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Path integral derivation of the Brown-Henneaux central charge

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We rederive the Brown-Henneaux commutation relation and central charge in the framework of the path integral. To obtain the Ward-Takahashi identity, we can use either asymptotic symmetry or its leading part. If we use asymptotic symmetry, the central charge arises from the transformation law of the charge itself. Thus, this central charge is clearly different from the quantum anomaly which can be understood as the Jacobian factor of the path integral measure. Alternatively, if we use the leading transformation, the central charge arises from the fact that the boundary condition of the path integral is not invariant under the transformation. This is in contrast with the usual quantum central charge which arises from the fact that the measure of the path integral is not invariant under the relevant transformation. Moreover, we discuss the implications of our analysis in relation to the black hole entropy.

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I. INTRODUCTION

To the black hole in general relativity, we can assign an entropy called the Bekenstein-Hawking entropy. There have been many attempts to understand the origin of this black hole entropy. In particular, there are some attempts which are based on the central charge in conformal field theory (CFT). Note that the density of states can be calculated from the central charge by using the Cardy formula [1,2] in conformal field theory. For example, in the framework of superstring theory, it has been shown that the black hole entropy can be calculated from the central charge of the effective superconformal field theory for D-branes [3,4].

On the other hand, Strominger [5] has used the Brown-Henneaux central charge instead of the usual central charges. In the case of (2+1)-dimensional gravity with a negative cosmological constant $\Lambda=-1/l^2$, Brown and Henneaux [6] have shown that the asymptotic symmetry of an asymptotically AdS_3 spacetime is the conformal group in two dimensions rather than the AdS_3 group, SO(2,2). Moreover, they have shown that this symmetry is canonically realized by the Poisson brackets algebra of the generators (or the Dirac brackets algebra of the charges) with a central extension. The central charge becomes

$$c=\frac{3l}{2G}. \tag{1.1}$$

By combining this central charge with the Cardy formula, Strominger has shown that the resultant entropy agrees with the Bekenstein-Hawking entropy of the Bañados-Teitelboim-Zanelli (BTZ) black hole [7,8]. Nevertheless, there remain some open questions in this approach [2]. In particular, the physical meaning of the Brown-Henneaux central charge is not clear from the original derivation.

Brown-Henneaux's central charge was also obtained by some approaches. Bañados and co-workers [9,10] used the Chern-Simons formulation of the (2+1)-dimensional gravity. See Refs. [11–13] in the context of AdS/CFT correspondence [14,15].

Natsuume, Okamura, and Sato [16] have generalized the Brown-Henneaux central charge to include a conformal scalar field and applied Strominger's approach to the Martínez-Zanelli (MZ) black hole [17]. However, since they have obtained the same charge and central charge as the case of the pure gravity, the density of states from the Cardy formula does not agree with the Bekenstein-Hawking entropy. (The functional form does agree but the over-all numerical factor does not.) Thus, they have considered that the Cardy formula gives the "maximum possible entropy" for a given mass. See also Ref. [18] for the extension to various theories.

Since the Brown-Henneaux central charge is based on the asymptotic symmetry at infinity in (2+1) dimensions, Carlip [19,20] has generalized to the symmetry near the black hole horizon in any dimensions. This symmetry contains a natural Virasoro subalgebra and the resulting central charge reproduces the Bekenstein-Hawking entropy by the Cardy formula. (However, since the generator of Ref. [19] is not "differentiable" [21,22], it has been proposed by Soloviev [23] to use a modified Poisson bracket which is applicable to the "nondifferentiable" functionals.) Similar conclusion was obtained by Solodukhin [24] with effective two-dimensional theories for the spherically symmetric metrics in higher dimensions. Moreover, Carlip [25] has calculated the prefactor of the Cardy formula and the logarithmic correction to the Bekenstein-Hawking entropy.

In this paper, we rederive the Brown-Henneaux commutation relation and central charge in terms of the path integral formulation since it was originally obtained by the canonical formulation. In view of the equivalence of these two approaches to the quantum theory, we must obtain the same result within the path integral. The anomalous commutators have been obtained also by using the path integral formulation in various gauge theories [26–28]. The central charge arises from the quantum anomaly which is understood as the Jacobian factor of the path integral measure in the usual case [27]. However, the Brown-Henneaux central charge is a classical one because it exists at the level of the Poisson brackets. We thus want to clarify the origin of this central charge in the path integral. Moreover, we want to answer the question if the Brown-Henneaux central charge is concerned with the degrees of freedom contained in the system as usual central charges. We also hope that this path integral derivation would be useful for exploring the relation between Strominger's approach and Gibbons-Hawking's approach [29,30]. This is because the topological consideration is an advantage of the path integral calculation.

One might think that the path integral would be irrelevant to the classical quantity. However, the classical quantity still exists after the quantization as the zeroth order term in $\hbar$. In order to calculate the whole (classical and quantum) central charge by the path integral, we need the derivation of its classical part within the framework of the path integral. Although the quantum part is difficult to calculate, its origin in terms of the path integral is quite clear. We thus ignore the quantum part and concentrate on the classical one.

To derive the commutation relation, we must first identify the charge. Since the original derivation was based on the Regge-Teitelboim method [31], the charge was obtained indirectly by integrating its variation. This process seems to be ad hoc in more general situations. Thus, we extract the charge from the variation of the action, directly. We then use two transformations to obtain the Ward-Takahashi identity. One is the asymptotic symmetry and the other is its leading part. If we use the asymptotic symmetry, we find that the central charge arises from the transformation law of the charge itself. Thus, we can see it as a classical central charge. On the other hand, if we use its leading transformation, we find that the central charge arises due to the change of the *boundary condition* of the path integral. This contrasts with the usual quantum central charge which arises due to the change of the *measure* of the path integral.

The paper is organized as follows. In Sec. II, we summarize the variation and transformation property of the action. In Sec. III, we review the asymptotically AdS_3 spacetime and calculate some basic quantities. In Sec. IV, we identify the Brown-Henneaux charge and derive its transformation law. In Sec. V, we combine all the results to obtain the commutation relation. In Sec. VI, we give another derivation where the origin of the central charge is more interesting. In Sec. VII, we discuss the implications of our result.

II. VARIATION OF ACTION

We consider the $(2+1)$-dimensional gravity with a negative cosmological constant $\Lambda<0$. We assume that the boundary of the spacetime is only at infinity Σ^∞ whose unit normal vector is u^a. We thus begin with the Hilbert action with the surface term [29,30],

$$S=\frac{1}{16\pi G}\int_M\sqrt{-g}(R-2\Lambda)d^3x+\frac{1}{8\pi G}\int_{\Sigma^\infty}\sqrt{-\gamma}\Theta\, d^2x, \tag{2.1}$$

where γ_{ab} is the induced metric on Σ^∞ defined by

$$\gamma_{ab}\equiv g_{ab}-u_a u_b, \tag{2.2}$$

and Θ^{ab} is the extrinsic curvature of Σ^∞ defined by

$$\Theta^{ab}=\gamma^{ac}\nabla_c u^b, \quad \Theta=g_{ab}\Theta^{ab}=\gamma^{ab}\nabla_a u_b. \tag{2.3}$$

As is well known, the variation of this action with the condition $\delta g_{ab}=0$ at Σ^∞ contains no surface terms and thus we can obtain Einstein equation. However, if we take a *generic* variation of this action, we obtain [32,33]

$$\delta S=-\frac{1}{16\pi G}\int_M\sqrt{-g}\tilde{G}^{ab}\delta g_{ab}d^3x -\frac{1}{16\pi G}\int_{\Sigma^\infty}\sqrt{-\gamma}\Pi^{ab}\delta\gamma_{ab}d^2x, \tag{2.4}$$

where

$$\tilde{G}^{ab}=R^{ab}-\tfrac{1}{2}g^{ab}R+\Lambda g^{ab}, \tag{2.5}$$

$$\Pi^{ab}=\Theta^{ab}-\Theta\,\gamma^{ab}. \tag{2.6}$$

(Note that $\sqrt{-\gamma}\Pi^{ab}$ has the same form as the canonical momentum in Arnowitt-Deser-Misner formulation.)

Next, we consider the coordinate transformation which is generated by a vector ζ^a, namely,

$$\delta_\zeta g_{ab}=\pounds_\zeta g_{ab}=\nabla_a\zeta_b+\nabla_b\zeta_a. \tag{2.7}$$

By using Eq. (2.4), we find that

$$\delta_\zeta S=-\frac{1}{8\pi G}\int_M\sqrt{-g}\tilde{G}^{ab}\nabla_a\zeta_b d^3x -\frac{1}{8\pi G}\int_{\Sigma^\infty}\sqrt{-\gamma}\Pi^{ab}\nabla_a\zeta_b d^2x, \tag{2.8}$$

where we have used

$$\Pi^{ab}\delta\gamma_{ab}=\Pi^{ab}\delta g_{ab}, \tag{2.9}$$

since $\Pi^{ab}u_a=0$. We then decompose ζ^a into $\tilde{\zeta}^a$ and $\hat{\zeta}^a$, where

$$\tilde{\zeta}^a\equiv\gamma^a{}_b\zeta^b=\zeta^a-\eta u^a, \tag{2.10}$$

$$\hat{\zeta}^a\equiv\eta u^a, \tag{2.11}$$

$$\eta=\zeta^a u_a. \tag{2.12}$$

Note that $\tilde{\zeta}^a$ is tangential to the boundary Σ^∞ since $\tilde{\zeta}^a u_a=0$ and $\hat{\zeta}^a$ is normal to the boundary because $\hat{\zeta}^a$ is proportional to the normal vector u^a. The tangential part becomes

$$\Pi^{ab}\nabla_a\tilde{\zeta}_b=\Pi^{cd}\gamma_c{}^a\gamma_d{}^b\nabla_a\tilde{\zeta}_b=\Pi^{cd}\mathcal{D}_c\tilde{\zeta}_d, \tag{2.13}$$

where $\mathcal{D}_a$ is the covariant derivative associated with γ_{ab}. On the other hand, the normal part becomes

$$\Pi^{ab}\nabla_a\hat{\zeta}_b=\eta\Pi^{ab}\nabla_a u_b=\eta\Pi^{ab}\Theta_{ab}=\eta(\Theta^{ab}\Theta_{ab}-\Theta^2). \tag{2.14}$$

Therefore, one finds that

$$\delta_\zeta S = -\frac{1}{8\pi G}\int_M \sqrt{-g}\tilde{G}^{ab}\nabla_a \zeta_b d^3x - \frac{1}{8\pi G}\int_{\Sigma^\infty}\sqrt{-\gamma}[\Pi^{ab}\mathcal{D}_a\tilde{\zeta}_b + \eta(\Theta^{ab}\Theta_{ab}-\Theta^2)]d^2x. \tag{2.15}$$

III. ASYMPTOTICALLY AdS_3 SPACETIME

From now on, we consider the asymptotically AdS_3 spacetime which is defined by the boundary condition,

$$\begin{aligned} g_{tt} &= -\frac{r^2}{l^2} + \mathcal{O}(1), \\ g_{tr} &= \mathcal{O}(1/r^3), \\ g_{t\phi} &= \mathcal{O}(1), \\ g_{rr} &= \frac{l^2}{r^2} + \mathcal{O}(1/r^4), \\ g_{r\phi} &= \mathcal{O}(1/r^3), \\ g_{\phi\phi} &= r^2 + \mathcal{O}(1). \end{aligned} \tag{3.1}$$

We treat the boundary Σ^∞ as the $r = r_*$ surface and then take the limit $r_* \to \infty$.

Brown and Henneaux [6] have shown that the asymptotic symmetry, namely the coordinate transformation which preserves the asymptotic boundary condition (3.1), becomes

$$\begin{aligned} \xi^t &= lT(t,\phi) + \frac{l^3}{r^2}\overline{T}(t,\phi) + \mathcal{O}(1/r^4), \\ \xi^r &= rR(t,\phi) + \frac{l^2}{r}\overline{R}(t,\phi) + \mathcal{O}(1/r^3), \\ \xi^\phi &= \Phi(t,\phi) + \frac{l^2}{r^2}\overline{\Phi}(t,\phi) + \mathcal{O}(1/r^4), \end{aligned} \tag{3.2}$$

where they satisfy

$$\begin{aligned} l\partial_t T(t,\phi) &= \partial_\phi \Phi(t,\phi) = -R(t,\phi), \\ l\partial_t \Phi(t,\phi) &= \partial_\phi T(t,\phi), \\ \overline{T}(t,\phi) &= -\frac{l}{2}\partial_t R(t,\phi), \\ \overline{\Phi}(t,\phi) &= \frac{1}{2}\partial_\phi R(t,\phi), \end{aligned} \tag{3.3}$$

but $\overline{R}(t,\phi)$ is arbitrary, and this is the conformal group in two dimensions. This fact can be understood easily by means of conformal infinity by Penrose [34,35]. Briefly, since the induced metric on the boundary $r = r_* \to \infty$ is formally written as

$$\infty \times (-dt^2 + l^2 d\phi^2), \tag{3.4}$$

the conformal transformation on the (t,ϕ) plane leaves this induced metric "invariant,"

$$\infty \times e^{\rho(t,\phi)}(-dt^2 + l^2 d\phi^2) = \infty \times (-dt^2 + l^2 d\phi^2). \tag{3.5}$$

We write the next leading terms of the metric as

$$\begin{aligned} g_{tt} &= -\frac{r^2}{l^2} + e_{tt} + \mathcal{O}(1/r^2), \\ g_{tr} &= \frac{l^3}{r^3}e_{tr} + \mathcal{O}(1/r^5), \\ g_{t\phi} &= e_{t\phi} + \mathcal{O}(1/r^2), \\ g_{rr} &= \frac{l^2}{r^2} + \frac{l^4}{r^4}e_{rr} + \mathcal{O}(1/r^6), \\ g_{r\phi} &= \frac{l^3}{r^3}e_{r\phi} + \mathcal{O}(1/r^5), \\ g_{\phi\phi} &= r^2 + l^2 e_{\phi\phi} + \mathcal{O}(1/r^2), \end{aligned} \tag{3.6}$$

where e_{ab} depend only on (t,ϕ). Then, one can obtain the expressions for some basic quantities,

$$\begin{aligned} \Pi^t{}_t &= -\frac{1}{l} + \frac{l}{r^2}\left(\frac{1}{2}e_{rr} + e_{\phi\phi}\right) + \mathcal{O}(1/r^4), \\ \Pi^t{}_\phi &= \frac{l}{r^2}e_{t\phi} + \mathcal{O}(1/r^4), \\ \Pi^\phi{}_t &= -\frac{1}{r^2 l}e_{t\phi} + \mathcal{O}(1/r^4), \\ \Pi^\phi{}_\phi &= -\frac{1}{l} + \frac{l}{r^2}\left(\frac{1}{2}e_{rr} - e_{tt}\right) + \mathcal{O}(1/r^4), \\ \Theta^{ab}\Theta_{ab} - \Theta^2 &= -\frac{2}{l^2} - \frac{2}{r^2}(e_{tt} - e_{rr} - e_{\phi\phi}) + \mathcal{O}(1/r^4). \end{aligned} \tag{3.7}$$

Note that the transformation law for e_{ab} is

$$\begin{aligned} \delta_\xi e_{rr} &= lT\partial_t e_{rr} + \Phi\partial_\phi e_{rr} - 2e_{rr}R - 4\overline{R}, \\ \delta_\xi e_{\phi\phi} &= lT\partial_t e_{\phi\phi} + \Phi\partial_\phi e_{\phi\phi} + \frac{2}{l}e_{t\phi}\partial_\phi T \\ &\quad + 2e_{\phi\phi}\partial_\phi\Phi + 2\overline{R} + 2\partial_\phi\overline{\Phi}, \end{aligned} \tag{3.8}$$

$$\delta_\xi e_{t\phi} = lT\partial_t e_{t\phi} + \Phi\partial_\phi e_{t\phi} + e_{tt} l\partial_\phi T + e_{t\phi}\partial_\phi \Phi + e_{t\phi} l\partial_t T + l^2 e_{\phi\phi}\partial_t \Phi - l\partial_\phi \overline{T} + l^2 \partial_t \overline{\Phi},$$

and so on.

Furthermore, the equations of motion $\tilde{G}_{ab}=0$ say that

$$\frac{l^2}{2}\partial_t e_{rr} + l^2 \partial_t e_{\phi\phi} = \partial_\phi e_{t\phi},$$
$$-\frac{1}{2}\partial_\phi e_{rr} + \partial_\phi e_{tt} = \partial_t e_{t\phi}, \tag{3.9}$$
$$e_{tt} - e_{rr} - e_{\phi\phi} = 0.$$

These equations arise from the tr, $r\phi$, and rr components. The other components become trivial up to this order.

IV. CURRENT AND CHARGE

We can obtain the Brown-Henneaux charge [6] from the tensor Π^{ab},

$$J[\xi] = -\frac{1}{8\pi G}\lim_{r_*\to\infty}\int_{r=r_*} d\phi\sqrt{\sigma}(\Pi^a{}_b - \hat{\Pi}^a{}_b)\tilde{\xi}^b n_a, \tag{4.1}$$

where n^a is the unit normal vector of the time slice and σ_{ab} is the induced metric on the boundary $r=r_*$ of the time slice. The hat means that it is evaluated by AdS_3 spacetime so that the charge becomes zero in AdS_3 spacetime. This is similar to the charge defined by Brown and York [33] in the context of the quasilocal energy,

$$J_{BY}[\xi] = -\frac{1}{8\pi G}\lim_{r_*\to\infty}\int_{r=r_*} d\phi\sqrt{\sigma}(\Pi^{ab} - \hat{\Pi}^{ab})\tilde{\xi}_b n_a. \tag{4.2}$$

However, these are different because $\hat{\Pi}^a{}_b\tilde{\xi}^b \neq \hat{\Pi}^{ab}\tilde{\xi}_b$ in the subtraction term.

By using the expansions (3.6) and (3.2), this becomes

$$J[\xi] = \frac{1}{8\pi G}\lim_{r_*\to\infty}\frac{r_*^2}{l}\int_{r=r_*} d\phi(\Pi^t{}_b - \hat{\Pi}^t{}_b)\tilde{\xi}^b = \frac{1}{8\pi G}\int d\phi\left\{\left[\frac{1}{2}(e_{rr}+1) + e_{\phi\phi}\right]lT + e_{t\phi}\Phi\right\}, \tag{4.3}$$

for the asymptotic symmetry of the asymptotically AdS_3 spacetime. In fact, one can check that, by using the same expansions, the Brown-Henneaux charge also becomes this expression.

The current for the transformation is considered as

$$j^a[\xi] = -\frac{1}{8\pi G}(\Pi^a{}_b - \hat{\Pi}^a{}_b)\tilde{\xi}^b|_{r=r_*}, \tag{4.4}$$

and the charge is then written as

$$J[\xi] = \lim_{r_*\to\infty}\int_{r=r_*} d\phi\sqrt{\sigma} j^a[\xi] n_a. \tag{4.5}$$

By using equations of motion (3.9), one can find that the charge is actually conserved,

$$\partial_t J[\xi] = 0, \tag{4.6}$$

where we have used the condition (3.3) for ξ^a.

The charge for the Lie bracket of two vectors becomes

$$J[[\xi_1,\xi_2]] = \frac{1}{8\pi G}\int d\phi\left\{\left[\frac{1}{2}(e_{rr}+1) + e_{\phi\phi}\right](l^2 T_1\partial_t T_2 + l\Phi_1\partial_\phi T_2 - l^2 T_2\partial_t T_1 - l\Phi_2\partial_\phi T_1) + e_{t\phi}(lT_1\partial_t\Phi_2 + \Phi_1\partial_\phi\Phi_2 - lT_2\partial_t\Phi_1 - \Phi_2\partial_\phi\Phi_1)\right\}. \tag{4.7}$$

On the other hand, by using Eqs. (3.8) and (3.3), one can obtain

$$\delta_{\xi_2} J[\xi_1] = \frac{1}{8\pi G}\int d\phi\left\{\left[\frac{1}{2}(\delta_{\xi_2}e_{rr}) + (\delta_{\xi_2}e_{\phi\phi})\right]lT_1 + (\delta_{\xi_2}e_{t\phi})\Phi_1\right\} = J[[\xi_1,\xi_2]] + K[\xi_1,\xi_2] + \cdots, \tag{4.8}$$

where $K[\xi_1,\xi_2]$ is the Brown-Henneaux central charge [6],

$$K[\xi_1,\xi_2] = (J[\xi_1] \quad \text{at} \quad g_{ab} = \hat{g}_{ab} + \pounds_{\xi_2}\hat{g}_{ab}) = -\frac{1}{8\pi G}\int d\phi [T_1(\partial_\phi + \partial_\phi^3) + \Phi_1(l\partial_t + l^3\partial_t^3)] l\Phi_2, \tag{4.9}$$

and "$\cdots$" means the terms which vanish by using the equations of motion (3.9). Note that there is the central term in the transformation law of the charge itself, which is explicitly derived from our definition of the charge without using the Dirac brackets algebra. However, we must supply the commutator by using the path integral in order to identify it as the central charge. This is because other contributions might arise from somewhere. Indeed, in Sec. VI, we will see that there is another possible source of the classical central charge.

V. COMMUTATION RELATION

To derive the commutation relation of two charges, we begin with the path integral,

$$\langle J[\xi_1]\rangle = \int_B d\mu J[\xi_1] e^{iS}, \tag{5.1}$$

where $d\mu$ and B denote the measure and boundary condition of the path integral, respectively. We first replace the integration variable g_{ab} everywhere in Eq. (5.1) with $\tilde{g}_{ab}$. This step is mathematically trivial, similar to the replacement

$$\int f(x)dx \to \int f(y)dy. \tag{5.2}$$

We recognize this new integration variable as the transformed metric by the infinitesimal asymptotic symmetry transformation ξ_2,

$$\tilde{g}_{ab} \equiv g_{ab} + \delta_{\xi_2} g_{ab}. \tag{5.3}$$

We then find that

$$\begin{aligned}\langle J[\xi_1]\rangle &= \int_{\tilde{B}} d\tilde{\mu} \tilde{J}[\xi_1] e^{i\tilde{S}} \\ &= \int_B d\mu (J[\xi_1] + \delta_{\xi_2} J[\xi_1])(1 + i\delta_{\xi_2} S) e^{iS} \\ &= \int_B d\mu (J[\xi_1] + \delta_{\xi_2} J[\xi_1] + iJ[\xi_1]\delta_{\xi_2} S) e^{iS} \\ &= \langle J[\xi_1]\rangle + \langle \delta_{\xi_2} J[\xi_1]\rangle + i\langle \mathrm{T}^* J[\xi_1]\delta_{\xi_2} S\rangle, \end{aligned} \tag{5.4}$$

where we have used the fact that the boundary condition is invariant under the asymptotic symmetry, $\tilde{B} = B$. We also assumed that the measure of the path integral is invariant under this transformation, $d\tilde{\mu} = d\mu$, since we want to see the classical central charge. Therefore, we can obtain the Ward-Takahashi identity

$$\langle \delta_{\xi_2} J[\xi_1]\rangle = -i\langle \mathrm{T}^* J[\xi_1]\delta_{\xi_2} S\rangle. \tag{5.5}$$

By using Eq. (2.15), we can evaluate the right-hand side of this identity. However, the result becomes infinite in the limit of $r = r_* \to \infty$. In order to get a finite result, it is usual to subtract a functional S_0 of the boundary data γ_{ab} from the action. In this case, we choose so that [33]

$$\delta_\xi S_0 = -\frac{1}{8\pi G}\int_{\Sigma^\infty} \sqrt{-\gamma}[\hat{\Pi}^a{}_b \mathcal{D}_a \tilde{\xi}^b + \eta(\hat{\Theta}^{ab}\hat{\Theta}_{ab} - \hat{\Theta}^2)]d^2x, \tag{5.6}$$

where the hats again mean that they are evaluated by AdS_3 spacetime. For notational simplicity, we write $S - S_0$ as S anew.

Then, one finds that

$$\begin{aligned}\delta_{\xi_2} S &= -\frac{1}{8\pi G}\lim_{r_*\to\infty}\int_{r=r_*} dt d\phi \sqrt{-\gamma}(\Pi^a{}_b - \hat{\Pi}^a{}_b)\mathcal{D}_a \tilde{\xi}_2^b \\ &\quad + \cdots \\ &= -\frac{1}{8\pi G}\lim_{r_*\to\infty}\frac{r_*^2}{l}\int_{r=r_*} dt d\phi \mathcal{D}_a[(\Pi^a{}_b - \hat{\Pi}^a{}_b)\tilde{\xi}_2^b] + \cdots \\ &= -\frac{1}{8\pi G}\lim_{r_*\to\infty}\frac{r_*^2}{l}\int_{r=r_*} dt d\phi \partial_t[(\Pi^t{}_b - \hat{\Pi}^t{}_b)\tilde{\xi}_2^b] + \cdots \\ &= -\int dt \partial_t J[\xi_2] + \cdots, \end{aligned} \tag{5.7}$$

where "· · ·" again means the terms which vanish by using the equations of motion (3.9). Note that

$$(\Theta^{ab}\Theta_{ab} - \Theta^2) - (\hat{\Theta}^{ab}\hat{\Theta}_{ab} - \hat{\Theta}^2) = \mathcal{O}(1/r^4),$$

$$\mathcal{D}_a(\Pi^a{}_b - \hat{\Pi}^a{}_b) = \mathcal{O}(1/r^4),$$

by the equations of motion.

Then, the right-hand side of the identity (5.5) is

$$\begin{aligned}\langle \mathrm{T}^* J[\xi_1]\delta_{\xi_2} S\rangle &= -\left\langle \mathrm{T}^* J[\xi_1]\int dt \partial_t J[\xi_2] + \cdots \right\rangle \\ &= -\int dt \partial_t \langle \mathrm{T}^* J[\xi_1] J[\xi_2] + \cdots \rangle \\ &= -\int dt \partial_t \langle \mathrm{T}\, J[\xi_1] J[\xi_2] + \cdots \rangle \\ &= \langle [J[\xi_1], J[\xi_2]] + \cdots \rangle, \end{aligned} \tag{5.8}$$

where we have simply identified the T* product as the T product after extracting the time derivative by the standard Bjorken-Johnson-Low argument [36,37].

Finally, by using Eq. (4.8) and the equations of motion (3.9), we can obtain the commutation relation,

$$\langle [J[\xi_1], J[\xi_2]]\rangle = \langle iJ[[\xi_1,\xi_2]] + iK[\xi_1,\xi_2]\rangle. \tag{5.9}$$

This is consistent with the result of Brown and Henneaux [6] if we identify the Dirac brackets as the commutator,

$$\{A,B\}_{\mathrm{D.B.}} \leftrightarrow \frac{1}{i}[A,B]. \tag{5.10}$$

Note that the classical central charge comes from the transformation law of the charge (4.8) rather than the Jacobian factor of the path integral measure. This is analogous to the central charge in $N = 2$ supersymmetric theory [28].

VI. ANOTHER DERIVATION

We can derive this central charge in another way [38] if we use only the leading part of the asymptotic symmetry,

$$\begin{aligned}\xi'^t &= lT(t,\phi), \\ \xi'^r &= rR(t,\phi), \\ \xi'^\phi &= \Phi(t,\phi), \end{aligned} \tag{6.1}$$

where T, R, Φ again satisfy the condition (3.3). This transformation is *not* the asymptotic symmetry since it breaks the boundary conditions for g_{tr} and $g_{r\phi}$,

$$\pounds_\xi g_{tr} = \frac{l^2}{r}\partial_t R + \mathcal{O}(1/r^3),$$

$$\pounds_{\xi} g_{r\phi} = \frac{l^2}{r} \partial_\phi R + \mathcal{O}(1/r^3).$$

Note that

$$J[\xi'] = J[\xi], \quad J[[\xi_1', \xi_2']] = J[[\xi_1, \xi_2]], \tag{6.2}$$

and

$$\delta_{\xi'} S = \delta_\xi S, \tag{6.3}$$

since the nonleading parts do not contribute to these quantities. On the other hand, we have

$$\delta_{\xi_2'} J[\xi_1'] = J[[\xi_1', \xi_2']] + K'[\xi_1', \xi_2'] + \cdots, \tag{6.4}$$

where

$$K'[\xi_1', \xi_2'] = -\frac{1}{8\pi G} \int d\phi [T_1 \partial_\phi + \Phi_1 l \partial_t] l \Phi_2. \tag{6.5}$$

This is not a nontrivial central charge since we can eliminate this term by adding a constant to the charge. (Actually, it can be achieved by choosing the $M = J = 0$ BTZ black hole as the background rather than AdS_3 spacetime [38].) This is the interesting aspect of this leading transformation (6.1). Since the remaining quantities in the Ward-Takahashi identity (5.5) are equal, one might think that one could obtain the commutator without the nontrivial central charge,

$$\langle [J[\xi_1], J[\xi_2]] \rangle \overset{??}{=} \langle iJ[[\xi_1, \xi_2]] + iK'[\xi_1', \xi_2'] \rangle, \tag{6.6}$$

if we use this leading transformation (6.1). However, this is not correct. Since the leading transformation breaks the boundary condition of the path integral B, we cannot obtain the Ward-Takahashi identity (5.5). Instead, the Ward-Takahashi identity is supplemented with an additional term due to the change of the boundary condition B, namely,

$$\langle \delta_{\xi_2'} J[\xi_1'] \rangle = -i \langle \mathrm{T}^* J[\xi_1'] \delta_{\xi_2'} S \rangle - \Delta[\xi_1', \xi_2'], \tag{6.7}$$

where

$$\Delta[\xi_1', \xi_2'] \equiv \left(\int_{B + \delta_{\xi_2'} B} - \int_B \right) d\mu J[\xi_1'] e^{iS}, \tag{6.8}$$

and the boundary condition $B + \delta_{\xi_2'} B$ denotes that the transformed metric $g_{ab} + \delta_{\xi_2'} g_{ab}$ must satisfy the asymptotically AdS_3 condition (3.1). Repeating the derivation as above, we find that

$$\langle [J[\xi_1'], J[\xi_2']] \rangle = \langle iJ[[\xi_1', \xi_2']] + iK'[\xi_1', \xi_2'] \rangle + i\Delta[\xi_1', \xi_2']. \tag{6.9}$$

In order to evaluate $\Delta[\xi_1', \xi_2']$, we perform the infinitesimal change of the integration variable corresponding to the inverse transformation of ξ_2' in the first integral and that of ξ_2 in the second integral, similar to the calculations in Eq. (5.4). The integrals then become

$$\int_{B + \delta_{\xi_2'} B} d\mu J[\xi_1'] e^{iS} = \int_B d\mu (J[\xi_1'] - iJ[\xi_1'] \delta_{\xi_2'} S - \delta_{\xi_2'} J[\xi_1']) e^{iS}, \tag{6.10}$$

$$\int_B d\mu J[\xi_1'] e^{iS} = \int_B d\mu (J[\xi_1'] - iJ[\xi_1'] \delta_{\xi_2} S - \delta_{\xi_2} J[\xi_1']) e^{iS}. \tag{6.11}$$

Note that the boundary condition of both of the path integral become the same. By using Eqs. (4.8) and (6.4), we can obtain that

$$\Delta[\xi_1', \xi_2'] = \langle \delta_{\xi_2} J[\xi_1'] - \delta_{\xi_2'} J[\xi_1'] \rangle = \langle K[\xi_1, \xi_2] - K'[\xi_1', \xi_2'] \rangle. \tag{6.12}$$

Thus, we can again obtain the Brown-Henneaux commutation relation,

$$\langle [J[\xi_1'], J[\xi_2']] \rangle = \langle iJ[[\xi_1', \xi_2']] + iK[\xi_1, \xi_2] \rangle, \tag{6.13}$$

by using the equations of motion.

Therefore, the nontrivial part of the central charge arises from the fact that the *boundary condition* of the path integral B is not invariant under the leading transformation ξ'. This phenomenon is in contrast to the usual quantum case where the anomaly arises from the fact that the *measure* of the path integral $d\mu$ is not invariant under the relevant transformation.

VII. DISCUSSION

We have reproduced the Brown-Henneaux commutation relation in the context of the path integral. The origin of the central charge is not the Jacobian factor of the path integral measure as the quantum anomaly. If we use the asymptotic symmetry to derive the Ward-Takahashi identity, it arises from the transformation law of the charge. Thus, it can be considered as a classical anomaly even though the path integral formulation has been used to obtain the commutator of two charges. This is similar to the central charge in the $N = 2$ supersymmetric theory. In order to apply to the black hole entropy, we want to relate this central charge with some ignorance. It would be considered as follows. We could see the tensor Π^{ab} as a tensor in two dimensions on the boundary Σ^∞ since it can be obtained as the conjugate variable to the induced metric γ_{ab} on the boundary. The charge $J[\xi]$ is made from this tensor and boundary value of ξ. In this sense, the charge is a quantity on the two-dimensional boundary. However, the transformation ξ is in the three-dimensional spacetime. This transformation consists of the two-dimensional transformation with some additional terms. Especially, $\bar{T}$ and $\bar{\Phi}$ terms, which are required to maintain the boundary condition, give rise to the central charge. This gap corresponds to the ignorance due to the limit $r_* \to \infty$.

Alternatively, in order to derive the Ward-Takahashi identity, we can use the leading part of the asymptotic symmetry.

Then, we can find that the central charge arises from the fact that the *boundary condition* of the path integral is not invariant under the leading transformation. This is in contrast to the quantum central charge which arises from the fact that the *measure* of the path integral is not invariant under the relevant transformation. From this picture, we could see that the Brown-Henneaux central charge does not count the degrees of freedom in the system. Moreover, we could understand other classical central charges, such as in the $N=2$ supersymmetric theory, as above in the path integral formulation. Our analysis also suggests the possibility that the classical central charge may arise in more general theories if the boundary condition of the path integral is nontrivial.

Finally, one of the advantages of the present analysis is that the charge is identified directly and it is thus easy to apply to more general situations. The past approaches were based on the Regge-Teitelboim method where the charge $J[\xi]$ was derived from its variation $\delta J[\xi]$. However, in more general cases [19,16], it is not straightforward to do this integration. On the other hand, since the present analysis can identify the charge directly, it would be straightforward to apply in such cases. The other advantage of the present analysis would be the topological consideration. We hope that we could relate Strominger's approach to the Gibbons-Hawking approach [29] by using this path integral derivation.

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