, Mod. Phys. Lett. A **9**, 3383 (1994).
[13] A. Das, G. Dunne, and J. Frenkel, Phys. Lett. B **472**, 332 (2000).
[14] I. Aitchison and C. Fraser, Phys. Lett. **146B**, 63 (1984); Phys. Rev. D **31**, 2605 (1985); **32**, 2190 (1985).
[15] A. Das and A. Karev, Phys. Rev. D **36**, 623 (1987); **36**, 3725 (1987).
[16] I. Aitchison and C. Fosco, Phys. Rev. D **57**, 1171 (1998).
[17] R. Gonzalez Felipe, Phys. Lett. B **417**, 114 (1998); L. L. Salcedo, Nucl. Phys. **B549**, 98 (1999); C. D. Fosco, G. L. Rossini, and F. A. Schaposnik, Phys. Rev. D **59**, 085012 (1999); M. Hott and G. Metikas, *ibid.* **60**, 067703 (1999); S. Feng and D. Zhu, *ibid.* **60**, 105015 (1999).
[18] J. Barcelos-Neto and A. Das, Phys. Rev. D **58**, 085022 (1998).
[19] A. Actor and K. Rothe, J. Phys. A **33**, 4585 (2000).
[20] J. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
[21] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).