Discrete symmetry enhancement in non-Abelian models and the existence of asymptotic freedom

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We study the universality between a discrete spin model with icosahedral symmetry and the $O(3)$ model in two dimensions. For this purpose we study numerically the renormalized two-point functions of the spin field and the four point coupling constant. We find that those quantities seem to have the same continuum limits in the two models. This has far reaching consequences, because the icosahedron model is *not* asymptotically free in the sense that the coupling constant proposed by Lüscher, Weisz, and Wolff [Nucl. Phys. **B359**, 221 (1991)] does not approach zero in the short distance limit. By universality this then also applies to the $O(3)$ model, contrary to the predictions of perturbation theory.

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I. INTRODUCTION

The subject of the enhancement of a discrete symmetry to a continuous one at large distances is important both theoretically and phenomenologically. Indeed, even if there are good grounds to expect that a certain material is well described by some model enjoying $O(N)$ symmetry, one may wonder what might be the effect of anisotropies [16]. This question was addressed in 1977 by José *et al.* [1] for the $O(2)$ nonlinear σ model. Nonrigorous renormalization group arguments led them to the conclusion that in two dimensions (2D) the discrete symmetry Z_N should be enhanced to full $O(2)$ invariance if $N\geqslant 5$ for β (inverse temperature) not too large. For N sufficiently large, the occurrence of this phenomenon was proven rigorously by Fröhlich and Spencer in 1981 [2], who showed that there exists a range of temperatures in which spin correlation functions decay algebraically and are $O(2)$ invariant. Fröhlich and Spencer proved also that a similar phenomenon of discrete Abelian symmetry enhancement occurs in 4D gauge theories.

For a long time, the consensus was that no symmetry enhancement should occur in non-Abelian models. The main reason appears to have been the belief that, for continous symmetries, these models exhibit asymptotic freedom (AF). The discrete models, known rigorously to undergo phase transitions at nonzero temperature, did not seem likely to be AF; hence, they had to be different. A proposal for non-Abelian symmetry enhancement came however from Newman and Schulman [3]. Their argument was based on the fact that if the discrete symmetry group is sufficiently large, any fourth order polynomial invariant under the discrete group is also invariant under the continuous group in which it is contained. While this is an undisputable mathematical fact, the question was why fourth order? Their heuristic answer was that the renormalization group flow was expected to be free of bifurcations as the dimension D was varied between 2 and 4. So if one started just below $D=4$, where the most one could have is a ϕ^4 interaction (higher powers being irrelevant), this symmetry enhancement should persist down to $D=2$. Since however in $D=2$ all polynomials in ϕ are relevant and it is easy to write down such polynomials possessing only a discrete symmetry, and to construct the corresponding $P(\phi)_2$ models, the validity of the argument of Newman and Schulman remains unclear.

A different heuristic argument in favor of symmetry enhancement for Abelian as well as non-Abelian groups was put forward by Patrascioiu in 1985 [4]. For spin models, his argument went as follows: at sufficiently low temperatures, there exists a phase with long-range order (LRO) because, as Peierls showed long ago, given an ordered state, there is not enough free energy to create a domain in which the spin points elsewhere. Now consider a model like Z_5. As one increases the temperature, clearly the first abundant domains to form would be those in which the spin pointed in a direction immediately neighboring the one chosen by the boundary conditions for the ordered state. For temperatures not too high, the system could form domains inside domains of neighboring spin values. This would be different from the phase at high temperature, where no such restriction between adjacent domains would be required. This scenario does not seem to have anything do with the model being Abelian or not, and Patrascioiu suggested that, since it was known to happen in Abelian models, it must happen also in non-Abelian cases.

Except for the papers quoted above, in the 1980's, while it was quite fashionable to replace continous groups with discrete ones in Monte Carlo simulations, everybody seemed to be convinced that the discrete and continuous models belonged to different universality classes. Our interest in the subject was rekindled in 1990 when, together with Richard, we derived a rigorous inequality relating correlation functions in the dodecahedron model to those of Z_{10} [5]. Our result was that for any β, the dodecahedron model is more ordered than Z_{10} at $0.607^2\beta$. Since it is was pretty well established that Z_{10} possess an extended intermediate phase, which is $O(2)$ invariant, our inequality implied that provided

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$\beta_m(D) > \beta_c(Z_{10})$ the dodecahedron must also possess an intermediate massless phase. Here $\beta_m(D)$ denotes the onset of the LRO phase in the dodecahedron and $\beta_c(Z_{10})$ the onset of algebraic decay in Z_{10}. We determined numerically these values [6] and concluded that the dodecahedron seemed to possess an intermediate massless phase for $2.15 < \beta < 2.8$. We conjectured that this phase must enjoy full $O(3)$ invariance.

Intrigued by our findings for the dodecahedron, in the early 1990's we looked numerically at the other regular polyhedra. While the cube is obviously equivalent to 3 uncoupled Ising models, and hence is not a good candidate for exhibiting $O(3)$ invariance, the other 3 regular polyhedra (platonic solids) *a priori*, are. Actually, in the scenario advocated by Patrascioiu in 1985 [4], the tetrahedron should not be able to simulate spin waves since its spin gradient can take only one nontrivial value (in fact it is nothing but the 4 state Potts model). The octahedron and the icosahedron could. Our numerics suggested that the octahedron had a first order transition. For the icosahedron, the Monte Carlo data suggested a second order transition from the high temperature phase to the low temperature phase exihibiting LRO; in particular, there did not seem to be an extended massless phase, as for the dodecahedron.

That the discrete icosahedral symmetry may be enhanced to $O(3)$ began to become manifest in 1998 when we began extensive numerical investigations of the contiuum limit of the spin 2-point function versus p/m [7] in the dodecahedron model. The original motivation of that study was to compare the lattice continuum limit with the form factor (FF) prediction of Balog and Niedermaier [8]. Since it was not to be expected that the latter could possibly describe the continuum limit of the dodecahedron model, we decided to take data on this model too. To our surprise, the $O(3)$ data seemed to agree with both the FF prediction and with the dodecahedron. Since Balog and Niedermaier had produced a convincing argument [9] that the FF approach incorporates AF and we could not see how a discrete model, freezing at nonzero temperature, could possibly exhibit AF, we decided to refine our data by concentrating on the region $p/m < 13$. Our results [10] showed small but statistically significant deviations between $O(3)$ and FF, but excellent agreement between $O(3)$ and the dodecahedron.

The comparison of the FF and $O(3)$ could have been marred by lattice artifacts, a problem to which we will return below. This is why a different comparison was performed by Patrascioiu [11], who computed the renormalized spin 2-point function versus the physical distance x/ξ. The results were very similar with the ones we will report here about the icosahedron. They suggested that the dodecahedron and $O(3)$ models share the same continuum limit.

Another reason to investigate the discrete spin models arose in 1999, while in collaboration with Balog *et al.*, we decided to compare the FF prediction for the renormalized coupling g_R (to be defined below) with its lattice continuum limit value. Although our first results [12] suggested excellent agreement, as we continued to reduce the error bars and especially to take data at larger correlation length $\xi \approx 167$, our original extrapolation to the continuum limit became dubious, and in our second paper [13] we stated that we could no longer give a reliable number for the lattice continuum limit value of g_R. In the hope of gaining insight into this issue, we turned again to the discrete models. Corroborating our previous findings regarding $G_r(p/m)$ and $G_r(x/\xi)$, the data, to be shown later, suggested that g_R also agreed in the dodecahedron, icosahedron, and $O(3)$ models.

In the course of the debates of our collaboration [8,9], Niedermayer raised the intriguing possibility that maybe indeed all these models do have the same continuum limit, which however is AF. The present paper is our reply to his suggestion. The results were communicated to him already in 1999, hence we were surprised by his recent paper with Hasenfratz [14]. In their paper, Hasenfratz and Niedermayer claim that indeed they find strong numerical evidence that the dodecahedron and icosahedron models have the same continuum limit as $O(3)$, but state that, since "*overwhelming evidence exists that the $O(3)$ model is AF,*" the dodecahedron and icosahedron models must be AF too. The authors do not mention which "overwhelming evidence" they have in mind. They might be thinking of the results of Ref. [13] that show some rough agreement between the results of Monta Carlo simulations of the $O(3)$ model and the form factor bootstrap (FFB) construction; since the latter most likely has AF, that would be a point supporting their claim. There are, however, some facts not mentioned by Hasenfratz and Niedermayer, which should have cautioned them:

The last report of Balog *et al.* [13] retracted the original prediction $g_R = 6.77(2)$ and stated instead that the lattice artifacts were not sufficiently under control to decide whether the continuum limit was in agreement with the FFB prediction. Thus it cannot be claimed that numerical evidence supports the FFB ansatz, which most likely incorporates AF.

In our recent paper [15], we combined mathematically rigorous arguments with some numerics to conclude that the $O(3)$ model must undergo a transition to a massless phase at finite β. We then proved rigorously that such a phase transition rules out the existence of AF in the massive continuum limit.

Moreover, the $1/\xi$ fit produced by Hasenfratz and Niedermayer, would also provide evidence against AF. But the data for for g_R at larger values of ξ, while in clear disagreement with a Symanzik type fit ($1/\xi^2$ with a possible multiplcative $\log \xi$), do not support their original $1/\xi$ fit either.

Since they are based on numerics, any of these statements could be false, but the paper of Hasenfratz and Niedermayer does not contain any evidence pertinent to these issues.

In this paper we will compare the continuum limit of the $O(3)$ model to that of the icosahedron model by comparing the values of g_R and of the renormalized spin 2-point function $G_r(x/\xi)$. The advantage of the icosahedron model is the existence of a rather well localizable critical point, whereas, as stated above, the dodecahedral model appears to have a soft intermediate phase. To address the issue of AF, we study the value of the Lüscher-Weisz-Wolff (LWW) coupling constant at the critical point. According to LWW, AF requires that the continuum limit of this observable vanishes at short distances. If in fact the icosahedron model has the same continuum limit as $O(3)$, then the LWW coupling constant should vanish at the critical point of the former model. We find excellent evidence that it does not. Thus our conclusion is that either, in spite of the excellent agreement observed for $\xi\approx 121$, $O(3)$ and the icosahedron model have different continuum limits, or neither has AF.

The paper is organized as follows: we first describe the critical properties of the icosahedon model and in particular locate its critical point. This allows us to determine the value of the LWW running coupling at the critical point. Then we move on to the comparison of the icosahedron and the $O(3)$ models. We compare the renormalized coupling constant and the renormalized spin-spin correlation function of the two models and present convincing evidence that they converge to the same continuum limit.

II. CRITICAL BEHAVIOR OF ICOSAHEDRON MODEL

We first describe the critical properties of the icosahedron model: we locate the critical point and give some estimates of critical exponents. This allows us to determine the value of the LWW coupling constant at the critical point, which is independent ot the size L of the lattice, as required by scaling. This (nonzero) value is also the short distance limit of the continuum value of the LWW coupling constant.

The icosahedron model is defined by the standard nearest neighbor coupling between the spins

$$H=-\sum_{\langle ij\rangle} s_i s_j \tag{1}$$

where the spins s_i are unit vectors forming the vertices of a regular icosahedron. In suitable coordinates, those 12 vertices are given by

$$e_k=\left(s\cos\frac{2\pi k}{10}, s\sin\frac{2\pi k}{10}, c\cos(\pi k)\right) \qquad (k=1,2,\ldots,10); \tag{2}$$

where

$$s=\frac{2}{\sqrt{5}}, \quad c=\frac{1}{\sqrt{5}} \tag{3}$$

and

$$e_{11}=(0,0,1) \quad \text{and} \quad e_{12}=(0,0,-1). \tag{4}$$

The model has at least two phases, a high temperature phase with exponential clustering and full symmetry under the icosahedral group Y, and a low temperature phase with spontaneous magnetization and 12 coexisting phases, with the magnetization pointing into one of the 12 directions of the icosahedron. At intermediate temperatures there could be, in principle, a phase with partial breaking of Y, but there is no reasonable candidate for a possible unbroken subgroup. There is also the possibility of an extended intermediate phase with no symmetry breaking, but actual enhancement of the symmetry from Y to $O(3)$, as well as only algebraic decay of correlations analogous to the Z_N models and the dodecahedron model (see above).

In the icosahedron model the situation seems to be simpler, however, it seems to have a single critical point separating the high and low temperature phases, as we will demonstrate.

To determine the critical point, we proceed as follows: we determine the mass gap $m(L)=1/\xi(L)$ in an (ideally infinite) strip of width L; in practice we use a finite strip of size $L\times L_t$ with $L_t \gg L$ and measure the effective correlation length defined by

$$\xi(L)=\frac{1}{2\sin(\pi/L_t)}\sqrt{\chi/G_1-1} \tag{5}$$

where

$$\chi=\frac{1}{LL_t}\sum_{i,j}\langle s_i\cdot s_j\rangle, \tag{6}$$

$$G_1=\frac{1}{LL_t}\sum_{i,j}\langle s_i s_j\rangle \exp[2\pi(i_1-j_1)/L_t]. \tag{7}$$

We then study the behavior of the quantity

$$\bar{g}\equiv\frac{2L}{(N-1)\xi(L)}, \tag{8}$$

which we consider as a function $\bar{g}(z,\xi)$ of $z=L/\xi(\infty)$ and the infinite volume correlation length $\xi(\infty)$. The continuum limit of this quantity for fixed z is the "running coupling constant" introduced by Lüscher, Weisz, and Wolff [16] for the $O(N)$ models (we refer readers worried about our slightly different definition of the correlation length, Eq. (5), to our recent paper [17], where the practical equivalence of the two definitions is demonstrated). In the high temperature phase, where the model has a mass gap $m(\infty)=1/\xi(\infty)$ in the infinite volume limit, $\bar{g}$ will grow linearly with L. On the other hand, in the low temperature magnetized phase, the

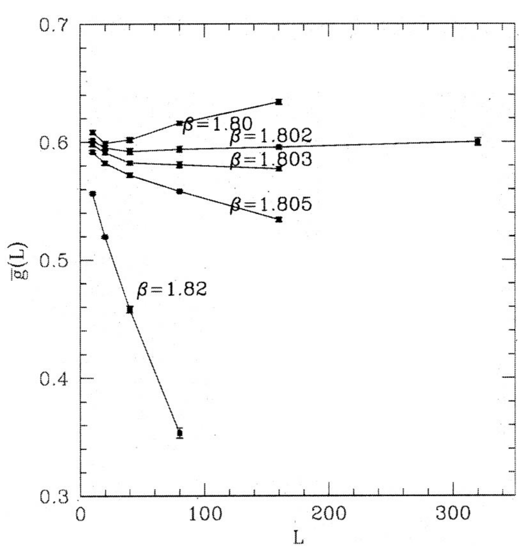

FIG. 1. The LWW coupling constant $\bar{g}$ as a function of L for various values of β.

TABLE I. The LWW running coupling $\overline{g}$ and the renormalized coupling g_R as a function of L for various values of β.

$\beta=1.665$						
L	10	20	40	80	160	320
$\overline{g}$	.9121(14)	1.0651(11)	1.3033(21)			
g_R	2.216(13)	2.501(12)	3.080(29)			
$\beta=1.707$						
$\overline{g}$	.8156(9)	.9312(12)	1.0882(23)	1.3276(35)		
g_R	1.898(6)	2.213(11)	2.593(23)	3.171(36)		
$\beta=1.75$						
$\overline{g}$	.7221(24)	.7902(21)				
g_R	1.695(14)	1.841(24)				
$\beta=1.80$						
$\overline{g}$	.6083(16)	.5989(17)	.6018(21)	.6158(17)	.6337(22)	
g_R	1.363(9)	1.329(9)	1.357(14)	1.395(9)	1.459(13)	
$\beta=1.802$						
$\overline{g}$	.6015(10)	.5951(14)	.5919(26)	.5938(24)	.5955(14)	.6000(32)
g_R	1.340(4)	1.3330(6)	1.328(5)	1.348(11)	1.360(7)	1.358(14)
$\beta=1.803$						
$\overline{g}$	.5979(16)	.5912(18)	.5823(14)	.5807(25)	.5772(17)	
g_R	1.335(10)	1.327(10)	1.307(9)	1.306(14)	1.302(10)	
$\beta=1.805$						
$\overline{g}$	.5918(13)	.5821(17)	.5721(19)	.5583(8)	.5341(19)	
g_R	1.308(7)	1.298(8)	1.274(10)	1.242(4)	1.173(11)	
$\beta=1.82$						
$\overline{g}$	.5566(8)	.5196(8)	.4578(28)	.3533(42)		
g_R	1.213(4)	1.119(4)	.964(10)	.652(8)		

mass gap in a finite volume goes to 0 faster than $1/L$, so $\overline{g}$ will decrease to 0 as $L\rightarrow\infty$. At a critical point, the model will be scale invariant for large distances and $\overline{g}$ should converge to a finite nonzero limit.

To determine $\overline{g}$, we took data on lattices of size $L\times L_t$ with $L_t=10L$; L was varied from 10 to 320. Figure 1 (Table I) shows clearly the dramatic change in behavior between $\beta=1.800$ and $\beta=1.805$. For $\beta=1.802$, $\overline{g}$ shows only some small variation for small L ($L<40$) and stabilizes for larger L. So we estimate

$$\beta_{crt}=1.802(1) \tag{9}$$

where the error is, of course, somewhat subjective.

To corroborate this determination of the critical point, we also measured, on the same lattices, the renormalized coupling defined as

$$g_R=\left(\frac{5}{3}-\frac{g_4}{g_2^2}\right)\frac{LL_t}{\xi^2}. \tag{10}$$

Here g_2 is the magnetic susceptibility multiplied by the volume of the lattice $L\times L_t$, i.e.

$$g_2=\sum_{i,j}\langle s_i s_j\rangle \tag{11}$$

and

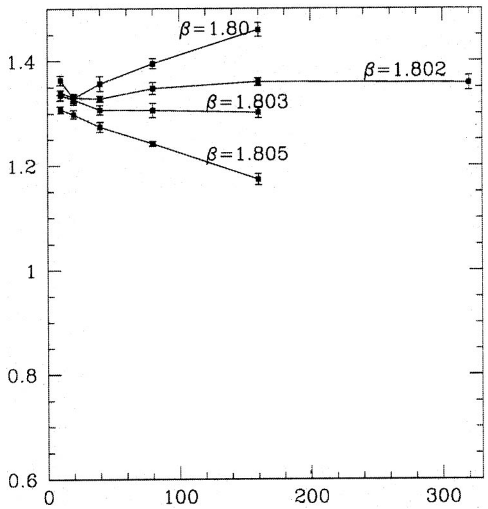

FIG. 2. The renormalized coupling for asymmetric $L\times L_t$ lattices as a function of L for various values of β.

$$g_4=\sum_{i_1,i_2,i_3,i_4}\langle(s_{i_1}s_{i_2})(s_{i_3}s_{i_4})\rangle. \tag{12}$$

This quantity is also a renormalization group (RG) invariant and therefore should also go to a constant for $\beta=\beta_{crt}$.

The behavior of this quantity as a funcction of β and L near β_{crt} is a little tricky; therefore we want to make a few remarks about it. The main point is that one has to be careful about the order of the limits $L\to\infty$ and $\beta\to\beta_{crt}$, which cannot be interchanged. For fixed finite L, $g_R(\beta,L)$ is a smooth funtion of β, which however gets steeper and steeper near β_{crt} as L increases. In the limit $L\to\infty$, it develops a jump at β_{crt}.

It is well known that in the high temperature phase, i.e. for $\beta<\beta_{crt}$, $\lim_{L\to\infty}g_R(\beta,L)$ is a nontrivial number (with some dependence on β) which has a nontrivial limit g^* if we send β to β_{crt} from below. This limit is the value of the four point coupling constant in the *massive* continuum limit and has a value above 6.7 (see Sec. III). In the magnetized low temperature phase, on the other hand, $\lim_{L\to\infty}g_R(\beta,L)=0$, because the factor $\xi(L)^2$ in the denominator of Eq. (10) is growing rapidly with L. So one expects for g_R a qualitatively similar picture as for $\bar{g}$; the values of g_R at different L for fixed $\beta<\beta_{crt}$ are growing with L, while for $\beta>\beta_{crt}$ they are decreasing towards 0. Since g_R is a RG invariant, right at $\beta=\beta_{crt}$, it should be essentially scale invariant and go to a constant less than g^*. In other words, we expect

$$\lim_{L\to\infty}\lim_{\beta\to\beta_{crt}} g_R(\beta,L)<\lim_{\beta\to\beta_{crt}-0}\lim_{L\to\infty} g_R(\beta,L)=g^* \tag{13}$$

whereas

$$\lim_{L\to\infty}\lim_{\beta\to\beta_{crt}} g_R(\beta,L)>\lim_{\beta\to\beta_{crt}+0}\lim_{L\to\infty} g_R(\beta,L)=0. \tag{14}$$

The data presented in Table I and displayed in Fig. 2 confirm this nicely and suggest a value of $\lim_{L\to\infty}g_R(\beta_{crt},L)\approx 1.3$; the data also corroborate our estimate of $\beta_{crt}=1.802(1)$ given above.

Finally we want to see if our determination of β_{crt} is consistent with a singularity in the thermodynamic values of the correlation length ξ and the susceptibility χ. We therefore measured ξ and χ on lattices with $L/\xi\approx 7$ for various values of $\beta<1.802$; our data are given in Table II. There is a row listing the number of runs; a run consists of 100 000 single cluster updates for thermalization (corresponding to between 1000 and 2000 lattice updates) followed by 20 000 sweeps of the lattice for measurements. Each run is started independently with a randomly chosen new configuration.

To describe the critical behavior of the data for ξ and χ, two types of fits were tried: first a Kosterlitz type fit with an exponential singularity

$$\xi=C_\xi\exp\left(\frac{-a_\xi}{\sqrt{\beta_{crt}-\beta}}\right),\quad \chi=C_\chi\exp\left(\frac{-a_\chi}{\sqrt{\beta_{crt}-\beta}}\right), \tag{15}$$

and second a power law fit of the type

$$\xi=C_\xi(\beta_{crt}-\beta)^{-\nu},\quad \chi=C_\chi(\beta_{crt}-\beta)^{-\gamma} \tag{16}$$

or similar ones in $1/\beta$. Both types of fit are not very good, with similar quite large values of χ^2, and they do not allow a very precise determination of the fit parameters. The reason seems to be the following: the asymptotic singular behavior seems to have significant subleading contributions, which however cannot be well determined with only 5 values of β. Trying to fit our very precise data with functions that do not describe the behavior with similar accuracy, necessarily leads to a poor fit quality, even though visually the data may be very well described (see Fig. 3).

TABLE II. Correlation length ξ, susceptibility χ and renormalized coupling g_R in the high temperature phase of the icosahedron model.

β	1.470	1.550	1.610	1.665	1.707
L	80	140	250	500	910
ξ	11.203(5)	19.627(12)	33.655(21)	63.628(33)	122.09(16)
χ	181.88(10)	479.43(36)	1228.04(91)	3774.5(2.1)	12009(17)
$g_R(z)$	6.404(28)	6.487(35)	6.546(39)	6.684(36)	6.715(88)
$g_R(7.25)$	6.412(28)	6.495(35)	6.535(39)	6.651(36)	6.702(88)
# runs	200	100	100	149	28

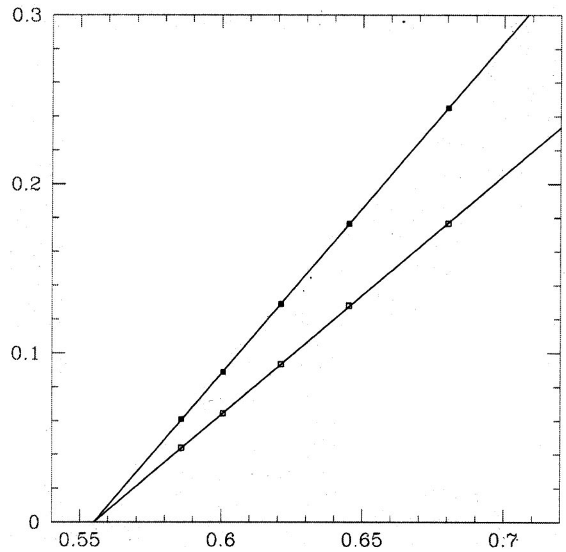

FIG. 3. Thermodynamic values of $\xi^{-1/\nu}$ and $\chi^{-1/\gamma}$ with $\nu=1.717$ and $\gamma=3.002$ as functions of β; the lines are fits.

The Kosterlitz type fit leads to predictions of $\beta_{crt}\approx 1.93$, which is unacceptably large—this value is deeply in the magnetized phase. The power law fits, on the other hand, give values of β_{crt} quite close to our preferred value 1.802. By playing with the number of parameters and fitting in $1/\beta$ as well as β, we obtain quite a spread in the values of the exponents; they fall into the intervals

$$\nu=1.6-2.0 \quad \text{and} \quad \gamma=2.9-3.5. \tag{17}$$

This is consistent with a value of

$$\eta=2-\frac{\gamma}{\nu}\approx .25 \tag{18}$$

which is also favored by the data for spin-spin correlation function (see below).

In Fig. 3 we use the best values produced by the power law fit in $1/\beta$ with no subleading corrections, namely,

$$\nu=1.717, \quad \gamma=3.002. \tag{19}$$

We plot $\xi^{-1/\nu}$ and $\chi^{-1/\gamma}$ vs β together with the fits, which are straight lines intersecting the abscissa at $\beta=1.802$. So even though the thermodynamic data for ξ and χ do not lead to a precise prediction of the critical point and the critical exponents, they are certainly consistent with our determination based on the LWW coupling constant.

We also investigated the possibility that the transition from the high temperature phase to the one with long range order is first order, but we did not find any signal for phase coexistence.

III. THE RENORMALIZED COUPLING IN THE ICOSAHEDRON AND $O(3)$ MODELS

To check whether the icosahedron and the $O(3)$ model define the same continuum limit, we also determined the

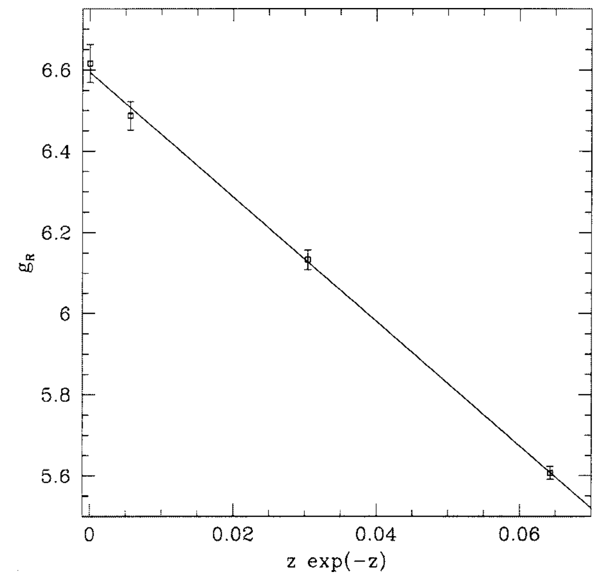

FIG. 4. Finite size scaling of g_R.

renormalized coupling constant g_R on lattices of fixed physical size $z=L/\xi\approx 7$. At this value of z, the finite size effects are already very weak; to account for the remaining small differences in z between the measurements at different values of β, we used a finite size scaling technique. Our procedure was the following.

First we took data at various values of $z=L/\xi$ at $\beta=1.550$ corresponding to $\xi\approx 19.7$. To describe the finite size effects we used ansatze of the form

$$g_R(z)=g_R(\infty)(1-dz^pe^{-z}). \tag{20}$$

This type of ansatz is suggested by the spherical model, where it holds with $p=1/2$. Since it is to be expected that p is model dependent, we tried to see if our data favored a certain value of that parameter; it turns out that the fit is best for $p=1$. We then used the above ansatz with $p=1$ and the parameters determined at $\beta=1.55$ to renormalize the values at other β values to $z=7.25$.

In Fig. 4 we show the data at $\beta=1.55$ together with the fit, which has a χ^2 per degree of freedom of 0.3; the constant d is determined to be

$$d=2.327\pm .075. \tag{21}$$

We then used this ansatz with $p=1$ and the parameters determined above to "renormalize" the values at other β values to $z=7.25$. In Table II we present our data together with the "renormalized" values.

For the comparison with the $O(3)$ model, we took data for this model at $\beta=1.6$ (corresponding also to $\xi\approx 19$) at different values of L; these data can also be described by the ansatz Eq. (20) and again $p=1$ is the preferred value of the exponent. We used the parameters determined in this way to renormalize the g_R values of the $O(3)$ model obtained earlier to physical size $z=7$ [14]. Those data together with the renormalized values are given in Table III.

TABLE III. Correlation length ξ and renormalized coupling g_R in the high temperature phase of the $(O3)$ model (from Ref. [14]).

β	1.5	1.6	1.7	1.8	1.9	1.95
L	80	140	250	500	910	1230
ξ	11.030(7)	18.950(14)	34.500(15)	64.790(26)	122.330(74)	167.71(17)
$g_R(z)$	6.553(16)	6.612(15)	6.665(14)	6.691(15)	6.737(21)	6.792(40)
$g_R(7.25)$	6.553(16)	6.603(15)	6.665(14)	6.663(15)	6.724(21)	6.786(40)
# runs	344	370	367	382	127	68

Figure 5 shows the renormalized data for the two models for various values of ξ. Even though the lattice artifacts are quite different for the two models, and in spite of the fact that we are not quite sure how one should extrapolate to the continuum limit, the data show that the two models approach each other with increasing ξ (decreasing lattice spacing) and suggest that they will have the same continuum limit.

IV. SPIN CORRELATION FUNCTION IN THE ICOSAHEDRON AND $O(3)$ MODELS

In this section we compare the renormalized spin-spin correlation functions of the icosahedron and the $O(3)$ models. They are defined as

$$G(i/\xi)\equiv\frac{\xi^2}{\chi}\langle s(0)\,s(i)\rangle. \tag{22}$$

The physical distance is

$$x=\frac{i}{\xi}. \tag{23}$$

We measured the two-point functions in the icosoahedron model at $\beta=1.47$, 1.55, 1.61, 1.665, and 1.707 corresponding to $\xi\approx11$, 20, 34, 64, 122 (see Table II). For the $O(3)$ model, we used the data from Ref. [12] at $\beta=1.8$, 1.9, 1.95 corresponding to $\xi\approx65$, 122, and 168. In Fig. 6 we show $G(x)x^{1/4}$ for the $O(3)$ model together with the 2-loop perturbation theory prediction

$$G(x)=\frac{1.000}{3\pi^3 1.002^2}\left[t+\ln t+1.116+\frac{1}{t}\ln t+\frac{.116^2}{t}\right] \tag{24}$$

where $t=-\ln(x/8)-1$. This expression was taken from Ref. [18], the constants in front were communicated to us by Balog and Niedermaier; they were computed via the form factor approach [9] and take in account our different definition of the correlation length. It can be seen that the data approach their continuum limit from above and deviate considerably from the PT prediction. For $x>0.4$, barring some very slow convergence to the continuum limit, our data suggest that the lattice artifacts are quite small for the large correlation lengths we are using.

In Fig. 7 we present $G(x)x^{1/4}$ for the icosahedron model together with the same expression for $O(3)$ at approximately the largest value of $\xi\approx122$. The lattice artifacts have the opposite sign at least for the lattices with $\xi<122$, i.e. the data are increasing with decreasing lattice spacing. At $\xi\approx122$, they are already quite close to the corresponding correlation function in the $O(3)$ model. It is therefore reasonable to expect that if we could further refine the lattice we would see that the data follow the behavior of the $O(3)$

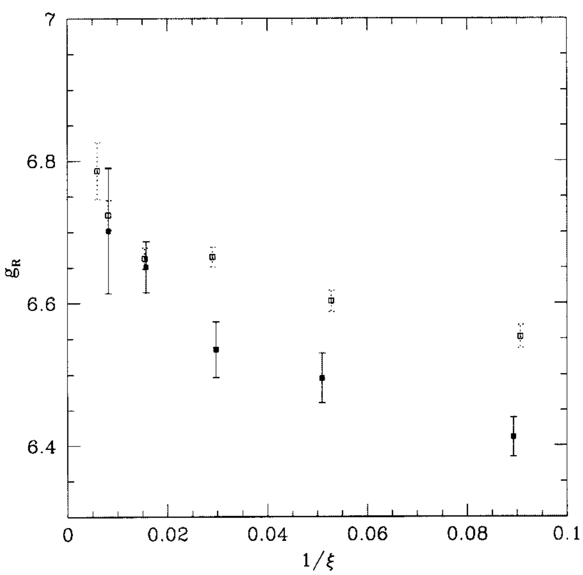

FIG. 5. Comparison of g_R in the icosahedron and $O(3)$ models. Full symbols: icosahedron; open symbols: $O(3)$.

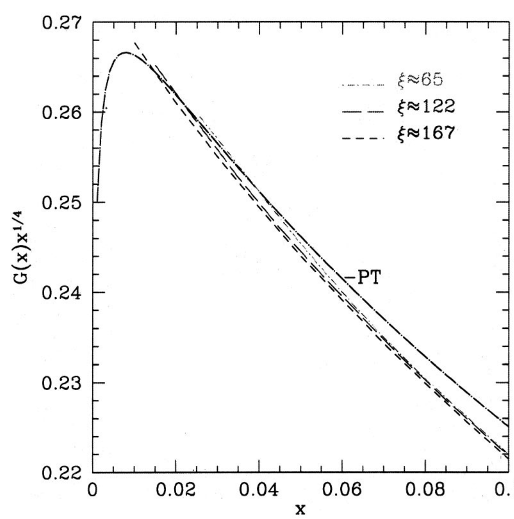

FIG. 6. The renormalized spin-spin correlation for the $O(3)$ model for various lattice spacings.

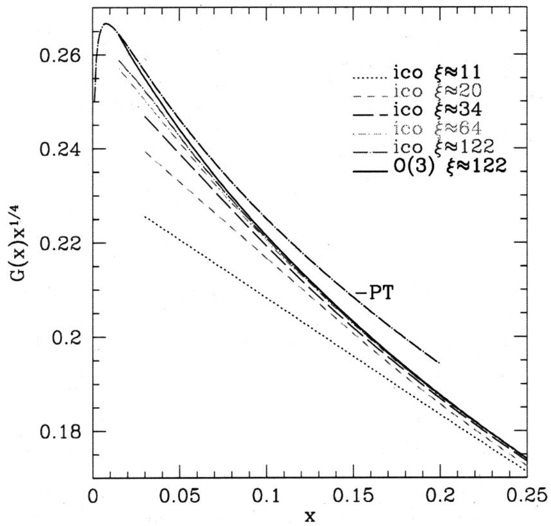

FIG. 7. The renormalized spin-spin correlation for the icosahedron model for various lattice spacings: solid line $O(3)$ model for $\xi\approx 122$.

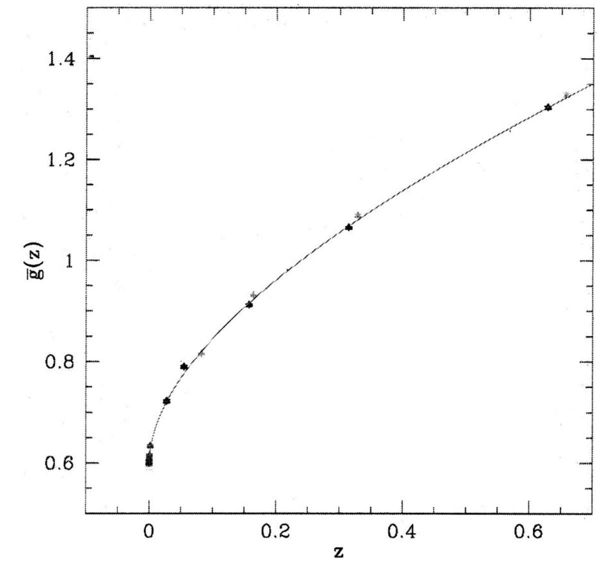

FIG. 8. The LWW running coupling for the iscosahedron model as a function of $L/\xi(\infty)$.

model, i.e. the lattice artifacts would change sign and the final approach to the continuum limit would be from above.

It should be stressed that already at $\xi\approx 122$, the icosahedron and $O(3)$ data are much closer to each other than the $O(3)$ data are to the PT prediction.

V. CONCLUSION: UNIVERSALITY BETWEEN THE ICOSAHEDRON AND $O(3)$ MODELS AND ASYMPTOTIC FREEDOM

We have accumulated strong evidence that the continuum limits of the discrete icosahedron model and the continuous classical Heisenberg $[O(3)]$ model describe the same quantum field theory.

As discussed in the introduction, this is one more fact which puts the asymptotic freedom of the $O(3)$ model severely into doubt. The point of view advocated by Hasenfratz and Niedermayer [14], namely, that the continuum limit of the discrete icosahedron model should be asymptotically free, is untenable in view of our results about the LWW running coupling $\bar{g}$; our data (see Fig. 1) indicate that $\bar{g}(L)$ runs to a fixed point value $g^*\approx .59$ at small distances. Actually to determine the true running coupling, one should take first the continuum limit at fixed $z=L/\xi$ and then the limit $z\to 0$. Since this is not feasible, we instead studied the finite size scaling at and around the critical point, and took as our estimate of g^* the apparent limit

$$\lim_{L\to\infty} m(L)L \tag{25}$$

at the critical point. Somebody might wonder if it is not possible that the continuum limit shows asymptotic freedom after all; in other words, if it is possible that

$$\lim_{z\to 0}\ \lim_{\xi\to\infty} \bar{g}(z,\xi)=0? \tag{26}$$

Our data make this extremely unlikely; for fixed $\beta<\beta_{crt}\approx 1.802$, $\bar{g}$ is *increasing* with L (except for very small lattices) and it is always larger than .59, even at $\beta=1.802$, our estimated critical point. To claim that in the continuum limit $\bar{g}(z,\infty)$ would go to 0, one would have to assume some truly bizarre L dependence at fixed β. This is illustrated in Fig. 8, which shows some of our data for $\bar{g}$ as a function of $z=L/\xi(\infty)$. We used data at $\beta=1.665$ and 1.707, where we know the correlation length $\xi(\infty)$ quite well, together with the data taken slightly below the estimated critical point, where we used the fit appearing in Fig. 3 to estimate $\xi(\infty)$. The solid curve is a fit of the form

$$\bar{g}(z)=g^*+az^{1/2}+bz. \tag{27}$$

Since we did not make any effort to control the lattice artifacts and estimate the precise continuum values, this figure should be taken with some caution. It does, however, illustrate nicely the qualitative behavior of the LWW running coupling near the critical point.

To sum up, the universality observed between the icosahedron and the $O(3)$ model gives strong evidence *against* asymptotic freedom of the latter.

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