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Stable black strings in anti–de Sitter space

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In five-dimensional Einstein gravity with a negative cosmological constant in the presence or absence of a *non-fine-tuned* 3-brane, we investigate the classical stability of black string solutions which are foliations of four-dimensional AdS/dS-Schwarzschild black holes. Such black strings are generically unstable as in the well-known Gregory-Laflamme instability. For AdS black strings, however, it turns out that they become stable if the longitudinal size of the horizon is larger than the order of the AdS_4 radius. Even in the case of unstable black strings, the AdS black strings have very different features of string fragmentation from those in the flat brane world. Some implications of our results for the Gubser-Mitra conjecture are also discussed.

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I. INTRODUCTION

The Schwarzschild black hole, which has a compact black hole horizon, is known to be stable in general relativity [1]. However, it is known that black string or brane solutions that are a foliation of lower dimensional Schwarzschild black holes are unstable, the so-called Gregory-Laflamme instability [2]. This instability extends to a much broader range of charged black branes in string theory with the exception of extremal or near extremal cases [3,4].

Gregory [5] showed for the first time that this black string instability persists in the presence of a cosmological constant. Recently, Randall and Sundrum [6] proposed an interesting model where gravity in higher dimensions with a negative cosmological constant is localized on a lower dimensional domain wall. In the flat brane world model where the tension of a 3-brane is fine tuned with the five-dimensional cosmological constant, any Ricci-flat four-dimensional metric can be embedded. For instance, a black string solution, which is simply a foliation of four-dimensional Schwarzschild black holes perpendicular to the 3-brane, can easily be constructed. The properties of such black strings were investigated in Refs. [7,8]. In particular, they argued that the black string is unstable near the AdS_5 horizon, but becomes stable in the vicinity of the 3-brane, indicating fragmentation into a cigar type (or pancakelike, more accurately [8]) black hole across the brane as a final state. Gregory [5] confirmed this conjecture by explicitly performing a linearized perturbation analysis of the black string background in this flat brane world scenario. The author also showed the instability of a black string embedded in the AdS_5 spacetime without the 3-brane.

As far as we know, however, all black string or brane instabilities mentioned above were shown within the context of asymptotically locally flat spacetimes in the sense that all slices orthogonal to the string or brane are asymptotically flat.[1] In fact, in the presence of a negative cosmological constant, it is also possible to construct black string solutions which are a foliation of black holes that are asymptotically *nonflat*. For example, the pure AdS_5 spacetime can be sliced into pure four-dimensional anti–de Sitter (AdS_4) or de Sitter (dS_4) spacetimes. Then it is straightforward to show that any four-dimensional metric satisfying four-dimensional Einstein equations with the same cosmological constant can be embedded into AdS_5 space. In particular, the AdS_4 (dS_4) Schwarzschild black hole can be embedded, resulting in a five-dimensional hypercylindrical AdS (dS) black string solution. Similarly, even in curved brane world models where a 3-brane with non-fine-tuned tension is introduced in AdS_5 backgrounds [10,11], this generalization holds (see Ref. [12] for the dS embedding).

Now it is of interest to see whether the Gregory-Laflamme instability still persists for these black string solutions in AdS backgrounds, because some naive arguments given below seem to indicate that the stability behavior for such black strings could be very different from the known Gregory-Laflamme instability. In addition, recently Gubser and Mitra [13] proposed an interesting conjecture about the relationship between the classical black string or brane instability and the local thermodynamic stability [4]. It states that a black string or brane with noncompact translational symmetry is classically stable if, and only if, it is locally thermodynamically stable. Since black string solutions in AdS space have warped geometries along the extra dimension, our study will show what happens when the assumption of translational symmetry is discarded in the Gubser-Mitra (GM) conjecture.

In this paper, we investigate the stability of the black string solutions that are asymptotically locally AdS_4/dS_4

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[1] Actually Emparan, Horowitz, and Myers have argued, based on entropy comparison, for the stability of Bañados-Teitelboim-Zanelli (BTZ) black string solutions in *four*-dimensional AdS space [9]. However, they used two 2-branes with a suitable identification of spacetime. It seems that, even if the separation of two 2-branes is taken to be infinitely large, this black string does not correspond to a one-dimension-fewer configuration of our AdS case with a single 3-brane due to the specific embedding of the 2-branes.

mentioned above. It turns out that such black strings are generically unstable as usual. Interestingly, however, AdS black strings become stable if the four-dimensional horizon radius is larger than the order of the AdS_4 radius. This stability can be understood naively since the geometry along the string produces a sort of effective compactification whose scale is determined by the AdS_4 radius. Even in the case of unstable AdS black strings, the features of the instability are very different near the conformal infinity. In fact, the black-strings become stable near the boundary of AdS_5. We will first present some naive arguments indicating this behavior by using entropy comparisons and the Gubser-Mitra conjecture. Then, linearized metric perturbation analysis and numerical results are shown explicitly. Finally, some discussions and physical implications of our results follow.

II. BLACK STRING SOLUTIONS AND NAIVE STABILITY ARGUMENTS

Let us consider the five-dimensional pure anti–de Sitter spacetime whose metric is given in the following form [10,11]:

$$ds^2 = H^{-2}(z)(\gamma_{\mu\nu}dx^\mu dx^\nu + dz^2). \tag{1}$$

Here the warping factors are

$$H(z) = \begin{cases} l_4/l_5 \sinh z/l_4, & dS_4(\Lambda_4 > 0) \quad (2) \\ z/l_5, & M_4(\Lambda_4 = 0) \quad (3) \\ l_4/l_5 \sin z/l_4, & AdS_4(\Lambda_4 < 0), \quad (4) \end{cases}$$

where $\Lambda_5 = -6/l_5^2$ and the four-dimensional cosmological constant $\Lambda_4 = \pm 3/l_4^2$ is arbitrary. The metric $\gamma_{\mu\nu}$ describes four-dimensional de Sitter, flat Minkowski, and anti–de Sitter spacetimes, respectively, depending on the warping factor. These metrics actually describe the same five-dimensional anti–de Sitter spacetime with radius l_5, and simply correspond to different ways of slicing it. If we introduce a 3-brane with uniform tension σ at $z=0$ perpendicular to the fifth direction, parts of the AdS_5 spacetime need cutting and gluing in order to make the geometry smooth around the 3-brane. The resulting geometries are still described by the metrics above with $z \to |z| + c$. Here c is an arbitrary integral constant which is related to the location of the 3-brane. This c and the tension of a 3-brane determine the cosmological constant Λ_4; $|\sigma| = \sigma_0 \cosh c/l_4$ for the dS brane, $|\sigma| = \sigma_0 = 3\sqrt{-\Lambda_5/6}/8\pi G_5$ for the flat brane, and $|\sigma| = \sigma_0 \cos c/l_4$ for the AdS brane [10,11].

Now the embedding of Ricci-flat metrics in the flat brane world can be generalized as follows. The metric given in Eq. (1) satisfies the five-dimensional Einstein equation with a negative cosmological constant Λ_5 if the metric $\gamma_{\mu\nu}$ is any solution of the four-dimensional Einstein equation with the cosmological constant Λ_4. In particular, one can easily construct AdS/dS black string solutions by taking the AdS_4/dS_4-Schwarzschild black holes for $\gamma_{\mu\nu}$ such as

$$ds^2 = H^{-2}(z)\left[-f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2 + dz^2\right], \tag{5}$$

where

$$f(r) = 1 - \frac{r_0}{r} - \frac{\Lambda_4}{3}r^2. \tag{6}$$

We wish to study the classical stability of the black string solutions constructed in Eq. (5) with the warping factors $H(z)$ in Eqs. (2)–(4) under linearized metric perturbations. In particular, we will focus on the case of AdS black strings because the nature of their stability is quite different from those of the other two cases and the usual black strings studied in Refs. [2,3,5,4]. Before going into the linearized analysis in detail, let us give several naive arguments revealing the basic nature of the stabilities under consideration.

The five-dimensional Riemann tensor squared of the black strings in Eq. (5) is given by

$$R_{MNPQ}(g)R^{MNPQ}(g) \simeq \frac{10}{9}\Lambda_5^2 + H^4(z)\left[R_{\mu\nu\alpha\beta}(\gamma)R^{\mu\nu\alpha\beta}(\gamma) - \frac{8}{3}\Lambda_4^2\right], \tag{7}$$

where $R_{\mu\nu\alpha\beta}(\gamma)$ is the four-dimensional Riemann tensor constructed from the metric $\gamma_{\mu\nu}$. Thus, for the case of dS-Schwarzschild black strings (2) and Schwarzschild black strings (3), there generically exist curvature singularities at $z=\infty$ in addition to the usual singularity at the centers of black strings (e.g., $r=0$). In fact, these are naked singularities, not surrounded by some event horizon [7]. For Schwarzschild black strings, in particular, Gregory [5] has shown that they are unstable, presumably indicating fragmentation into an array of black holes. If these black strings were stable and so had no tendency to fragmentation, the full 5D spacetimes would be pathological due to such naked singularities. For AdS black strings (4), however, the function $H(z)$ is finite everywhere and so there is no naked or curvature singularity other than the usual ones at $r=0$. Therefore, we expect dS black strings to be unstable at least near the "Rindler horizon" $z=\infty$, but AdS black strings do not necessarily have to be so.

A common and more convincing argument is based on entropy comparison between different black hole configurations. The existence of black string instability is often explained by arguing that there exists a length for a segment of black string above which a compact black hole with the same mass becomes entropically favorable [2,3]. This possibly indicates that a black string decays into an array of black holes. We will compare the entropy contained in a segment of the black string with that contained in a five-dimensional compact black hole of the same mass. Since there is no known exact five-dimensional black hole solution in the presence of a 3-brane, we consider only the case with no 3-brane. However, we believe the result obtained applies equally to the case with a 3-brane because the presence of a 3-brane does not change the important nature of the Kaluza-Klein mass spectrum relevant in the linearized perturbation analysis, as will be shown below.

The entropy contained within a segment of AdS/dS black string can be obtained by integrating the area of the horizon as follows:

$$S_{\mathrm{b.s.}}=\frac{A}{4}=\frac{1}{4}\int_a^b \mathcal{A}(z)\,\frac{dz}{H(z)}=\pi r_+^2 L,\quad L=\int_a^b \frac{dz}{H^3(z)}. \tag{8}$$

Here $\mathcal{A}(z)=4\pi r_+^2(z)=4\pi r_+^2/H^2(z)$ with $f(r_+)=0$ is the area of the AdS_4/dS_4-Schwarzschild black hole measured by an observer at $z=$const. Notice that we can set $l_4=l_5=l$ by using diffeomorphism. The mass contained in the segment can also be obtained by integrating the first law of black hole thermodynamics $\delta M=T\delta S$ as in Ref. [8]:

$$M=\int_0^{r_+} T\frac{\partial S}{\partial r_+}dr_+=\frac{r_0 L}{2}. \tag{9}$$

Now the *five*-dimensional AdS-Schwarzschild black hole is described by

$$ds^2=-f(R)dt^2+\frac{1}{f(R)}dR^2+R^2 d\Omega_3^2,$$

$$f(R)=1+\frac{R^2}{l^2}-\frac{R_0^2}{R^2}. \tag{10}$$

Thus, the black hole entropy and mass become

$$S_{\mathrm{b.h.}}=\frac{\pi}{3}R_+^3,\qquad M=\frac{R_0^2}{4}, \tag{11}$$

respectively. By identifying the mass with that of the string segment, one can express $\Delta S=S_{\mathrm{b.s.}}-S_{\mathrm{b.h.}}$ as a function of r_+, l, and L. Surprisingly, this difference can be positive, independent of the ''length'' of the black string segment L provided that

$$\frac{\sqrt{13}-3}{2}l\simeq 0.30l<r_+,\quad \mathrm{dS}(\Lambda_4>0), \tag{12}$$

$$\frac{3-\sqrt{5}}{2}l\simeq 0.38l<r_+<\frac{3+\sqrt{5}}{2}l\simeq 2.62l,\quad \mathrm{AdS}(\Lambda_4<0). \tag{13}$$

Note that, for the case of the dS black string, the event horizon should be inside its cosmological horizon, $r_+\leqslant l/\sqrt{3}\simeq 0.58l$ (i.e., $r_0\leqslant 2l/3\sqrt{3}\simeq 0.38l$).

Thus, it appears that black strings are entropically more favorable than a five-dimensional AdS-Schwarzschild black hole with the same mass no matter how far the hypercylindrical horizon of the black string is extended, possibly indicating stability, if the size of the four-dimensional horizon r_+ lies in the range shown above. For black strings with the horizon radius r_+ not belonging to this range, the black hole becomes entropically favorable as the ''length'' of the string segment L increases, as usual. Interestingly, however, the black string segment again becomes entropically favorable if its ''length'' increases further. Here, however, we would like to point out that this sort of ''global'' thermodynamic stability argument should not be taken seriously since, from the viewpoint of the classical black hole area theorem, this argument indicates only some plausibility for the classical decay of black strings.

The Gubser-Mitra [13] conjecture can be regarded as a refinement of the ''global'' entropy argument given above, and is proved in Ref. [4]. Although the black string solutions we consider do not have translational symmetry due to the warping factors in Eqs. (2)–(4), it is interesting to apply this conjecture to our case. The local thermodynamic stability of a segment of AdS/dS black string will be determined by the sign of the heat capacity given by

$$\frac{dM}{dT}=-2\pi\frac{1-\Lambda_4 r_+^2}{1+\Lambda_4 r_+^2}r_+^2 L. \tag{14}$$

For the AdS case, one can easily see that the heat capacity is negative for $r_+<1/\sqrt{-\Lambda_4}=l_4/\sqrt{3}$, but becomes positive for $r_+>l_4/\sqrt{3}$ [14]. Thus, we expect AdS black strings to become classically stable when $r_+>l_4/\sqrt{3}$ according to the GM conjecture. On the other hand, dS black strings are expected to be unstable classically since they are locally thermodynamically unstable for $r_+<l_4/\sqrt{3}$ and the cosmological horizon is located at $r_+=l_4/\sqrt{3}$.

III. LINEARIZED PERTURBATION ANALYSIS

So far, we have given three naive arguments that possibly indicate that AdS black strings are stable when the four-dimensional horizon radius becomes large. Now let us perform a classical stability analysis explicitly. We consider small metric perturbations about AdS/dS black string background spacetimes and see whether or not there exists any mode that is regular spatially but grows exponentially in time. By choosing the Randall-Sundrum gauge [5,6,15], vacuum metric perturbations can be written as follows:

$$ds^2=H^{-2}(z)[(\gamma_{\mu\nu}+h_{\mu\nu})dx^\mu dx^\nu+dz^2],\quad \nabla^\mu h_{\mu\nu}=0,$$

$$h=\gamma^{\mu\nu}h_{\mu\nu}=0, \tag{15}$$

where ∇ is the covariant derivative operator compatible with the four-dimensional AdS/dS-Schwarzschild black hole metric $\gamma_{\mu\nu}(x)$ in Eq. (5). Then the linearized Einstein equations for vacuum metric fluctuations become simply

$$\Box h_{\mu\nu}(x,z)+2R_{\mu\rho\nu\tau}h^{\rho\tau}(x,z)-\left(-\partial_z^2+\frac{3\partial_z H}{H}\partial_z\right)h_{\mu\nu}(x,z)$$
$$=0, \tag{16}$$

where $\Box\equiv\gamma^{\rho\tau}\nabla_\rho\nabla_\tau$ and $h^{\mu\nu}\equiv\gamma^{\mu\rho}\gamma^{\nu\tau}h_{\rho\tau}$. Putting $h_{\mu\nu}(x,z)=H^{3/2}(z)\xi(z)h_{\mu\nu}(x)$, this equation can be decomposed into a four-dimensional part (e.g., the massive Lichnerowicz equation) and a fifth part as usual:

$$\Delta_L h_{\mu\nu}(x) \equiv \Box h_{\mu\nu}(x) + 2R_{\mu\rho\nu\tau}h^{\rho\tau}(x) = m^2 h_{\mu\nu}(x), \tag{17}$$

$$[-\partial_z^2 + V(z)]\xi(z) = m^2\xi(z),$$
$$V(z) = -\frac{3}{2}\frac{H''}{H} + \frac{15}{4}\left(\frac{H'}{H}\right)^2. \tag{18}$$

Thus, one sees that essentially the fifth dimension gives massive gravitons as usual from the Kaluza-Klein (KK) point of view, and their mass spectrum can be read off from the form of the effective potential $V(z)$. For the flat case in Eq. (3), $V(z)$ vanishes as $z \to \infty$ and so the KK mass spectrum is continuous starting at $m=0$. For the dS case in Eq. (2), $V(z)$ goes to a nonzero constant $9/4l_4^2$, and the KK mass spectrum is again continuous, but has a nonzero minimum mass $m_{\text{min}}=3/2l_4$. For the AdS case, however, $V(z)$ grows infinitely, making effectively a confining box due to the AdS nature of the spacetime. Thus, the z direction is effectively compactified even if it is still infinite in proper length. Consequently, the KK mass spectrum now becomes discrete and its lowest mass is $m_{\text{min}}=4/l_4$. Here one can observe that the scale of effective compactification is l_4 instead of l_5. Note that this effective compactification does not happen when the AdS_5 spacetime is sliced into four-dimensional Minkowski or dS_4 submanifolds. When a 3-brane is introduced, it gives a delta-function-like potential well at the position of the 3-brane, producing an attractive force. Hence the KK mass spectrum for a black string in brane world scenarios has essentially the same features as in the case of no 3-brane, but its magnitude is somewhat reduced [10,16]. Here we should point out that the nonzero finiteness of the lowest KK mass for AdS/dS cases plays an important role in the stability of black strings, as will be shown below explicitly.[2]

As explained above, we wish to find any instability mode that is a solution of the massive Lichnerowicz equation in Eq. (17) with suitable reference to gauge and boundary conditions at the future event horizon and spatial infinity. Since higher angular momentum fluctuation modes are more stable in general, we will consider a zero angular momentum mode only, an *s*-wave mode [2,3]. General, spherically symmetric perturbations that cause instability can be written in canonical form as [1,2]

$$h_{\mu\nu}(x) = e^{\Omega t}\begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 \\ 0 & 0 & K(r) & 0 \\ 0 & 0 & 0 & K(r)\sin^2\theta \end{pmatrix}, \tag{19}$$

with $\Omega > 0$. Using transverse traceless (TTF) gauge conditions in Eq. (15), we can eliminate all but one variable, say H_{tr}, from the Lichnerowicz equation, obtaining a second order ordinary differential equation as follows [3,17]:

$$\left[\Omega^2 + m^2 f + \frac{ff''}{2} - \frac{f'^2}{4} + \frac{ff'}{r}\right]H''_{tr} + \left[\frac{\Omega^2}{f}\left(3f' + \frac{2f}{r}\right) + m^2\left(2f' + \frac{2f}{r}\right) + \left(\frac{3f'f''}{2} + \frac{ff''}{r} + \frac{3f'^2}{2r} - \frac{3f'^3}{4f} + \frac{2ff'}{r^2}\right)\right]H'_{tr} + \left[-\left(\frac{\Omega^2}{f} + m^2\right)^2 + \frac{\Omega^2}{f} \times\left(-\frac{2f}{r^2} + \frac{f'}{r} + \frac{f''}{2} + \frac{5f'^2}{4f}\right) + m^2\left(-\frac{2f}{r^2} + \frac{2f'}{r} + \frac{f''}{2} - \frac{f'^2}{4f}\right) + \left(\frac{5f'^2}{2r^2} - \frac{f'^3}{rf} + \frac{f'^2f''}{4f} - \frac{ff''}{r^2} + \frac{3f'f''}{r} + \frac{f''^2}{2} - \frac{2ff'}{r^3} - \frac{f'^4}{4f^2}\right)\right]H_{tr} = 0. \tag{20}$$

Asymptotically, H_{tr} has the solutions

$$H_{tr} \sim \begin{cases} r^{-5/2 \pm \sqrt{9/4 + (ml_4)^2}} & \text{for } r \to \infty, \\ (r - r_+)^{-1 \pm \Omega/2\kappa} & \text{for } r \to r_+ \end{cases} \tag{21}$$

for the AdS case, and

$$H_{tr} \sim \begin{cases} (r_{++} - r)^{-1 \pm \Omega/2\kappa_{++}} & \text{for } r \to r_{++}, \\ (r - r_+)^{-1 \pm \Omega/2\kappa} & \text{for} r \to r_+ \end{cases} \tag{22}$$

for the dS case. Here $\kappa = f'(r_+)/2$ and $\kappa_{++} = -f'(r_{++})/2$ are surface gravities of the event and cosmological horizons, respectively, and r_{++} in the dS case denotes the cosmological horizon. As emphasized in Refs. [1–3], it is very important to impose right boundary conditions on the perturbations. Since our analysis is based on linearized equations, any fluctuations should remain "small." This seemingly excludes asymptotic solutions with negative roots near the horizon. However, one can see that even asymptotic solutions with positive roots diverge near the horizon when $\Omega/2\kappa < 1$. This is probably because the Schwarzschild coordinates are not good near the horizon. In fact, it turns out that, if we use some regular coordinate system such as Kruskal coordinates, the only asymptotic solution suitable for our linearized analysis is the one with positive roots with any Ω (>0) as pointed out in Refs. [2,3]. At spatial infinity for the AdS case, if we require vanishing boundary conditions as usual [18], even the positive root satisfies this condition provided $ml_4 < 2$. Actually, asymptotic AdS spacetimes are not globally hyperbolic. Thus, one needs to impose some extra condition by hand in order to make the dynamics of metric fluctuations well posed [19]. As imposed usually for matter fields in the pure AdS background spacetime [20], we require that the total energy of gravitational fluctuations on this AdS background should be conserved. This requirement is satisfied

[2]Actually, there exist massless KK modes as well for both AdS and dS cases. They are not normalizable except for the dS case with a 3-brane. However, these massless modes are irrelevant for seeking instability modes as will be explained in more detail below.

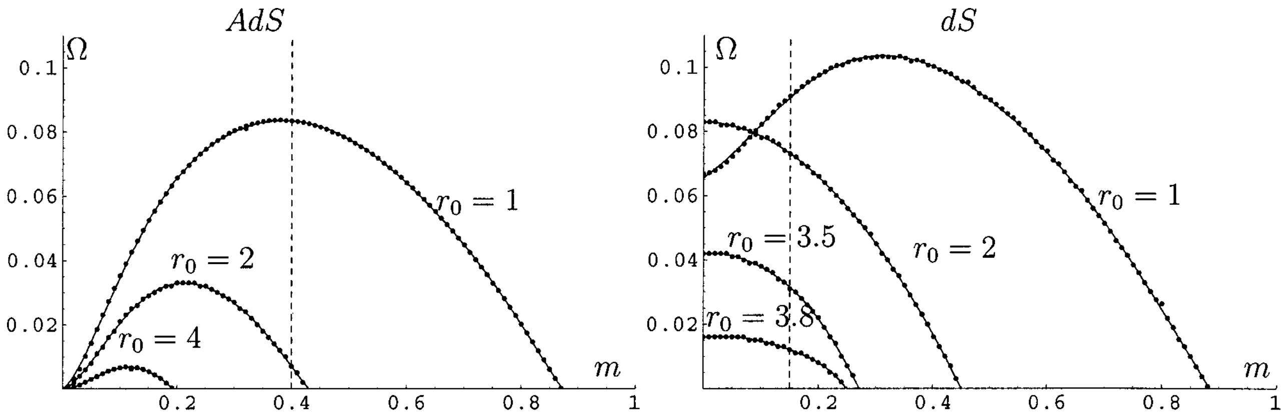

FIG. 1. The left figure is for the AdS case with $r_0=1$, 2, and 4. The right figure is for the dS case with $r_0=1$, 2, 3.5, and 3.8. The Nariai solution corresponds to $r_0\simeq 3.85$. The fixed AdS and dS radius is $l_4=10$. The straight vertical lines denote the lowest KK masses, 0.4 for AdS and 0.15 for dS.

only if $\lambda<-5/2$ when $H_{tr}\sim r^{\lambda}$ at $r\sim\infty$. Hence only the negative root is satisfactory in Eq. (21). We will give the details in [17].

With these boundary conditions described above, we now search for instability modes characterized by (Ω,m) which are solutions of Eq. (20) for given r_0 and l_4. In other words, for the AdS case we start from a solution with $H_{tr}\sim r^{-5/2-\sqrt{9/4+(ml_4)^2}}$ at $r\sim\infty$, and find the Ω that makes H_{tr} extrapolate to $H_{tr}\sim(r-r_+)^{-1+\Omega/2\kappa}$ near the horizon through Eq. (20). Similarly, we start from $H_{tr}\sim(r_{++}-r)^{-1+\Omega/2\kappa_{++}}$ near the cosmological horizon for the dS case. Since Eq. (20) is quite complicated to handle analytically, we solve it numerically. However, it is worthwhile to observe some scaling symmetries in Eq. (20) as follows:

$$r\to\alpha r,\ \ r_0\to\alpha r_0,\ \ l_4\to\alpha l_4(\Lambda_4\to\alpha^{-2}\Lambda_4),$$
$$\Omega\to\alpha^{-1}\Omega,\ \ m\to\alpha^{-1}m. \tag{23}$$

So we can fix one of these parameters, say $l_4=1$. Moreover, when $r_0\gg l_4$, $f\sim -r_0/r\pm r^2/l_4^2$, and so there exists another approximate scaling symmetry for large black holes given by [18]

$$r\to\alpha r,\ \ r_0\to\alpha^3 r_0,\ \ l_4\to l_4,\ \ \Omega\to\alpha\Omega,\ \ m\to m. \tag{24}$$

Then we find $\Omega\sim r_0^{1/3}$ for the case of large black holes.

Using MATHEMATICA and the Gear method for solving differential equations, we obtained the results in Fig. 1. For both AdS and dS cases, one can see that the instability shrinks in parameter space as the mass parameter r_0 increases with fixed AdS/dS radius. More precisely, the mass of the so-called threshold unstable mode [4] $(\Omega,m)=(0,m_*)$ decreases quickly down to zero as the horizon radius r_+ increases for the AdS case, whereas it approaches some nonzero finite value as r_+ increases toward the cosmological horizon for the dS case. Thus, it appears that there always exist instability modes. However, the KK mass m cannot be arbitrary, but is determined by the geometry in the fifth direction through Eq. (18) as explained before. The lowest KK masses are $4/l_4$ and $3/2l_4$ for the AdS and dS cases without 3-brane, respectively, as denoted in Fig. 1 by straight vertical lines. So, if the threshold mass becomes smaller than this lowest KK mass for a certain r_0, there exists no unstable mode indeed. As can be seen in Fig. 1, this happens when $0.21 l_4\lesssim r_0$ (i.e., $0.20 l_4\lesssim r_+$) for the AdS case. Therefore, we find that the AdS black string is unstable when its four-dimensional horizon size is small, but it becomes stable when the horizon size is larger than the order of the AdS_4 radius (i.e., $r_+^{\rm cr}\simeq 0.20 l_4$). On the other hand, the presence of a 3-brane reduces the lowest KK mass. Consequently, it increases the value of the critical horizon radius for stable black strings in the AdS brane world model. In particular, in the vicinity of the flat brane world (i.e., $\Lambda_4\sim 0$ or $l_4\sim\infty$), AdS black strings almost always become unstable since $m_{\rm min}\simeq 0$, which can be expected from the results in Ref. [10].

For the dS case, on the other hand, although the threshold mass decreases as the horizon radius increases up to the cosmological one, they all still seem to remain larger than the lowest KK mass (see also Fig. 3 below). Therefore, dS black strings seem to be always unstable. In particular, the instability seems to persist all the way down to the Nariai solution in which the event horizon coincides with the cosmological horizon. However, it should be pointed out that the Nariai limit must be treated separately since boundary conditions become invalid and the numerical error in our analysis increases near this extremal case. As argued in Ref. [12], the stability behavior of this case might be very different from that of nonextremal cases. The presence of a 3-brane in dS black strings again makes the system more unstable since it reduces the lowest KK mass in units of l_4. For the flat case (i.e., $\Lambda_4=0$), we have confirmed the results obtained in Ref. [5]. That is, since the KK mass spectrum is continuous with zero lowest mass and the threshold mass asymptotes to zero as $r_0\to\infty$ (as can be seen in Fig. 3 below), all black strings are unstable in this case.

Figure 2 illustrates how the threshold mass changes for a given r_0 as the cosmological constant Λ_4 varies away from zero. It shows that the instability in parameter space shrinks as Λ_4 becomes negative (i.e., the AdS case), but expands as Λ_4 becomes positive (i.e., the dS case). In other words, add-

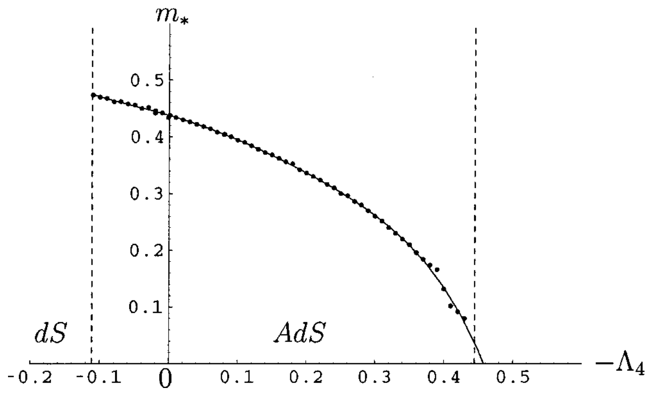

FIG. 2. The threshold masses m_* for varying Λ_4 with given $r_0 = 2$. The left vertical dashed line denotes the Nariai limit and the right one the critical Λ_4 predicted by the GM conjecture.

ing a negative cosmological constant has a stabilizing effect as in the case of adding charge to black strings [3], whereas adding a positive cosmological constant has a destabilizing influence.

It is interesting to see how well the results obtained by explicit perturbation analysis agree with those in the naive arguments given before. For the AdS case, critical values for stable black strings were predicted as $r_+ \simeq 0.38 l_4$ and $0.58 l_4$ in Eqs. (13) and (14) by the entropy comparison and by the GM conjecture, respectively. The numerical results predict $r_+ \simeq 0.20 l_4$ which agrees within the order of 1. The entropy comparison, however, also predicts another critical horizon radius, $r_+ \simeq 2.62 l_4$, across which black strings become unstable again. We have searched various parameters around this critical value, but could not find any unstable black string. Thus, our numerical results agree well, at least qualitatively, with the GM conjecture, but with the entropy argument only in part. For the dS case, on the other hand, only the prediction in the GM conjecture agrees well with the numerical results.

Figure 3 shows how the threshold mass for a given Λ_4 decreases as the black hole becomes large. These numerical results agree well qualitatively with those in Refs. [21] obtained analytically with some approximation and different gauge choices in a different context. Both flat (i.e., $\Lambda_4 = 0$) and AdS (i.e., $\Lambda_4 < 0$) cases give almost the same decreasing pattern for small r_0, but they start to deviate as the black hole becomes large, around $r_0 \simeq 4$. As $r_0 \to \infty$ for the flat case, m_* denoted by the dashed curved line in Fig. 3 asymptotes to zero ($\sim 1/r_0$), but never touches it. Consequently, since the continuum KK mass spectrum starts at $m = 0$, one can see again that all black strings are unstable no matter how large r_0 is. If the black string is compactified, however, the continuum KK mass spectrum becomes discrete. The massless mode is not a real instability mode, but presumably a gauge artifact [3,2] since the Lichnerowicz equation with $m = 0$ becomes that of pure four-dimensional black holes. So the lowest instability mode will start at nonzero m. Thus Fig. 3 shows that compactified black strings in the flat case will become stable if r_0 is larger than some critical value determined by the compactification scale. The stability of AdS black strings can be understood similarly from this point of view. This is because the AdS_5 nature of geometry in the fifth direction with AdS_4 slicing gives an effective compactification whose scale is determined by l_4 instead of l_5 as explained above. However, we point out there is another interesting feature in this case. As can be seen in Fig. 3 for the AdS case, although numerical error increases as m_* becomes small, the curve for m_* seems to touch the horizontal axis if the data points are extrapolated further. Moreover, this terminating point seems to agree with the critical value $r_0 \simeq 0.77 l_4$ (i.e., $r_+ \simeq 0.58 l_4$) obtained by the GM conjecture, the horizon radius across which the heat capacity changes its sign. Consequently, one might expect that the black string will be stable at least if r_0 is larger than this terminating value, no matter what the KK mass spectrum is. Therefore, in addition to the stabilization due to effective compactification, AdS black string solutions seem to have a sort of intrinsic tendency for stabilization probably due to the AdS_4 nature of the longitudinal four-dimensional geometries.

It can be seen that the critical value $r_+ \simeq 0.20 l_4$ obtained for AdS black strings does not exactly agree with but occurs slightly "earlier" than that of the GM conjecture, r_+

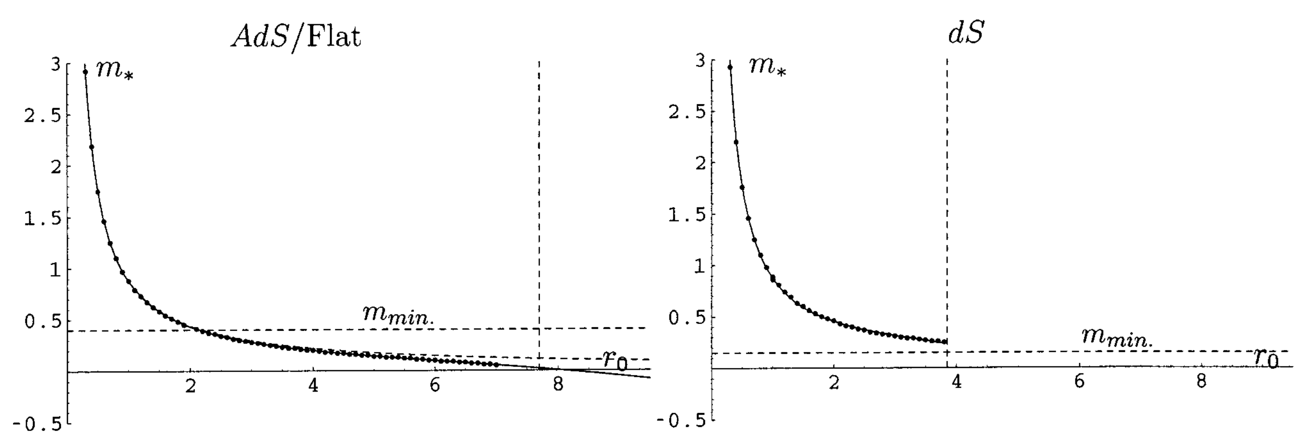

FIG. 3. The left figure: threshold masses for varying r_0 with given $l_4 = 10$ in the AdS case. The vertical dashed line denotes the critical $r_0 \simeq 7.7$ predicted from the Gubser-Mitra conjecture. The numerical data stop at $r_0 \simeq 7.0$. The right figure: same diagram for the dS case. The vertical dashed line denotes the Nariai limit $r_0 \simeq 3.85$. Note that $m_*(r_0 \simeq 3.85) = 0.29 > m_{\rm min} \simeq 0.15$.

$\simeq 0.58 l_4$. This discrepancy, however, is expected because the GM conjecture assumes noncompact translational symmetry. Actually, this condition can easily be replaced in the proof of the GM conjecture [4] as follows: a black string or brane, as long as its KK mass spectrum is continuous starting at zero mass, is classically stable if, and only if, it is locally thermodynamically stable. One can see that the flat case satisfies this modified GM conjecture. The GM conjecture then predicts that a terminating point must exist if the system is locally thermodynamically stable. In fact, our numerical results show not only that the terminating point exists, but also that it agrees with the critical value of local thermodynamic stability in the GM conjecture. Thus, it is to be expected that a black string having discrete KK mass spectrum will become stable, if it happens, before the terminating point as in the AdS case.

Finally, we have so far concentrated on the features of stability with special emphasis on stable black string configurations in AdS_5 spacetimes. Now it will also be interesting to see what the final states would be for unstable black strings. In order to answer this question, we just need to know how the eigenfunction $\xi(z)$ in Eq. (18) with given m behaves along the fifth coordinate z. For black strings in the flat case, it has been argued in Ref. [5] that the interval of successive wiggles in proper length becomes exponentially tiny toward the AdS_5 horizon, and so the string is somewhat stable near the 3-brane but quickly becomes unstable away from it, generating an accumulation of ''mini'' black holes toward the AdS_5 horizon. For the dS case with a 3-brane, the shape of the potential $V(z)$ in Eq. (18) is a volcano type and similar to that of the flat case. The only difference is that $V(z)$ approaches a nonzero constant as $z \to \infty$ (e.g., the ''Rindler'' horizon) instead of vanishing. Then $\xi(z)$ will be similar to that of the flat case, which is a Bessel function, but goes to zero more quickly. Accordingly, the features of fragmentation will be almost the same as in the flat case, with a slightly stronger instability. For the AdS case, however, it turns out to be very different. The potential $V(z)$ is again a volcano type around the 3-brane, but diverges at the boundary of AdS_5 (e.g., the conformal infinity), effectively creating a box. Thus, $\xi(z)$ will behave like a Hermite function, which is an eigenfunction of a harmonic oscillator with slight modifications in the vicinity of the 3-brane. Consequently, the black string again becomes stable near the boundary of AdS_5 as well as in the vicinity of the 3-brane, generating multiple black holes in between. This is why a segment of AdS black string becomes entropically favorable again when its length L in Eq. (8) becomes large enough.

IV. CONCLUSION

To conclude, we have shown that, although black strings in AdS spacetimes that are not locally asymptotically flat are generically unstable classically under linearized metric fluctuations, the AdS black string solutions are stable when the longitudinal size of the horizon is larger than the order of the AdS_4 radius. Generically, adding a negative cosmological constant has a stabilization effect whereas adding a positive cosmological constant has a destabilizing influence. It will be straightforward to extend our study to higher dimensional cases. We believe the essential features of stability for AdS black string or brane solutions in higher dimensions will be the same.

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[1] C.V. Vishveshiwara, Phys. Rev. D **1**, 2870 (1970); T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957); R.M. Wald, J. Math. Phys. **20**, 1056 (1979).

[2] R. Gregory and R. Laflamme, Phys. Rev. Lett. **70**, 2837 (1993).

[3] R. Gregory and R. Laflamme, Nucl. Phys. **B428**, 399 (1994).

[4] H. S. Reall, Phys. Rev. D **64**, 044005 (2001).

[5] R. Gregory, Class. Quantum Grav. **17**, L125 (2000).

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[7] A. Chamblin, S.W. Hawking, and H.S. Reall, Phys. Rev. D **61**, 065007 (2000).

[8] R. Emparan, G.T. Horowitz, and R.C. Myers, J. High Energy Phys. **01**, 007 (2000); A. Chamblin, H.S. Reall, H. Shinkai, and T. Shiromizu, Phys. Rev. D **63**, 064015 (2001); A. Chamblin, C. Csaki, J. Erlich, and T.J. Hollowood, *ibid.* **62**, 044012 (2000).

[9] R. Emparan, G.T. Horowitz, and R.C. Myers, J. High Energy Phys. **01**, 021 (2000).

[10] A. Karch and L. Randall, Int. J. Mod. Phys. A **16**, 780 (2001).

[11] O. DeWolfe, D.Z. Freedman, S.S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000); H.B. Kim and H.D. Kim, *ibid.* **61**, 064003 (2000); N. Kaloper, *ibid.* **60**, 123506 (1999).

[12] J. Garriga and M. Sasaki, Phys. Rev. D **62**, 043523 (2000).

[13] S.S. Gubser and I. Mitra, ''Instability of Charged Black Holes in Anti–de Sitter Space,'' hep-th/0009126; ''The Evolution of Unstable Black Holes in Anti–de Sitter Space,'' hep-th/0011127.

[14] S.W. Hawking and D.N. Page, Commun. Math. Phys. **87**, 577 (1983).

[15] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000).

[16] A. Miemiec, ''A Power Law for the Lowest Eigenvalue in Localized Massive Gravity,'' hep-th/0011160.

[17] T. Hirayama and G. Kang (in preparation).

[18] G. Horowitz and V.E. Hubeny, Phys. Rev. D **62**, 024027

(2000).
[19] R.M. Wald, J. Math. Phys. **21**, 2802 (1980).
[20] S.J. Avis, C.J. Isham, and D. Storey, Phys. Rev. D **18**, 3565 (1978); P. Breitenlohner and D. Freedman, Ann. Phys. (N.Y.) **144**, 249 (1982); Phys. Lett. **115B**, 197 (1982).
[21] T. Prestidge, Phys. Rev. D **61**, 084002 (2000); R. Garattini, Class. Quantum Grav. **17**, 3335 (2000); **18**, 571 (2001); M.S. Volkov and A. Wipf, Nucl. Phys. **B582**, 313 (2000).