Choptuik scaling and quasinormal modes in the anti–de Sitter space/conformal-field theory correspondence

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We establish an exact connection between the Choptuik scaling parameter for the three-dimensional Bañados-Teitelboim-Zanelli black hole, and the imaginary part of the quasinormal frequencies for scalar perturbations. Via the anti–de Sitter space/conformal-field theory correspondence, this leads to an interpretation of Choptuik scaling in terms of the time scale for return to equilibrium of the dual conformal field theory.

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I. INTRODUCTION

Within the context of numerical relativity, one of the most significant recent results is the evidence for universal scaling behavior in black hole formation [1,2]. In particular, one considers a generic smooth one-parameter family of initial data (labeled by p), such that a black hole is formed for values of p greater than a critical value p^*, while no black hole is formed for $p<p^*$. The mass of the black hole then satisfies the scaling relation [1]

$$M \sim (p-p^*)^{\gamma}, \tag{1}$$

where γ is a universal exponent known as the Choptuik scaling parameter. In [3], it was shown that the Gott time machine [4], namely a two-body collision process, gives a precise algebraic mechanism for the formation of the $(2+1)$-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole. This led to an exact analytic determination of Choptuik scaling.

In [5–13], the quasinormal modes of scalar fields in the background of anti–de Sitter black holes were studied. The associated complex quasinormal frequencies describe the decay of the scalar perturbation, and depend only on the parameters of the black hole. In terms of the anti–de Sitter space/conformal-field theory (AdS/CFT) correspondence [14–18], an off-equilibrium configuration in the bulk AdS space is related to an off-equilibrium state in the boundary conformal field theory. The time scale for the decay of the scalar perturbation is given by the imaginary part of the quasinormal frequencies. Thus, by virtue of the AdS/CFT correspondence, one obtains a prediction of the time scale for return to equilibrium of the dual conformal field theory. Interestingly, it was shown numerically [7] that the imaginary part of the quasinormal frequencies for intermediate-sized black holes in four dimensions, $\omega_{\rm Im}$, scaled with the horizon radius, r_+. In particular, it was found that

$$\omega_{\rm Im} \sim \frac{1}{\gamma} r_+ , \tag{2}$$

where γ is the Choptuik scaling parameter. This relation, although not understood, suggested a deeper connection between black hole critical phenomena and quasinormal modes.

In this paper, we compute exactly the quasinormal modes of massive scalar fields in the background of the BTZ black hole, see also [11,12]. It is shown that the imaginary part of the quasinormal frequencies has a universal scaling behavior precisely of the form (2). This leads to a conformal field theory interpretation of Choptuik scaling within the context of the AdS/CFT correspondence.

II. QUASINORMAL MODES OF THE BTZ BLACK HOLE

To begin, we recall that the BTZ black hole is a solution of the vacuum Einstein equations of three-dimensional anti–de Sitter gravity, i.e., with negative cosmological constant $\Lambda = -1/l^2$. The line element can be written in the form [19,20]

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2. \tag{3}$$

The mass and angular momentum of the black hole can be expressed in terms of the inner and outer horizon radii, $r_\pm$, as

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+ r_-}{l}, \tag{4}$$

and we choose units for Newton's constant such that $8G = 1$.

We wish to study the properties of a massive scalar field in the background geometry of the BTZ black hole. A special feature of this $(2+1)$-dimensional case is that the corresponding wave equation can be solved exactly in terms of hypergeometric functions [21,22]. By choosing appropriate boundary conditions, we are led to an exact determination of the quasinormal modes for the scalar field. The scalar wave equation takes the form

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$$\left(\nabla^2-\frac{\mu}{l^2}\right)\Phi=0, \tag{5}$$

where μ is the mass parameter. Using the ansatz

$$\Phi=R(r)e^{-i\omega t}e^{im\phi}, \tag{6}$$

with the change of variables

$$z=\frac{r^2-r_+^2}{r^2-r_-^2}, \tag{7}$$

we are led to the radial equation

$$z(1-z)\frac{d^2R}{dz^2}+(1-z)\frac{dR}{dz}+\left(\frac{A}{z}+B+\frac{C}{1-z}\right)R=0. \tag{8}$$

Here,

$$A=\frac{l^4}{4(r_+^2-r_-^2)^2}\left(\omega r_+-\frac{m}{l}r_-\right)^2,$$
$$B=-\frac{l^4}{4(r_+^2-r_-^2)^2}\left(\omega r_--\frac{m}{l}r_+\right)^2, \tag{9}$$
$$C=-\frac{\mu}{4}.$$

We now define

$$R(z)=z^{\alpha}(1-z)^{\beta}F(z). \tag{10}$$

The radial equation then assumes the standard hypergeometric form [23]

$$z(1-z)\frac{d^2F}{dz^2}+[c-(1+a+b)z]\frac{dF}{dz}-abF=0, \tag{11}$$

where

$$c=2\alpha+1,$$
$$a+b=2\alpha+2\beta, \tag{12}$$
$$ab=(\alpha+\beta)^2-B,$$

and

$$\alpha^2=-A,$$
$$\beta=\frac{1}{2}(1\pm\sqrt{1+\mu}). \tag{13}$$

Without loss of generality, we take $\alpha=-i\sqrt{A}$ and $\beta=\frac{1}{2}(1-\sqrt{1+\mu})$.

In the neighborhood of the horizon, $z=0$, the two linearly independent solutions of Eq. (11) are given by [23] $F(a,b,c,z)$ and $z^{1-c}F(a-c+1,b-c+1,2-c,z)$. The quasinormal modes are defined as solutions which are purely ingoing at the horizon, and which vanish at infinity [7]. The solution which has ingoing flux at the horizon is given by

$$R(z)=z^{\alpha}(1-z)^{\beta}F(a,b,c,z). \tag{14}$$

To implement the vanishing boundary condition at infinity, $z=1$, we use the linear transformation formula [23]

$$R(z)=z^{\alpha}(1-z)^{\beta}(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$
$$\times F(c-a,c-b,c-a-b+1,1-z)+z^{\alpha}(1-z)^{\beta}$$
$$\times\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-z). \tag{15}$$

Clearly, the first term in Eq. (15) vanishes. However, the vanishing of the second term imposes the restriction

$$c-a=-n,\quad \text{or}\quad c-b=-n, \tag{16}$$

where $(n=0,1,2,\dots)$. This condition leads directly to an exact determination of the quasinormal modes. From Eq. (12), we have

$$a=\alpha+\beta+i\sqrt{-B},$$
$$b=\alpha+\beta-i\sqrt{-B}. \tag{17}$$

Thus, we find that the left and right quasinormal modes, denoted by ω_L and ω_R, are given by

$$\omega_L=\frac{m}{l}-2i\left(\frac{r_+-r_-}{l^2}\right)\left(n+\frac{1}{2}+\frac{1}{2}\sqrt{1+\mu}\right),$$
$$\omega_R=-\frac{m}{l}-2i\left(\frac{r_++r_-}{l^2}\right)\left(n+\frac{1}{2}+\frac{1}{2}\sqrt{1+\mu}\right). \tag{18}$$

It is important to stress that this is an exact calculation of all quasinormal modes for the scalar field in a general BTZ background. The result (18) agrees with the special cases considered in [11,12], for $\mu=0$ and $J=0$; quasinormal modes for the BTZ black hole were first studied in [5]. We also note that the imaginary parts of the quasinormal modes scale linearly with the left and right temperatures, defined by [24] $T_L=(r_+-r_-)/2\pi l^2$ and $T_R=(r_++r_-)/2\pi l^2$.

III. CHOPTUIK SCALING AND THE AdS/CFT CORRESPONDENCE

The aim now is to determine the precise connection between these quasinormal modes and the Choptuik scaling parameter of the BTZ black hole. Let us first recall that three-dimensional anti–de Sitter space, AdS_3, can be characterized in terms of the flat space $\mathbf{R}^{2,2}$, with coordinates (X_1,X_2,T_1,T_2), and line element [20]

$$ds^2=dX_1^2+dX_2^2-dT_1^2-dT_2^2. \tag{19}$$

The induced metric on the submanifold

$$X_1^2+X_2^2-T_1^2-T_2^2=-l^2, \tag{20}$$

then corresponds to the AdS_3 metric. It is convenient to combine the coordinates (X_1,X_2,T_1,T_2) into an $SL(2,\mathbf{R})$ matrix,

$$X=\frac{1}{l}\begin{pmatrix} T_1+X_1 & T_2+X_2 \\ -T_2+X_2 & T_1-X_1 \end{pmatrix}, \tag{21}$$

with unit determinant. Thus, AdS_3 may be viewed as the group manifold of $SL(2,\mathbf{R})$, with isometry group $[SL(2,\mathbf{R})\times SL(2,\mathbf{R})]/Z_2$. Thus, for $X\in SL(2,\mathbf{R})$, the isometry group acts by left and right multiplication, $X\rightarrow\rho_L X\rho_R$, with the identification $(\rho_L,\rho_R)\sim(-\rho_L,-\rho_R)$.

The essential point to note is that the BTZ black hole spacetime is locally isometric to AdS_3. As a result, it can be obtained as a quotient of the universal covering space of AdS_3 by a discrete group of isometries of AdS_3. For the region $r\geqslant r_+$, this can be seen by defining the coordinates of AdS_3 by [20,25]

$$\begin{aligned}
X_1&=l\sqrt{\alpha}\sinh\left(\frac{r_+}{l}\phi-\frac{r_-}{l^2}t\right),\\
X_2&=l\sqrt{\alpha-1}\cosh\left(\frac{r_+}{l^2}t-\frac{r_-}{l}\phi\right),\\
T_1&=l\sqrt{\alpha}\cosh\left(\frac{r_+}{l}\phi-\frac{r_-}{l^2}t\right),\\
T_2&=l\sqrt{\alpha-1}\sinh\left(\frac{r_+}{l^2}t-\frac{r_-}{l}\phi\right),
\end{aligned} \tag{22}$$

where

$$\alpha(r)=\left(\frac{r^2-r_-^2}{r_+^2-r_-^2}\right), \tag{23}$$

and $\phi\in(-\infty,\infty)$, $t\in(-\infty,\infty)$. It is then straightforward to show that the AdS_3 metric (19) transforms into the BTZ metric (3). However, the coordinate ϕ in Eq. (22) has an infinite range, and thus to identify the BTZ black hole, we must impose periodicity in the ϕ coordinate. This identification is an isometry of AdS_3, and corresponds to the element (ρ_L,ρ_R), with [20]

$$\begin{aligned}
\rho_L&=\begin{pmatrix} e^{\pi(r_+-r_-)/l} & 0 \\ 0 & e^{-\pi(r_+-r_-)/l} \end{pmatrix},\\
\rho_R&=\begin{pmatrix} e^{\pi(r_++r_-)/l} & 0 \\ 0 & e^{-\pi(r_++r_-)/l} \end{pmatrix}.
\end{aligned} \tag{24}$$

The BTZ black hole is then defined as the quotient of the universal covering space of AdS_3 by the group generated by (ρ_L,ρ_R). The other regions with $r\leqslant r_+$ can be dealt with in a similar fashion [20,25].

It is important to note that elements of $SL(2,\mathbf{R})$ are classified according to the value of their trace, namely [26]

$$\begin{aligned}
|\mathrm{Tr}T|&<2 \quad \text{elliptic},\\
|\mathrm{Tr}T|&=2 \quad \text{parabolic},\\
|\mathrm{Tr}T|&>2 \quad \text{hyperbolic}.
\end{aligned} \tag{25}$$

Thus, we see that the left and right generators of the BTZ black hole are hyperbolic elements.

In order to discuss the formation process for the BTZ black hole, we first recall the algebraic construction of particle spacetimes with vanishing cosmological constant [27]. In this case, the Lorentz group $SO(2,1)$ is locally equivalent to $SL(2,\mathbf{R})$. A point particle spacetime in $(2+1)$ dimensions is then defined via identifications by an elliptic generator of $SL(2,\mathbf{R})$. Specifically, the spacetime for a single static point particle is obtained by removing a wedge of deficit angle α, and then identifying opposite sides of the wedge. The particle spacetime is thus defined via the rotation generator with angle α: namely,

$$R(\alpha)=\begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}. \tag{26}$$

The mass of the particle is given by $m=\alpha/\pi$, in units with $8G=1$, and the resulting spacetime has a naked conical singularity.

A moving particle spacetime is obtained by boosting to the rest frame of the particle, rotating, and then boosting back. The corresponding boost matrix is given by

$$B(\boldsymbol{\xi})=\begin{pmatrix} \cosh\frac{\xi}{2}-\sinh\frac{\xi}{2}\sin\phi & \sinh\frac{\xi}{2}\cos\phi \\ \sinh\frac{\xi}{2}\cos\phi & \cosh\frac{\xi}{2}+\sinh\frac{\xi}{2}\sin\phi \end{pmatrix}, \tag{27}$$

where $\boldsymbol{\xi}$ is the boost vector with $\xi=|\boldsymbol{\xi}|$, and ϕ is the polar angle. Thus, the generator for a moving particle takes the form

$$T=B(\boldsymbol{\xi})R(\alpha)B^{-1}(\boldsymbol{\xi}), \tag{28}$$

where

$$T_{11}=\cos\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cosh\frac{\xi}{2}\sinh\frac{\xi}{2}\cos\phi, \tag{29}$$

$$\begin{aligned}
T_{12}=&-\sin\frac{\alpha}{2}\left[\cosh^2\frac{\xi}{2}+\sinh^2\frac{\xi}{2}\right]\\
&+2\sin\frac{\alpha}{2}\cosh\frac{\xi}{2}\sinh\frac{\xi}{2}\sin\phi,
\end{aligned}$$

$$T_{21}=\sin\frac{\alpha}{2}\left[\cosh^2\frac{\xi}{2}+\sinh^2\frac{\xi}{2}\right]$$
$$+2\sin\frac{\alpha}{2}\cosh\frac{\xi}{2}\sinh\frac{\xi}{2}\sin\phi,$$

$$T_{22}=\cos\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cosh\frac{\xi}{2}\sinh\frac{\xi}{2}\cos\phi.$$

To generalize this construction to AdS_3, we simply note that the static and moving particle spacetimes can be defined in an analogous fashion. The essential difference is that the particle spacetimes are now defined through left and right generators. In [26,28,29], the formation of BTZ black holes from point particle collisions was investigated. In particular, it was shown [3] that the Gott time machine [4] (a two-body collision process) suitably generalized to anti–de Sitter space, provides a precise mechanism for the formation of the BTZ black hole. Moreover, this purely algebraic process, in which a product of two elliptic generators becomes a hyperbolic generator, leads to an exact analytic determination of the Choptuik scaling parameter.

The Gott time machine is defined as a two-body collision process, with particles labeled by A and B, such that the mass and boost parameters obey a certain constraint, known as the Gott condition [4]. The elliptic generator for each particle is defined in terms of its mass and boost parameters, denoted by α and ξ. Moreover, the effective generator for the two particles is given by the product [27,30,31], namely $T^G=T_BT_A$. The order parameter of interest is the trace of this generator, which takes the form [3]

$$\frac{1}{2}\mathrm{Tr}T^G=-\cos\frac{\alpha_A}{2}\cos\frac{\alpha_B}{2}-\sin\frac{\alpha_A}{2}\sin\frac{\alpha_B}{2}$$
$$+\sin\frac{\alpha_A}{2}\sin\frac{\alpha_B}{2}\left[\cosh^2\left(\frac{\xi_A+\xi_B}{2}\right)+\cosh^2\left(\frac{\xi_A-\xi_B}{2}\right)\right]$$
$$-\sin\frac{\alpha_A}{2}\sin\frac{\alpha_B}{2}\cos(\phi_A-\phi_B)\left[\cosh^2\left(\frac{\xi_A+\xi_B}{2}\right)\right.$$
$$\left.-\cosh^2\left(\frac{\xi_A-\xi_B}{2}\right)\right]. \tag{30}$$

The original Gott time machine is recovered by choosing particles with equal masses, and equal and opposite boosts, namely $\alpha_A=\alpha_B=\alpha,\ \xi_A=\xi_B=\xi,\ \phi_A-\phi_B=\pi$. Thus, when the Gott condition is satisfied, namely $\sin^2\alpha/2\cosh^2\xi>1$, we see that T^G is a hyperbolic generator. When the Gott condition is not satisfied, we have an elliptic generator.

To construct the BTZ black hole, we simply take the independent left and right generators ρ_L,ρ_R to be defined in terms of two-particle Gott generators. Thus, we take $\rho_L=T^G$ in Eq. (30) with $\alpha_A=\alpha_B=\alpha,\ \phi_A-\phi_B=0$. This gives

$$\frac{1}{2}\mathrm{Tr}\rho_L=\cosh\left(\frac{\pi}{l}(r_+-r_-)\right)=-1+2\sin^2\frac{\alpha}{2}\cosh^2\left(\frac{\xi_A-\xi_B}{2}\right)$$
$$\equiv p_L. \tag{31}$$

For the right generator, we choose $\rho_R=T^G$ with $\alpha_A=\alpha_B=\alpha,\ \phi_A-\phi_B=\pi$, leading to

$$\frac{1}{2}\mathrm{Tr}\rho_R=\cosh\left(\frac{\pi}{l}(r_++r_-)\right)=-1+2\sin^2\frac{\alpha}{2}\cosh^2\left(\frac{\xi_A+\xi_B}{2}\right)$$
$$\equiv p_R. \tag{32}$$

We see that both ρ_L and ρ_R become hyperbolic if the input parameters α,ξ_A,ξ_B satisfy the appropriate Gott conditions, namely $p_L>1$ and $p_R>1$. Thus, the critical value of the input parameters is $p_L^*=p_R^*=1$. As shown in [3], the Choptuik scaling parameter can now be simply read off from Eqs. (31) and (32), by using the formula, $\operatorname{arccosh} p=\ln[p+\sqrt{p^2-1}]$. Writing $p_L=p_L^*+\epsilon$, and $p_R=p_R^*+\epsilon$, we find, to leading order,

$$\frac{r_+-r_-}{l}=\frac{\sqrt{2}}{\pi}(p_L-p_L^*)^{1/2},$$
$$\frac{r_++r_-}{l}=\frac{\sqrt{2}}{\pi}(p_R-p_R^*)^{1/2}. \tag{33}$$

Thus, the Choptuik scaling parameter for $(r_+\pm r_-)$ is $\gamma=1/2$. One can equally well express the scaling behavior in terms of the mass M and angular momentum J, by using Eq. (4). If the Gott condition is not satisfied, then one has an effective particle spacetime with an elliptic generator. Therefore, the nature of the transition described above is between a BTZ black hole spacetime and a particle spacetime with a naked conical singularity. By definition, the BTZ black hole is defined in terms of a hyperbolic generator. Thus, irrespective of which type of matter is used to produce such a black hole, the ultimate result is that the order parameter is defined as the trace of this hyperbolic generator. As we have seen, as long as the spacetime on the other side of the transition is a particle spacetime, the formation process will be characterized by a scaling parameter of 1/2. Indeed, such a scaling exponent of 1/2 was found for collapsing dust shells in [32]. Other aspects of Choptuik scaling for the BTZ black hole have been investigated in [33–36], although in these cases the nature of the transition is different from the above.

We can now compare this result with the quasinormal frequencies (18). We see immediately that the negative of the imaginary part of ω_L and ω_R, denoted by $(\omega_L)_{\mathrm{Im}}$ and $(\omega_R)_{\mathrm{Im}}$, scales with (r_+-r_-) and (r_++r_-), respectively. In particular, we have

$$(\omega_L)_{\mathrm{Im}}=\frac{1}{\gamma}\left(\frac{r_+-r_-}{l^2}\right)\left(n+\frac{1}{2}+\frac{1}{2}\sqrt{\mu+1}\right),$$
$$(\omega_R)_{\mathrm{Im}}=\frac{1}{\gamma}\left(\frac{r_++r_-}{l^2}\right)\left(n+\frac{1}{2}+\frac{1}{2}\sqrt{\mu+1}\right). \tag{34}$$

We have thus established an exact connection between the Choptuik scaling parameter and the imaginary part of the quasinormal modes. It is satisfying that in the $(2+1)$-dimensional case, these exact calculations lead to a result precisely of the form noticed in [7]. The Choptuik scaling parameter of $\gamma=1/2$ used to establish this connection will be present for any type of matter, when the transition is between a BTZ black hole phase (defined by a hyperbolic generator) and a particle spacetime (defined by an elliptic generator). It should be stressed that we have verified the connection between this Choptuik scaling parameter and the quasinormal modes of a scalar perturbation of the black hole. Of course, one can also consider quasinormal modes associated with other forms of matter, as well as those associated with the gravitational perturbations of the black hole. It remains to be seen if there is a similar connection between the Choptuik scaling parameter and the imaginary part of these modes. The quasinormal modes of electromagnetic and Weyl perturbations have been calculated in [12]. For the electromagnetic case, the modes are the same as those of a massless scalar field, and thus exhibit the connection with Choptuik scaling. For the Weyl perturbation, the modes have been calculated numerically, and exhibit a similar connection.

By virtue of the AdS/CFT correspondence, the imaginary part of the quasinormal modes has a direct interpretation in the dual conformal field theory. In the case at hand, the boundary conformal field theory of AdS_3 contains both left-moving and right-moving sectors [37], with Virasoro generators $\overline{L}_0=(r_+-r_-)^2/2l$ and $L_0=(r_++r_-)^2/2l$, respectively. Thus, the return to equilibrium of the conformal field theory is specified in terms of the left and right time scales given by $\tau_L=1/(\omega_L)_{\rm Im}$ and $\tau_R=1/(\omega_R)_{\rm Im}$. Further analysis of BTZ black hole formation within the context of the AdS/CFT correspondence has been presented in [38–41].

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