Renormalization of an effective light-cone QCD-inspired theory for the pion and other mesons

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The renormalization of the effective QCD Hamiltonian theory for the quark-antiquark channel is performed in terms of a renormalized or fixed-point Hamiltonian that leads to subtracted dynamical equations. The fixed-point Hamiltonian gives the renormalization conditions as well as the counterterms that render the theory finite. The approach is renormalization group invariant. The parameters of the renormalized effective QCD Hamiltonian come from the pion mass and radius for a given constituent quark mass. The $1s$ and excited $2s$ states of $\bar{u}q$ are calculated as a function of the mass of the quark q being s, c, or b, and compared to the experimental values.

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I. INTRODUCTION

The effective mass operator equation for the lowest light-front Fock-state component of a bound system of a constituent quark and antiquark of masses m_1 and m_2, obtained in the effective one-gluon-exchange interaction approximation, which is our starting point, was reviewed in Ref. [1]. The breakthrough to simplify this equation in the spin-zero channel was achieved in [2] by formulating the $\uparrow\downarrow$ model, which is reduced to

$$M^2\psi(x,\vec{k}_\perp)=\left[\frac{\vec{k}_\perp^2+m_1^2}{x}+\frac{\vec{k}_\perp^2+m_2^2}{1-x}\right]\psi(x,\vec{k}_\perp)-\int\frac{dx'd\vec{k}'_\perp\,\theta(x')\,\theta(1-x')}{\sqrt{x(1-x)x'(1-x')}}\times\left(\frac{4m_1m_2}{3\pi^2}\,\frac{\alpha}{Q^2}-\lambda\right)\psi(x',\vec{k}'_\perp),\quad(1)$$

where M is the mass of the bound state and ψ is the projection of the light-front wave function in the quark-antiquark Fock state. The mean four-momentum transfer is [3]

$$Q^2(x,\vec{k}_\perp,x',\vec{k}'_\perp)=-\frac{1}{2}[(k_1-k'_1)^2-(k_2-k'_2)^2].\quad(2)$$

The coupling constant α for the Coulomb-like potential and λ is the bare coupling constant of the Dirac-delta hyperfine interaction. The Dirac-delta hyperfine potential between constituent quarks in Eq. (1) is one aspect of chiral symmetry breaking in QCD; another one is the constituent quark mass itself. On the light front, chiral symmetry corresponds to helicity conservation, which is broken in the $\uparrow\downarrow$ model by the hyperfine potential and by the constituent quark mass as well. The hyperfine interaction, which causes quark helicity flips, is a clear candidate to create the pi-rho mass difference. The delta function is an approximation to this hyperfine potential, which also involves zero modes or vacuum degrees of freedom.

For the purposes of this work, we leave out the energy transfer in Eq. (2) that leads to additional singularities for the tridimensional momentum going to infinity that we are not going to treat here. The effective mass operator of Eq. (1) acting in the quark-antiquark sector has also been obtained by iterated resolvents, which were derived in [4] and presented in greater detail in [5], which allows us to express systematically the higher Fock-state components of the wave function in functionals of the lower ones. In this way, the higher Fock-state components can be retrieved from the $q\bar{q}$ projection, and the full complexity of the QCD theory is in principle described by the effective Hamiltonian acting in the lowest Fock-state component [4].

However, the $\uparrow\downarrow$ model of Eq. (1) is a rather drastic approximation to a severe truncation of the Fock space in the effective theory. The initial truncation of QCD involves only one-gluon exchange, which keeps Fock states with up to $q\bar{q}$ plus one gluon, and so the rich nonlinear structure of QCD is not directly present here (this is a "colored QED"). Moreover, the spin dependence and momentum dependence in the hyperfine interaction are greatly simplified to get Eq. (1). Also, confinement is absent in the model, but this is not a main drawback, at least for the pion, which is strongly bound in the constituent picture and has in practice a wave function confined to short distances.

For convenience, the Sawicki transformation, first derived for equal masses [6] and consistently formulated for unequal masses in Ref. [3], is applied to Eq. (1) which allows us to rewrite it in the instant-form momentum basis. It is useful in this case since the momentum transfer is approximated by a rotational invariant form given by Eq. (2),

$$x(k_z)=\frac{(E_1+k_z)}{E_1+E_2},\quad(3)$$

and the Jacobian of the transformation of $(x,\vec{k}_\perp)$ to $\vec{k}$ is

$$dx d\vec{k}_\perp = \frac{x(1-x)}{m_r A(k)} d\vec{k}, \tag{4}$$

with the dimensionless function

$$A(k) = \frac{1}{m_r} \frac{E_1 E_2}{E_1 + E_2} \tag{5}$$

and the reduced mass $m_r = m_1 m_2/(m_1+m_2)$. The individual energies are $E_i = \sqrt{m_i^2 + k^2}$ $(i=1,2)$ and $k \equiv |\vec{k}|$.

The mass operator equation in instant-form momentum variables is given by

$$M^2 \varphi(\vec{k}) = [E_1 + E_2]^2 \varphi(\vec{k}) - \int d\vec{k}' \left(\frac{4 m_s}{3\pi^2} \frac{\alpha}{\sqrt{A(k)A(k')} Q^2} - \frac{\lambda}{m_r \sqrt{A(k)A(k')}} \right) \varphi(\vec{k}'), \tag{6}$$

where $m_s = m_1 + m_2$, the phase-space factor is included in the factor $1/\sqrt{A(k)A(k')}$ and $\sqrt{x(1-x)}\psi(x,\vec{k}_\perp) = \sqrt{A(k)}\varphi(\vec{k})$.

The mass operator equation (6) needs to be regularized and renormalized in order to give physical results. In Ref. [2], the delta function was smeared out to a Yukawa form, regularizing Eq. (6) and the parameters found from the pion mass and radius. In principle the size parameter in momentum space of the Yukawa potential should be allowed to go to infinity while the physical input is kept constant. To make the dependence on the size parameter vanish is a nontrivial task, which is the purpose of this work. Here we obtain a renormalized form of the equation for the bound state mass (i) which is invariant under renormalization group transformations, (ii) the physical input is given by the pion mass and radius, and (iii) there is no regularization parameter.

We are going to apply the recent renormalization techniques developed in the context nonrelativistic Hamiltonian theory [7–9] to Eq. (6), since it defines an effective Hamiltonian for the quark-antiquark dynamics. The renormalized T matrix is the solution of a subtracted scattering equation, the physical input of which is given by the T matrix at some reference scale μ. The scheme is invariant under a change of the arbitrary scale μ and consequently the inhomogeneous term of the subtracted scattering equation satisfies a renormalization group equation, which expresses the matching of the theories at the scales μ and $\mu + d\mu$ [8]. We will find the mass of the bound state from the pole of the renormalized scattering matrix defined from the mass operator, Eq. (6).

This work is organized as follows. In Sec. II, the operators for the Coulomb-like and for the singular interactions appearing in the mass operator are defined, and the Lippman-Schwinger equation for the T matrix related to the given mass operator is written. In Sec. III, we show how to renormalize the effective theory defined by the mass operator in Sec. II through the definition of a renormalized or fixed-point singular interaction. This procedure is equivalent to using subtracted scattering equations. We discuss the renormalization group invariance of the method. The explicit form of the renormalized T matrix is obtained (the Appendix) and the physical input is the pion mass which determines the excited states as well as the mass of the other mesons. The physical observables of the renormalized effective theory do not depend on the subtraction point. In Sec. IV, the calculation of the pion charge radius is discussed, and although we have simplified the spin dependence in the dynamical equation, it is important in the evaluation of the radius. For this purpose we have used an effective pseudoscalar Lagrangian to construct the spin part of the pion wave function, and turning off the Coulomb-like interaction we retrieve a well-known result valid in the soft pion limit [14]. To gain insight we also write down the expressions where the quark spin is neglected. The numerical results are presented in Sec. V. We have solved a nonrelativistic example for a Coulomb plus a Dirac-delta interaction compared to a model in which the Dirac delta is substituted by a Yukawa potential. We also show how this effective theory, including a Dirac delta, can mimic a finite range theory, calibrated to the pion mass. Then, we present results for the pion charge radius and for the mass of the excited state, which we identified with the isovector vector mesons. Finally, in Sec. VI, we present our conclusions.

II. DEFINITIONS

For our purposes it is convenient to work in an operator form of Eq. (6):

$$(M_0^2 + V + V^\delta)|\varphi\rangle = M^2 |\varphi\rangle, \tag{7}$$

and the free mass operator M_0 is the sum of the energies of quarks 1 and 2, V the Coulomb-like potential, and V^δ the Dirac-delta interaction in the nonrelativistic limit. The matrix elements of these operators are given by

$$\langle \vec{k}|V|\vec{k}'\rangle = -\frac{4m_s}{3\pi^2} \frac{\alpha}{\sqrt{A(k)} Q^2 \sqrt{A(k')}} \tag{8}$$

and the short-range singular interaction

$$\langle \vec{k}|V^\delta|\vec{k}'\rangle = \langle \vec{k}|\chi\rangle \frac{\lambda}{m_r} \langle \chi|\vec{k}'\rangle = \frac{\lambda}{m_r} \frac{1}{\sqrt{A(k)}} \frac{1}{\sqrt{A(k')}}. \tag{9}$$

The phase-space factor $A(k)$ is defined by Eq. (5), and the square momentum transfer Q^2 comes from Eq. (2). For convenience in the formal manipulations of the next section, the form factor of the separable singular interaction is introduced and defined by $\langle \vec{k}|\chi\rangle = 1/\sqrt{A(k)}$.

The T matrix is obtained from the Lippman-Schwinger equation

$$T(M^2) = V + V^\delta + (V + V^\delta) G_0^{(+)}(M^2) T(M^2), \tag{10}$$

where M is the mass of the scattering state and the Green's function with outgoing wave boundary condition is

$$G_0^{(+)}(M^2) = \frac{1}{M^2 - M_0^2 + i\varepsilon}. \tag{11}$$

In the next section we will obtain a renormalized form of $T(M^2)$.

III. RENORMALIZATION OF THE EFFECTIVE THEORY

The ideas that have been developed in Refs. [7–9] to construct a renormalized T matrix in nonrelativistic Hamiltonian theory can be applied in the case where the two-body interaction can be split in a regular potential of finite range (V) and a Dirac-delta singularity as in Eq. (10). In this case, the physical information at the subtraction point will be introduced through the renormalized mass operator as well as all the counterterms that render finite the T-matrix equation (10). The physical information at the subtraction point is the T matrix corresponding to the Dirac-delta interaction. This renormalization approach has been applied in the two-nucleon system to calculate the T matrix of the one-pion-exchange potential, without the necessity of regularization or form factors [7].

The renormalized interaction is given by

$$V_{\mathcal{R}}=V+V_{\mathcal{R}}^{\delta}, \tag{12}$$

where the renormalized Dirac-delta interaction is given by

$$\begin{aligned} V_{\mathcal{R}}^{\delta}&=\frac{1}{1+T_{\mathcal{R}}^{\delta}(\mu^2)G_0^{(+)}(\mu^2)}T_{\mathcal{R}}^{\delta}(\mu^2)\\ &=T_{\mathcal{R}}^{\delta}(\mu^2)\frac{1}{1+G_0^{(+)}(\mu^2)T_{\mathcal{R}}^{\delta}(\mu^2)}\\ &=T_{\mathcal{R}}^{\delta}(\mu^2)\sum_{n=0}^{\infty}[-G_0^{(+)}(\mu^2)T_{\mathcal{R}}^{\delta}(\mu^2)]^n, \end{aligned} \tag{13}$$

and $T_{\mathcal{R}}^{\delta}(\mu^2)$ is the renormalized T matrix of the Dirac-delta interaction, with matrix elements given by

$$\langle\vec{p}|T_{\mathcal{R}}^{\delta}(\mu^2)|\vec{q}\rangle=\langle\vec{p}|\chi\rangle\lambda_{\mathcal{R}}(\mu^2)\langle\chi|\vec{q}\rangle, \tag{14}$$

where $\lambda_{\mathcal{R}}(\mu^2)$ is the renormalized strength of the Dirac-delta interaction at the mass scale μ^2. In the nonrelativistic limit the form factor $\chi(q)=1$, and the renormalized interaction becomes the Dirac-delta interaction. The scattering equation with the renormalized interaction appears in a subtracted form [7–9] after a little rearrangement of terms, in which all the divergent momentum integrals are removed, and it is written as

$$\begin{aligned} T_{\mathcal{R}}(M^2)=T_{\mathcal{R}}(\mu^2)+T_{\mathcal{R}}(\mu^2)[G_0^{(+)}(M^2)-G_0^{(+)}(\mu^2)]\\ \times T_{\mathcal{R}}(M^2). \end{aligned} \tag{15}$$

It is presented here in a general way, where we have dropped the δ superscript, just to remind the reader that for a regular potential Eq. (15) is completely equivalent to the traditional Lippman-Schwinger scattering equation.

The renormalized interaction is independent of the subtraction point; i.e., the physics expressed by the renormalized interaction is invariant by changes in the arbitrary renormalization point. This physical requirement is given by

$$\frac{d}{d\mu^2}V_{\mathcal{R}}^{\delta}=0, \tag{16}$$

qualifying the interaction as the fixed point of Eq. (16), which implies that the T matrix found from solution of

$$T_{\mathcal{R}}(M^2)=V+V_{\mathcal{R}}^{\delta}+(V+V_{\mathcal{R}}^{\delta})G_0^{(+)}(M^2)T_{\mathcal{R}}(M^2) \tag{17}$$

is invariant under dislocations of the subtraction point. Consequently, the renormalized coupling constant of the Dirac-delta interaction changes as the subtraction point moves, according to the Callan-Symanzik equation

$$\frac{d}{d\mu^2}T_{\mathcal{R}}^{\delta}(\mu^2)=-T_{\mathcal{R}}^{\delta}(\mu^2)\frac{1}{(\mu^2+i\varepsilon-M_0^2)^2}T_{\mathcal{R}}^{\delta}(\mu^2), \tag{18}$$

obtained from Eq. (16).

Although the renormalized interaction is not well defined for singular interactions, the resulting T matrix obtained by solving Eq. (17) is finite. This gives *a posteriori* justification for the formal manipulations used in Eqs. (13) and (17). The sum in the expression of the renormalized interaction (13) makes explicit all the counterterms which exactly cancels the infinities in the momentum integrals of the scattering equation (17), while introducing physical information through the value of the renormalized strength of the Dirac-delta interaction. We choose $\lambda_{\mathcal{R}}(\mu^2)$, in the following, in accordance with the physical value of the pion mass. We observe that, instead of working formally with the operator $V_{\mathcal{R}}^{\delta}$, we could use an ultraviolet momentum cutoff (Λ) by defining in this way a regularized interaction. After the construction of the T-matrix regularized equation one could perform the limit $\Lambda\to\infty$, arriving at the same results as the ones obtained directly with the use of the renormalized interaction.

The solution of the scattering equation (17) is found by using the two potential formulas in terms of the T matrix of the regular potential V, $T^V(M^2)$ and the renormalized T matrix of the Dirac-delta interaction (see the Appendix), which results in

$$T_{\mathcal{R}}(M^2)=T^V(M^2)+\frac{[1+T^V(M^2)G_0^{(+)}(M^2)]|\chi\rangle\langle\chi|[G_0^{(+)}(M^2)T^V(M^2)+1]}{\lambda_{\mathcal{R}}^{-1}(\mu^2)-\langle\chi|[1/(M^2+i\varepsilon-M_0^2)-1/(\mu^2+i\varepsilon-M_0^2)]|\chi\rangle-\langle\chi|G_0^{(+)}(M^2)T^V(M^2)G_0^{(+)}(M^2)|\chi\rangle}, \tag{19}$$

where $T^V(M^2)$ is a solution of the Lippman-Schwinger equation

$$T^V(M^2)=V+VG_0^{(+)}(M^2)T^V(M^2). \tag{20}$$

The structure of Eq. (19) allows one more subtraction in the denominator, which turns the convergence of the momentum integral faster in the term where $T^V(M^2)$ is present. This subtraction is appropriate if the potential V has a Coulomb or Yukawa form:

$$\langle\vec{p}|V|\vec{q}\rangle=\frac{1}{\eta^2+|\vec{p}-\vec{q}|^2}. \tag{21}$$

Thus, we define the renormalized strength of the Dirac-delta interaction at the subtraction point such that

$$\lambda_{\mathcal{R}}^{-1}(\mu^2)=\overline{\lambda}_{\mathcal{R}}^{-1}(\mu^2)+\langle\chi|G_0^{(+)}(\mu^2)T^V(\mu^2)G_0^{(+)}(\mu^2)|\chi\rangle \tag{22}$$

and introduce the physical information in the renormalized T matrix (19) through the value of $\overline{\lambda}_{\mathcal{R}}^{-1}(\mu^2)$.

Substituting Eq. (22) into Eq. (19), we obtain the renormalized T matrix written as

$$T_{\mathcal{R}}(M^2)=T^V(M^2)+[1+T^V(M^2)G_0^{(+)}(M^2)]|\chi\rangle t_{\mathcal{R}}(M^2)$$
$$\times\langle\chi|[G_0^{(+)}(M^2)T^V(M^2)+1], \tag{23}$$

where

$$t_{\mathcal{R}}^{-1}(M^2)=\overline{\lambda}_{\mathcal{R}}^{-1}(\mu^2)-\langle\chi|[G^{V(+)}(M^2)-G^{V(+)}(\mu^2)]|\chi\rangle, \tag{24}$$

and the interacting Green's function is

$$G^{V(+)}(M^2)=G_0^{(+)}(M^2)+G_0^{(+)}(M^2)T^V(M^2)G_0^{(+)}(M^2). \tag{25}$$

We use the renormalization condition that at the pion mass $M=m_\pi$ the T matrix, for $m_1=m_2$ and $m_1=m_u=m_d$, has the bound-state pole; consequently,

$$t_{\mathcal{R}}^{-1}(m_\pi^2)=0 \tag{26}$$

and choosing $\mu=m_\pi$ for convenience, which implies that

$$\overline{\lambda}_{\mathcal{R}}^{-1}(m_\pi^2)=0. \tag{27}$$

The invariance of the renormalized T matrix (23) under dislocation of the subtraction point just reads as

$$\frac{d}{d\mu^2}t_{\mathcal{R}}(M^2)=0 \tag{28}$$

and, from Eq. (28),

$$\overline{\lambda}_{\mathcal{R}}^{-1}(\mu'^2)=\overline{\lambda}_{\mathcal{R}}^{-1}(\mu^2)-\langle\chi|[G^{V(+)}(\mu'^2)-G^{V(+)}(\mu^2)]|\chi\rangle. \tag{29}$$

At the general subtraction point $t_{\mathcal{R}}^{-1}(\mu^2)=\overline{\lambda}_{\mathcal{R}}^{-1}(\mu^2)$, and the renormalized T matrix at μ^2 is given by

$$T_{\mathcal{R}}(\mu^2)=T^V(\mu^2)+[1+T^V(\mu^2)G_0^{(+)}(\mu^2)]|\chi\rangle\overline{\lambda}_{\mathcal{R}}(\mu^2)$$
$$\times\langle\chi|[G_0^{(+)}(\mu^2)T^V(\mu^2)+1]. \tag{30}$$

The full renormalized interaction can be written in a form analogous to Eq. (13),

$$V_{\mathcal{R}}=T_{\mathcal{R}}(\mu^2)\sum_{n=0}^{\infty}[-G_0^{(+)}(\mu^2)T_{\mathcal{R}}(\mu^2)]^n, \tag{31}$$

and with that, one could obtain the equation for the renormalized T matrix (17) in the subtracted form (15), displayed again below:

$$T_{\mathcal{R}}(M^2)=T_{\mathcal{R}}(\mu^2)+T_{\mathcal{R}}(\mu^2)[G_0^{(+)}(M^2)-G_0(\mu^2)]T_{\mathcal{R}}(M^2).$$

We observe that one could equally well construct the Callan-Symanzik equation for $(d/d\mu^2)T_{\mathcal{R}}(\mu^2)$, by performing the limit of $M\rightarrow\mu$ in Eq. (15), finding

$$\frac{d}{d\mu^2}T_{\mathcal{R}}(\mu^2)=-T_{\mathcal{R}}(\mu^2)\frac{1}{(\mu^2+i\varepsilon-M_0^2)^2}T_{\mathcal{R}}(\mu^2), \tag{32}$$

with the boundary condition given by Eq. (30). The solution of Eq. (32) gives the dependence of $\overline{\lambda}_{\mathcal{R}}$ on the subtraction point μ as expressed by Eq. (29).

Now comes a subtle point: It is important to realize the renormalization condition given by $\overline{\lambda}_{\mathcal{R}}^{-1}(m_\pi^2)=0$, and considering Eq. (24), the bare strength of the Dirac-delta interaction is given by

$$m_r\lambda_{bare}^{-1}=[\langle\chi|G^{V(+)}(m_\pi^2)|\chi\rangle]_{(m_u,m_{\bar{u}})}, \tag{33}$$

where the Green's function of the Coulomb-like interaction is calculated for an equal mass constituent quark and antiquark, i.e., $m_1=m_2=m_u=m_{\bar{d}}$. The bare coupling constant in the form given by Eq. (33) is sufficient to render finite the T matrices from Eq. (23) calculated with different quark masses.

The bound-state mass M_b of quark-antiquark states with also different quark masses is the position of the poles of the renormalized T matrix (23), which implies

$$m_r^{-1}t_{\mathcal{R}}^{-1}(M_b^2)=\left[\frac{1}{m_r}\langle\chi|G^V(m_\pi^2)|\chi\rangle\right]_{(m_u,m_{\bar{u}})}$$
$$-\left[\frac{1}{m_r}\langle\chi|G^V(M_b^2)|\chi\rangle\right]_{(m_1,m_2)}=0. \tag{34}$$

The zeros of $t_{\mathcal{R}}^{-1}$ give the position of the zero angular momentum states of the bound quark-antiquark systems, for different quark masses and excitation. It is easy to imagine that in the vicinity of a bound state of the Coulomb-like potential, because of the presence of the pole in G^V, the function $t_{\mathcal{R}}^{-1}$ is rapidly varying and it is infinity at the Coulomb bound state, and because of the change in sign of G^V, it will necessarily present a zero and a bound state of the whole potential.

IV. PION CHARGE RADIUS

The renormalized effective theory defined by Eqs. (23),(24) with the bound-state mass equation (34), once the pion mass is known, has two free parameters: the constituent quark mass and α, the strength of the effective one-gluon-exchange potential. The values of α and the constituent quark mass can be found from the pion charge radius and the ρ-meson mass. The pion wave function in the effective theory comes from the residue of the T matrix, Eq. (23), at the pion pole, such that

$$\psi_\pi(x,\vec{k}_\perp)=\sqrt{\frac{A(k)}{x(1-x)}}\varphi_\pi(\vec{k}), \tag{35}$$

where

$$\varphi_\pi(\vec{k})=\langle\vec{k}|G_0(m_\pi^2)[1+T^V(m_\pi^2)G_0(m_\pi^2)]|\chi\rangle \tag{36}$$

and

$$x(k_z)=\frac{1}{2}+\frac{k_z}{2E}, \tag{37}$$

from Eq. (3). The absolute normalization of the $q\bar{q}$ Fock component of the pion wave function (35) is such that the asymptotic form is given by the first term of Eq. (36), with the residue at the pion pole equal to 1. We impose such a normalization condition to be consistent with the soft pion limit ($m_\pi=0$) for the electromagnetic form factor when the Coulomb-type interaction goes to zero, as will be shown below.

A. Including quark spin

The pion electromagnetic form factor is obtained from the impulse approximation of the plus component of the current ($j^+=j^0+j^3$) in the Breit frame with momentum transfer $q^+=0$ and $q^2=-\vec{q}^2$. The leptonic decay constant of π^+ (f_π) is a physical quantity which depends directly on the probability of finding the quark-antiquark Fock state in the pion wave function and consequently properly normalizes it once the empirical value of f_π is given. Computing the pion form factor from an effective Lagrangian, described below, the value of f_π gives the normalization of the form factor. In this case the $q\bar{q}$ component of the pion wave function is normalized such that, in the vanishing limit of the Coulomb-type interaction, it retrieves the asymptotic form, Eq. (43) [12]. The coupling of the pion field to the quark field is taken from an effective Lagrangian with pseudoscalar coupling with the pion quark coupling constant given by the Goldberger-Treiman [11] relation at the quark level,

$$\mathcal{L}_{eff}=-i\frac{m}{f_\pi}\vec{\pi}\cdot\bar{q}\gamma^5\vec{\tau}q, \tag{38}$$

and $f_\pi=93$ MeV is the pion weak decay constant.

The general structure of the $q\bar{q}$ bound state forming the pion comes from the pseudoscalar coupling, and we will use such spin structure in the computation of the Feynman triangle diagram which expresses the impulse approximation to compute the pion electromagnetic current,

$$\begin{aligned}(p_\pi^\mu+p_\pi'^\mu)F_\pi(q^2) &= i2\frac{m^2}{f_\pi^2}N_c\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\Bigg[\frac{\not{k}+m}{k^2-m^2+i\varepsilon} \\ &\quad\times\gamma^5\frac{\not{k}-\not{p}'_\pi+m}{(k-p'_\pi)^2-m^2+i\varepsilon} \\ &\quad\times\gamma^\mu\frac{\not{k}-\not{p}_\pi+m}{(k-p_\pi)^2-m^2+i\varepsilon}\gamma^5\Bigg],\end{aligned} \tag{39}$$

where $F_\pi(q^2)$ is the pion electromagnetic form factor. The pion momenta in the initial and final states are defined by $p_\pi^0=p_\pi'^0$ and $\vec{p}'_{\pi\perp}=-\vec{p}_{\pi\perp}=\vec{q}_\perp/2$ in the Breit frame. $N_c=3$ is the number of colors.

The $+$ component of the current is calculated from Eq. (39). It is chosen because after integration over $k^-=k^0-k^3$ the suppression of the pair diagram is maximal for this component in the frame where $q^+=0$ and just the valence wave function enters in the form factor [12,13]. Although we are going to compute the integration in the $-$ component of the moment assuming a constant vertex, one can identify in the expression how the wave function corresponding to the non-constant vertex of Eq. (35) should be introduced in the expression. The result is [12]

$$\begin{aligned}F_\pi(q^2)&=\frac{2}{(2\pi)^3}\frac{m^2}{f_\pi^2}N_c\int_0^1dx\int d^2K_\perp M_0^2 \\ &\quad\times\left(1+\frac{(1-x)\vec{q}_\perp\cdot\vec{K}_\perp}{K_\perp^2+m^2}\right)\psi_\pi(x,\vec{K}_\perp)\psi_\pi(x,\vec{K}'_\perp),\end{aligned} \tag{40}$$

where the momentum fraction $x=(p^+-k^+)/p^+$. The relative transverse momentum is given by

$$\vec{K}_\perp=(1-x)(\vec{p}_\perp-\vec{k}_\perp)-x\vec{k}_\perp, \tag{41}$$

and $\vec{K}'_\perp=\vec{K}_\perp+(1-x)\vec{q}_\perp$. The free mass operator for the $q-\bar{q}$ is written in terms of the momentum fraction and the relative perpendicular momentum

$$M_0^2=\frac{K_\perp^2+m^2}{x(1-x)}, \tag{42}$$

and M'_0 is written as a function of $K'_\perp$. The expression for the pion form factor gives the standard Drell-Yan formula once the bound-state wave function of the constant vertex model, the asymptotic form, is recognized:

$$\psi_\pi(x,\vec{K}_\perp)=\frac{1}{\sqrt{x(1-x}(m_\pi^2-M_0^2)}, \tag{43}$$

which is the first term of the pion wave function in Eq. (36). The second term in Eq. (36) comes from the Coulomb-like potential, and implies a contribution from the effective theory to the pion radius. The other factors in Eq. (40) comes from the Melosh rotations of the individual spin wave function of the quarks.

The pion charge radius from Eq. (40) in the soft-pion limit with constant vertex, corresponding to the wave function of Eq. (43) with $m_\pi=0$, gives the well-known result of $r_\pi = [6(d/dq^2)F_\pi(q^2)|_{q^2=0}]^{1/2} = \sqrt{3}/(2\pi f_\pi)$ from Ref. [14]. The form factor (40) in the soft-pion limit with constant vertex, for $q^2=0$, reduces to the expression for f_π [12],

$$f_\pi = \frac{-2}{(2\pi)^3}\frac{m^2}{f_\pi}N_c\int_0^1 \frac{dx}{\sqrt{x(1-x)}}\int d^2K_\perp \psi_\pi(x,\vec{K}_\perp), \tag{44}$$

obtained from computation of the leptonic decay transition amplitude of π^+ with the effective Lagrangian (38) and the wave function given by Eq. (43).

B. Neglecting quark spin

In order to gain insight into the importance of the inclusion of the quark spin in the computation of the form factor, we have simplified the numerator of Eq. (40), which is the result from Dirac algebra, by taking the limit of $m\to\infty$. In this way the quark spin is neglected and the form factor reduces to the formula found for scalar particles, and only the overall normalization, which depends on f_π, is maintained. The wave function is in the form expressed by Eq. (35) with $A(k)=1$. With the above approximations the pion electromagnetic form factor is

$$F_\pi(q^2) = \frac{1}{\pi^3}\frac{m^4}{f_\pi^2}N_c\int_0^1 dx\int d^2K_\perp \psi_\pi(x,\vec{K}_\perp)\psi_\pi(x,\vec{K}'_\perp), \tag{45}$$

with relative transverse momentum

$$\vec{K}_\perp = (1-x)(\vec{p}_\perp - \vec{k}_\perp) - x\vec{k}_\perp, \tag{46}$$

and $\vec{K}'_\perp = \vec{K}_\perp + (1-x)\vec{q}_\perp$. In case of the $q\bar{q}$ wave function being supposed to be the complete pion wave function, the form factor is written as [1]

$$F_\pi(q^2) = \mathcal{N}\int_0^1 dx\int d^2K_\perp \psi_\pi(x,\vec{K}_\perp)\psi_\pi(x,\vec{K}'_\perp), \tag{47}$$

and from $F_\pi(0)=1$ the normalization $\mathcal{N}$ is determined.

The difference between the form factors defined by Eqs. (45) and (47) is the normalizations of the $q\bar{q}$ Fock component of the pion wave function. In the first case the normalization is defined by f_π, while for the second case the $q\bar{q}$ Fock-state component is normalized such that $F_\pi(0)=1$. As we will see, Eq. (47) gives a too small pion radius, while from Eq. (45) or (40) the pion radius can be described reasonably. The absolute normalization of the form factor computed with the inclusion of the quark spin as expressed by Eq. (40) is undetermined, since the integral diverges. However, the pion radius is finite with the magnitude of this Fock component of the pion wave function known from the empirical value of f_π, number of colors, and attributed constituent quark mass.

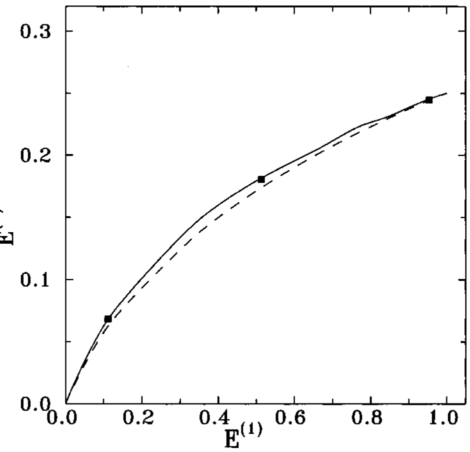

FIG. 1. Excited-state binding energy ($\epsilon^{(2)}$) as a function of the ground-state binding energy ($\epsilon^{(1)}$) for the nonrelativistic model. Attractive Coulomb plus repulsive Yukawa nonrelativistic model of the Ref. [10] (solid squares). Our numerical calculation of Eq. (48) is given by the solid line, and the solution of the effective model, Eq. (49) (Coulomb plus Dirac-delta), is given by the dashed line.

To close this section, we observe that our aim is to fit the strength α of the Coulomb-like potential using the pion charge radius. For this purpose, we introduce the pion wave function from Eq. (35) in the form-factor expression, Eq. (40), and calculate the charge radius. Although the form factor diverges, the charge radius is finite and for $m_\pi=0$ and $\alpha=0$ it retrieves the soft-pion limit. The pion mass and the effective Coulomb-like interaction give a correction to the soft-pion limit, $r_\pi^{soft}=0.58$ fm, towards the experimental result of 0.67 ± 0.02 fm [15].

V. NUMERICAL RESULTS

A. Test case

We begin this section on the numerical calculations comparing our results for the *s*-wave bound-state energies which are consistent with [10] for the nonrelativistic Coulomb plus repulsive Yukawa model:

$$[\epsilon - p^2]\varphi(p) = \frac{1}{\pi}\int_0^\infty dp'\frac{p'}{p}\left[\ln\frac{(p-p')^2}{(p+p')^2} - \ln\frac{(p-p')^2+\eta^2}{(p+p')^2+\eta^2}\right]\varphi(p'). \tag{48}$$

In our numerical procedure we checked the results with up to 200 Gaussian-Legendre quadrature points and the interval $-1<z<1$ was transformed to $0<k<\infty$ through the variable transformation $k=c(1-z)/(1+z)$ with c about 1.

In Fig. 1, we show our results for the first excited state $\epsilon^{(2)}$ as a function of the ground state $\epsilon^{(1)}$ and compare with

TABLE I. Results for S-wave binding energies of the nonrelativistic model of Eq. (48), our calculation with $N=100$ [C-Y], compared to the effective Coulomb plus Dirac-delta interaction [C-δ].

η	$\epsilon^{(1)}$ [C-Y]	$\epsilon^{(2)}$ [C-Y]	$\epsilon^{(2)}$ [C-δ]
0.1	0.1109	0.06781	0.06237
1	0.5119	0.1813	0.1736
10	0.9495	0.2449	0.2439

the calculation of [10] for $\eta=0.1$, 1, and 10. For these values of η we present results in Table I. Our precision is about 0.5%, which will suffice for our purposes. Also the calculation with the renormalized Coulomb plus Dirac delta is shown in Fig. 1 and Table I. The calculation with nonrelativistic renormalized model corresponds to finding the zeros of the nonrelativistic form of Eq. (34),

$$\int d\vec{q}d\vec{p}\langle\vec{q}|[G^{Vnr}(-\epsilon^{(n)})-G^{Vnr}(-\epsilon^{(1)})]|\vec{p}\rangle=0, \tag{49}$$

with the nonrelativistic resolvent operator $G^{Vnr}(\epsilon)$ for negative energies obtained from the solution of

$$G^{Vnr}(\epsilon)=G_0^{nr}(\epsilon)+G_0^{nr}(\epsilon)V_C G^{Vnr}(\epsilon), \tag{50}$$

where $\langle\vec{p}|V_C|\vec{q}\rangle=[\pi^2(|\vec{p}-\vec{q}|^2)]^{-1}$ and the free resolvent operator $G_0^{nr}(\epsilon)=[\epsilon-k^2]^{-1}$.

In momentum space Eq. (49) is given by

$$4\pi\int_0^\infty dp\, p^2\left[\frac{1}{-\epsilon^{(n)}-p^2}-\frac{1}{-\epsilon^{(1)}-p^2}\right] +8\pi^2\int_0^\infty dq\, q^2\int_0^\infty dp\, p^2\left[\frac{t_C(p,q;-\epsilon^{(n)})}{(-\epsilon^{(n)}-p^2)(-\epsilon^{(n)}-q^2)} -\frac{t_C(p,q;-\epsilon^{(1)})}{(-\epsilon^{(1)}-p^2)(-\epsilon^{(1)}-q^2)}\right]=0, \tag{51}$$

and s-wave projected T-matrix in Eq. (51) is the solution of

$$t_C(p,q;\epsilon)=\frac{1}{2\pi^2}\frac{1}{pq}\ln\frac{(p-q)^2}{(p+q)^2}+\frac{1}{\pi}\int_0^\infty dp'\frac{p'}{p} \times\ln\frac{(p-p')^2}{(p+p')^2}\frac{t_C(p',q;\epsilon)}{\epsilon-p'^2}. \tag{52}$$

We solve Eq. (49) for $n=2$ for each given $\epsilon^{(1)}$. Now, the motivation for plotting in Fig. 1 the binding energy of the excited state against the ground-state energy is clear. In Table I as well as in Fig. 1, the renormalized model reproduces with less than 10% accuracy the model results of Eq. (48). In the limit of $\eta\to\infty$, the renormalized model should work better; however, in the test case of Eq. (48) this limit corresponds to a vanishing Yukawa interaction, and the results tend trivially to the Coulomb value of the first excited state of 0.25 for the ground-state value of 1, which in fact is observed in Fig. 1 and Table I.

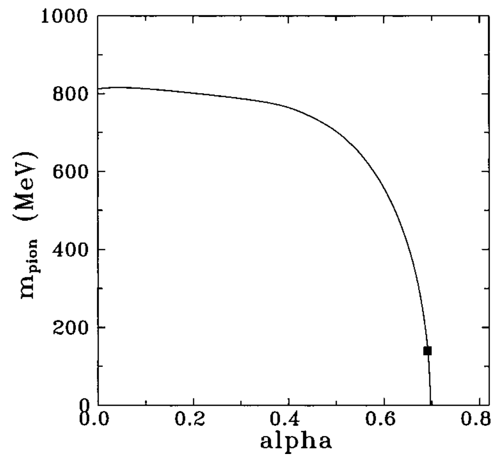

FIG. 2. Pion mass as a function of the strength of the Coulomb interaction α. Nonrelativistic model for Coulomb plus Yukawa interaction, $m=406$ MeV, solid line. Yukawa range parameter $\eta=1330$ MeV [2]. The solid square is the calculation of Ref. [2].

B. Effective pion model

The effective model of [2] corresponds to using the nonrelativistic phase space $A(k)=1$ in Eq. (6) and a smeared delta interaction of a Yukawa form:

$$m_\pi^2\varphi(\vec{k})=[4m^2+4k^2]\varphi(\vec{k})-\frac{4}{3\pi^2}\alpha\int\frac{d\vec{k}'}{m} \times\left(\frac{2m^2}{(\vec{k}-\vec{k}')^2}+\frac{\eta^2}{\eta^2+(\vec{k}-\vec{k}')^2}\right)\varphi(\vec{k}'), \tag{53}$$

which was solved with parameters adjusted to fit the pion mass and the rho-meson mass, resulting in $m=406$ MeV, $\alpha=0.6904$, and $\eta=1330$ MeV [2]. The value of η was found from the condition that the first excited bound-state mass m_π^* satisfy the Strutinsk requirement at the extremum:

$$\frac{d}{d\eta}m_\pi^*=0. \tag{54}$$

However, the pion charge radius calculated according to the nonrelativistic formula in [2] is about one-half of the empirical value. We will confirm this fact in the renormalized effective model while using Eq. (47) and φ_π from Eq. (36) with $A(k)=1$ to compute the pion radius.

In Fig. 2, we present our results for the pion mass as a function of α for $\eta=1330$ MeV. Our agreement with the calculation of [2] is within 10%. In Fig. 3, we show results for our calculation of the ground- and excited-state masses from Eq. (53), for α varying and $\eta=1330$ MeV, compared to the renormalized model for the Coulomb plus Dirac-delta interaction. In the last case, the bound-state masses of the pion ground and excited s-wave states, are found numerically from the zeros of Eq. (34) with the Green's function of the

Coulomb-like potential obtained from the solution of the integral scattering equation (20). In both equations, which define the nonrelativistic renormalized model Coulomb plus Dirac-delta interaction, $A(k)=1$ is used. We have disregarded the effect of the energy transfer in Q^2 of Eq. (2) as being a short-range effect parametrized by the value of the pion mass, which is input in this calculation. In momentum space Eq. (34) is given by

$$4\pi\int_0^\infty dp\,\frac{p^2}{A(p)}\left[\frac{1}{m_\pi^{*2}-4m^2-4p^2}-\frac{1}{m_\pi^2-4m^2-4p^2}\right]$$
$$+8\pi^2\int_0^\infty dq\,\frac{q^2}{\sqrt{A(q)}}\int_0^\infty dp\,\frac{p^2}{\sqrt{A(p)}}$$
$$\times\left[\frac{t^V(p,q;m_\pi^{*2})}{(m_\pi^{*2}-4m^2-4p^2)(m_\pi^{*2}-4m^2-4q^2)}\right.$$
$$\left.-\frac{t^V(p,q;m_\pi^2)}{(m_\pi^2-4m^2-4p^2)(m_\pi^2-4m^2-4q^2)}\right]=0, \qquad (55)$$

and the s-wave projected T matrix in Eq. (55) is

$$t^V(p,q;M^2)=\int_{-1}^1 d\cos(\theta)\langle\vec{p}|T^V(M^2)|\vec{q}\rangle, \qquad (56)$$

which is the solution of

$$t^V(p,q;M^2)=\frac{4m}{3\pi^2}\frac{\alpha}{pq}\frac{\ln\frac{(p-q)^2}{(p+q)^2}}{\sqrt{A(p)A(q)}}$$
$$+\frac{8m}{3\pi}\alpha\int_0^\infty\frac{dp'}{\sqrt{A(p)A(p')}}\frac{p'}{p}$$
$$\times\ln\frac{(p-p')^2}{(p+p')^2}\frac{t^V(p',q;M^2)}{M^2-4m^2-4p^2}, \qquad (57)$$

the momentum space representation of the s-wave projection of Eq. (20).

The agreement between the renormalized model and the smeared delta model is within a fraction of a percent, which still improves as the weakly bound limit of the ground state is approached, as shown in Fig. 3. The calculation of [2] is a few percent below ours. The inclusion of the relativistic phase space in the renormalized model of the Coulomb plus Dirac-delta interaction, Eq. (34), makes less bound the excited state for a given ground-state mass. As the pion mass grows the relativistic phase-space effect tends to vanish.

In the other study performed, $\alpha=0.6904$ was kept unchanged, while varying η between 350 and 1350 MeV. The results for the excited bound-state mass against the ground-state mass are shown in Fig. 4. The renormalized Coulomb plus Dirac-delta calculation agrees within a fraction of a percent with the smeared Dirac-delta calculation. The relativistic phase space in the renormalized interaction makes the excited state less bound for a given ground-state mass.

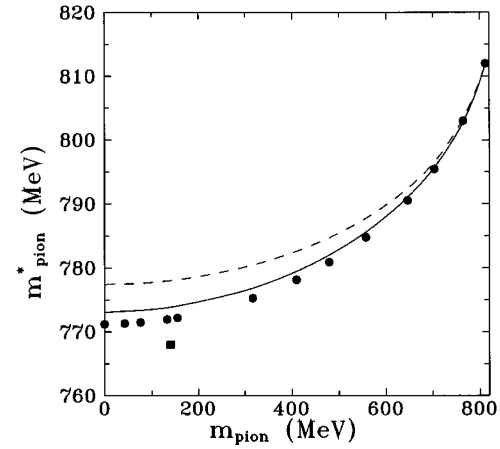

FIG. 3. Mass of the excited state as a function of the ground-state pion mass. Nonrelativistic model for Coulomb plus Yukawa interaction, $m=406$ MeV, solid line. Strength of the Coulomb potential within $0<\alpha<0.7$ and Yukawa range parameter $\eta=1330$ MeV [2]. The solid square is the calculation of Ref. [2]. The nonrelativistic effective model (Coulomb plus Dirac-delta) is given by the solid circles. The relativistic effective model for the Coulomb plus Dirac-delta interaction is given by the dashed line.

C. Effective meson model

The bound-state masses of the meson ground and excited s-wave states, are found numerically from the zeros of Eq. (34) with the Green's function of the Coulomb-like potential obtained from solution of the integral scattering equation (20). The energy transfer in Q^2 in Eq. (2) was neglected.

The renormalized strength of the singular interaction, from Eq. (22), is

$$\lambda_{\mathcal{R}}^{-1}(m_\pi^2)=\langle\chi|G_0(m_\pi^2)T^V(m_\pi^2)G_0(m_\pi^2)|\chi\rangle, \qquad (58)$$

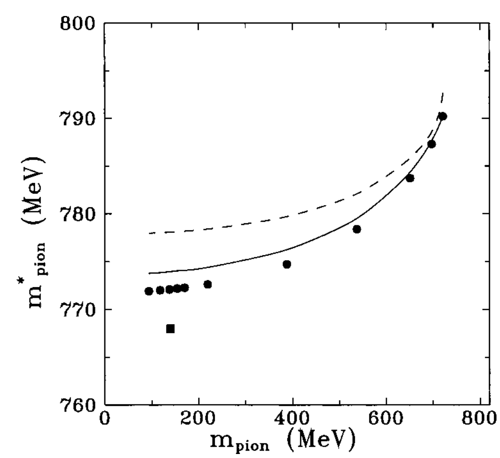

FIG. 4. Mass of the excited state as a function of the ground-state pion mass. Nonrelativistic model for Coulomb plus Yukawa interaction, $m=406$ MeV, solid line. Strength of the Coulomb potential $\alpha=0.6904$ and Yukawa range parameter within $0<\eta<1350$ MeV. The solid square is the calculation of Ref. [2]. The nonrelativistic effective model (Coulomb plus Dirac-delta) is given by the solid circles. The relativistic effective model for the Coulomb plus Dirac-delta interaction is given by the dashed line.

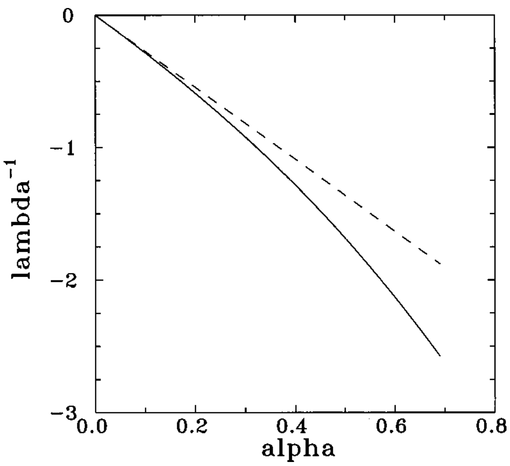

FIG. 5. Inverse renormalized strength $\lambda_{\mathcal{R}}^{-1}(m_\pi^2)$ in units of m_r as a function of the Coulomb intensity potential parameter α for a pion mass of 140 MeV. Nonperturbative calculation (solid line) and first-order perturbative calculation (dashed line); see the text for an explanation.

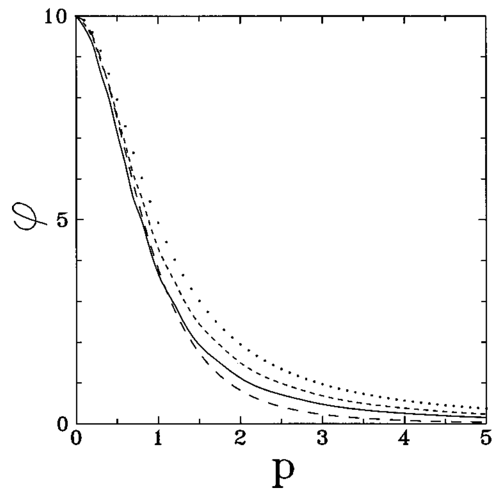

FIG. 6. Pion wave function (φ) with arbitrary normalization as a function of momentum in units of quark mass. Calculations performed with $A(k)=1$. Pion model wave function model of Eq. (60) for $\alpha=0.6904$ and $m=406$ MeV (solid line), first-order perturbative calculation from Eq. (61) (short-dashed line), asymptotic form Eq. (62) (dotted line), and fit from Eq. (63) (dashed line).

with the condition, Eq. (27), at the physical pion mass is shown in Fig. 5 for $m=406$ MeV and compared to the perturbative calculation

$$\lambda_{\mathcal{R}}^{-1}(m_\pi^2)|_{pert}=\langle\chi|G_0(m_\pi^2)VG_0(m_\pi^2)|\chi\rangle, \tag{59}$$

with V from Eq. (8) and T^V the solution of Eq. (20). For values of α below 0.2 the agreement between the perturbative and nonperturbative calculations of the renormalized strength is quite good, giving confidence to our numerical calculations.

The wave function φ_π of the pion from Eq. (36) in momentum space is written as

$$\varphi_\pi(p)=\frac{1}{m_\pi^2-4m^2-4p^2}\left(\frac{1}{\sqrt{A(p)}}+2\pi\int_0^\infty dq\,\frac{q^2}{\sqrt{A(q)}}\,\frac{t^V(p,q;m_\pi^2)}{m_\pi^2-4m^2-4p^2}\right). \tag{60}$$

The first-order perturbative pion wave function is

$$\varphi_\pi(p)|_{pert}=\frac{1}{\sqrt{A(p)}}\,\frac{1}{m_\pi^2-4m^2-4p^2}\times\left(1+\frac{8m}{3\pi}\alpha\int_0^\infty\frac{dq}{A(q)}\,\frac{q}{p}\,\frac{\ln\frac{(p-q)^2}{(p+q)^2}}{m_\pi^2-4m^2-4p^2}\right), \tag{61}$$

and for $\alpha=0$ the pion wave function has the asymptotic form

$$\varphi_\pi^0(\vec{p})=\frac{1}{\sqrt{A(p)}}\,\frac{1}{m_\pi^2-4m^2-4p^2}. \tag{62}$$

In [2], the pion wave-function eigenfunction of the effective square mass operator of the Coulomb plus Yukawa model, Eq. (53), was approximated by the following analytical form:

$$\varphi_\pi^a(\vec{p})=\frac{\mathcal{N}}{(1+p^2/p_a^2)^2}, \tag{63}$$

where $\mathcal{N}$ is an arbitrary normalization and the fit is performed for $\alpha=0.6904$, $\eta=1330$ MeV, and $m=406$ MeV, and $p_a=515$ MeV is adjusted to the numerical solution of Eq. (53).

The various nonrelativistic models of the pion wave function, with $A(k)=1$, are plotted in Fig. 6. In this figure, a comparison between φ_π, $\varphi_\pi|_{pert}$, φ_π^a, and φ_π^0 is performed. The model parameters are $\alpha=0.6904$ and $m=406$ MeV and the normalization is arbitrary. The perturbative calculation reproduces $\varphi_\pi(p)$ for p below m. The asymptotic wave function $\varphi_\pi^0(p)$ overestimates $\varphi_\pi(p)$, as it should be for an attractive Coulomb-type interaction, and for small p it approaches the nonperturbative eigenfunction. The analytical approximation works quite good for momentum up to about $1.5m$. More results on the pion wave function are shown in Fig. 7, where the results for $A(k)=1$ with the full calculation are compared. The effect of $A(p)$ diminishes the magnitude of $\varphi(p)$, as one could anticipate from Eq. (60). Also we plot $\varphi_\pi(p)$ for $\alpha=0.18$ and $m=386$ MeV, which is above the curve of the previous case due to the decrease of the Coulomb attraction (this parameter fits the empirical pion radius). We point also that, for very high momentum, again the asymptotic form dominates, as the second term of Eq. (60) tends to zero faster than the first one.

We are going to compute the pion radius in the effective renormalized model of the pion using Eqs. (40), (45), and (47). In the last two cases, the pion wave function from Eq.

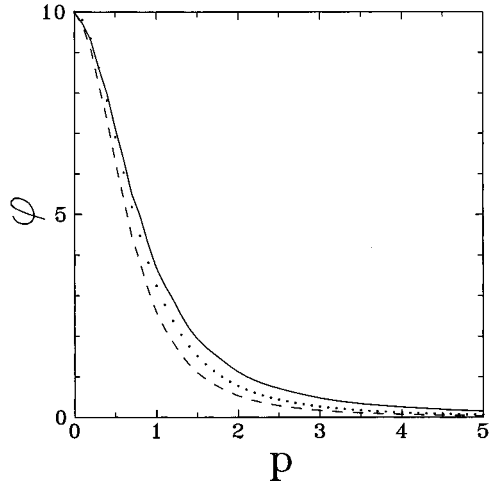

FIG. 7. Pion wave-function (φ) as a function of momentum in units of quark mass. Pion model for $\alpha=0.6904$ and $m=406$ MeV, calculation with Eq. (60) (dashed line), considering $A(k)=1$ (solid line). Results for $\alpha=0.18$ and $m=386$ MeV (dotted line).

(60) with $A(k)=1$ is used. The pion wave function is known from Eqs. (35) and (60) from which the charge radius including the effect of quark spin is obtained from integration by Gaussian quadrature of the difference

$$\Delta F_\pi(q^2)=F_\pi(q^2)-F_\pi(0), \tag{64}$$

obtained from Eq. (40), and the pion radius is calculated from

$$r_\pi=\left[6\frac{d}{dq^2}\Delta F_\pi(q^2)\bigg|_{q^2=0}\right]^{1/2}. \tag{65}$$

In the case the quark spin is neglected, two possibilities of the calculation of the pion radius are considered: one through Eq. (45) in which only the spin factors are simplified in the limit of the quark mass being infinity, while the normalization is defined as in Eq. (40), known from the empirical value of f_π and the number of colors. The second possibility is to declare normalized to 1 the $q\bar{q}$ Fock component of the pion wave function and use the formula for the form factor in which the quark has no spin, Eq. (47).

The results for the pion charge radius as a function of the strength α of the Coulomb-like interaction, Eq. (8), are shown in Fig. 8. For $r_\pi=0.67$ fm we found $\alpha=0.18$ using Eq. (40). Our calculations obtain the pion charge radius from the relativistic expression (40) which is known to give about twice the nonrelativistic radius [12]. The attractive Coulomb-like interaction increases the radius over the soft-pion limit with $\alpha=0$ which is below its experimental value. In that sense consistency is found with the effective theory which has an attractive Coulomb-like interaction. The repulsion would be completely inconsistent with the pion radius.

The pion charge radius obtained from first-order perturbative calculations of the pion wave function

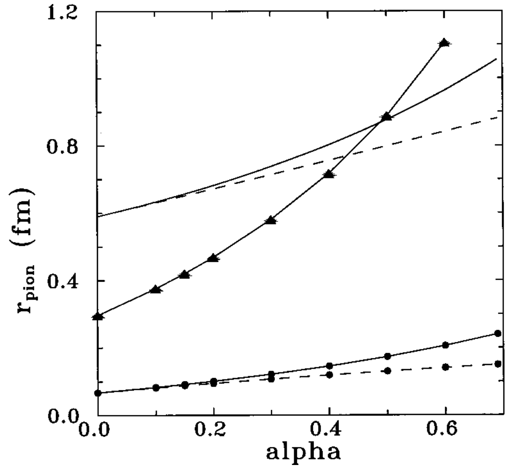

FIG. 8. Pion charge radius as a function of the strength α of the Coulomb potential. Quark mass of 406 MeV. Results for Eq. (40): nonperturbative calculation of the wave function (solid line) and first-order perturbative calculation (dashed line). Results for Eq. (45) and wave function obtained with $A(k)=1$ (solid line with triangles). Results for Eq. (47) with the wave function obtained with $A(k)=1$: nonperturbative calculation of the wave function (solid line with dots) and first-order perturbative calculation (dashed line with dots).

$$\psi_\pi(x,\vec{k}_\perp)|_{pert}=\sqrt{\frac{A(k)}{x(1-x)}}\varphi_\pi(k)\bigg|_{pert} \tag{66}$$

is also shown in Fig. 8. Consistent with the strength calculation presented in Fig. 5, we observe that for α below 0.2 the perturbative calculations match the nonperturbative results.

The results for the scalar quarks form factors with φ from Eq. (60) calculated in the effective renormalized theory for $A(k)=1$ are also shown in Fig. 8. The calculation of r_π with Eq. (45) shows values above the ones calculated with Eq. (47), which are too small compared to the empirical value, indicating the importance of the physical normalization of the $q\bar{q}$ Fock component of the wave function using f_π. We also performed a first-order perturbative calculation of r_π for the wave function normalized to 1. The agreement between the perturbative and nonperturbative calculations is reasonable for α below 0.2.

The plot of the pion charge radius against the mass of the first excited state is shown in Fig. 9. For decreasing values of α, the pion charge radius diminishes and consistently the excited-state mass increases; i.e., this state becomes less bound. It is clear from this figure that to simultaneously fit the radius and the mass of the rho meson (768 MeV) we are obliged to use a different quark mass from the value of 406 MeV. For comparison we also show the results from Eq. (45).

The experimental pion radius of 0.67 ± 0.02 fm is fitted with $m_u=m_{\bar{u}}=386$ MeV and $\alpha=0.18$, resulting $r_\pi=0.67$ fm and the mass of the singlet-$2s$ state of 768 MeV. The singlet-$2s$ excited state mass of the $u\bar{u}$ system is identified with the ρ-meson mass in the present effective QCD model. We remind the reader that for a given pion mass, the free

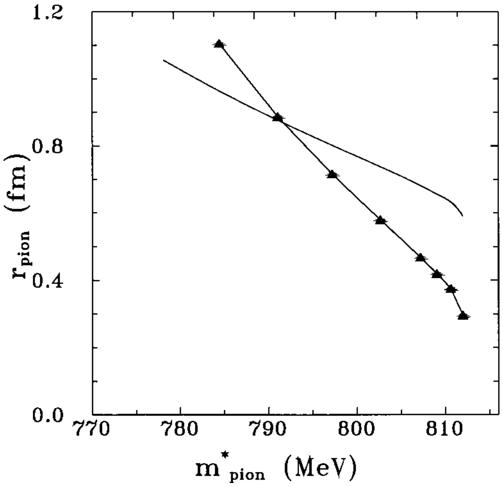

FIG. 9. Pion charge radius as a function of the mass of the excited $q\bar{q}$ state for a constrained pion mass of 140 MeV. The strength α is within the range of Fig. 6 and the quark mass is 406 MeV. Results for Eq. (40) (solid line) and for Eq. (45) with the wave function obtained considering $A(k)=1$ (solid line with triangles).

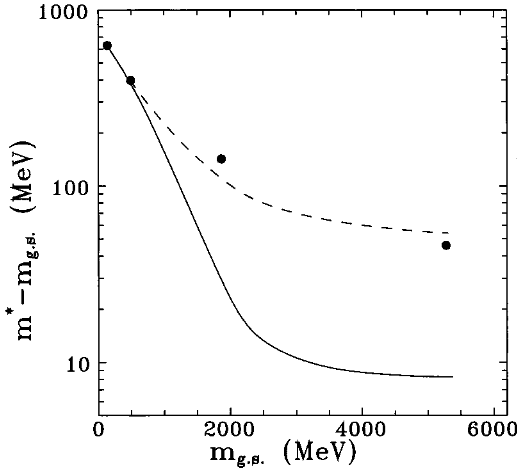

FIG. 10. The difference between the masses of the excited and ground state of the $q\bar{q}$ system as a function of the ground-state mass in the effective relativistic model. The quark mass is varied in the range between 386 and 5000 MeV. The other quark mass is fixed to 386 MeV. Calculations with $\alpha=0.18$ (solid line) and $\alpha=0.5$ (dashed line). The experimental values from Table II are given by the solid circles.

model parameters are two: α and the constituent quark mass. As was mentioned at the beginning of Sec. IV, they are fitted through the values of the pion charge radius and rho-meson mass. Then, the quark mass is varied to form mesons with one up antiquark together with strange, charm, or bottom quarks. The masses of the constituent quarks were within the range of 500–5000 MeV. The results are shown in Table II and compared to the experimental data. The singlet-2s states are identified with the lowest-mass vector mesons states, since the Dirac-delta interaction is the effective hyperfine interaction, the reason for the splitting between the pseudoscalar and vector mesons. Although in the singlet channel the hyperfine interaction is attractive, which is not valid for the spin-1 mesons, we believe that the Dirac-delta interaction parametrizes the short-range physics beyond that by taking care of the empirical value of the pion mass.

It is clear that the split between the heavy-meson masses will not be adjusted in the present calculation for α small. This is reasonable, since these mesons are weakly bound and the wave function spreads out in the region where the confinement potential, not present in our model, should be important. In Fig. 10, we study the difference of the excited- and ground-state masses as a function of the ground-state mass. As we have seen in Table II, the difference is underestimated for $\alpha=0.18$ above the kaon mass. For $\alpha=0.5$ the experimental data are reproduced. This is in fact reasonable if we think that α should on average increase with the size scale, indicating a confining behavior, which is stronger for the heavy mesons since they are less bound than the pion and the kaon.

TABLE II. Results for S-wave meson masses: singlet-1s ($M^{(1s)}_{b,th}$) and singlet-2s ($M^{(2s)}_{b,th}$) and experimental values ($M^{(1s,2s)}_{b,expt}$). All masses in MeV. $\alpha=0.18$.

$\bar{q}q$	m_1	m_2	$M^{(1s)}_{b,th}$	$M^{(1s)}_{b,expt}$	$M^{(2s)}_{b,th}$	$M^{(2s)}_{b,expt}$
$\bar{u}d$	386	386	140	$\pi^{\pm}$:140	768	ρ:768
$\bar{u}s$	386	500	511	$K^{\pm}$:494	882	K^*:892
$\bar{u}c$	386	1500	1852	D^0:1865	1882	D^{*0}:2007
$\bar{u}b$	386	5000	5375	$B^{\pm}$:5279	5383	B^*:5325

VI. CONCLUSION

The essential development made in this work is the renormalization of the effective QCD-inspired Hamiltonian theory with a singularity at zero range and its consequent application to the pion and other mesons. The method is an example of the Hamiltonian renormalization procedure and it is equivalent to a subtracted equation for the transition matrix. The physical renormalization condition is given to the two-particle model at the subtraction point and in the cases discussed here it is the ground-state binding energy or mass. The treatment is shown to be renormalization group invariant, i.e., independent of the arbitrary subtraction point. This independence is expressed by a fixed-point Hamiltonian that brings physical input to the theory—the pion mass or ground-state binding energy—as well as the necessary counterterms that render all the momentum integrations finite.

First, we have studied in an example the renormalization method applied to a two-body model with a Coulomb plus Dirac-delta interaction, where we have calculated the excited-state energy for a given ground-state binding energy. The results are compared to calculations with a Coulomb plus repulsive Yukawa interaction and the renormalized model reproduces with less than 10% accuracy the model results of Eq. (48). The success of this result drives us to the solution of the renormalization problem of the ↑↓ model [2].

The effective mass operator equation in the ↑↓ model, Eq.

(1), has as eigenstate the lowest Fock-state component ($\bar{q}q$) of the light-cone wave function of a meson bound system of constituent quarks or dressed quarks (not to be confused with the bare quark). This model picks out one particular aspect of the gluon exchange between quarks, namely, the strong attraction of the spin-spin interaction in the singlet channel. Previously [2], the renormalization was carried out by first regularizing the Dirac-delta interaction through a Yukawa form and then its parameter was found by the Strutinsky requirement that the mass of the excited state be stationary with respect to the variation of the regularization parameter. The step forward in this work was the use of the renormalization group invariant approach, in which the regularization parameter is not necessary to solve the model. All the short-range physics is parametrized by one parameter: the renormalized strength of the Dirac-delta interaction, which is determined by the mass of the pion. We showed that the results for the mass of the excited state obtained with the renormalized model and the smeared delta regularized model are in agreement within a fraction of a percent, for the same ground-state mass, which was varied either by changing the Coulomb interaction intensity or the Yukawa range. The concordance still improves as the weakly bound limit of the ground state is approached. The effect of the relativistic phase space in the renormalized model of the Coulomb plus Dirac-delta interaction, Eq. (34), makes the excited state, for a given ground-state mass, less bound. This difference tends to vanish as the pion becomes weakly bound.

The various models of the $\bar{q}q$ Fock component of the pion light-cone wave function, considering $A(k)=1$ and the relativistic phase space [$A(k)$ from Eq. (5)], were calculated and compared with the analytical form and perturbative result. We obtained the reduced wave function φ_π, Eq. (60), solution of the renormalized $\uparrow\downarrow$ model, $\varphi_\pi|_{pert}$, Eq. (61), the analytical form φ_π^a, Eq. (63), and the asymptotic form φ_π^0, Eq. (62). We have used model parameters $\alpha=0.6904$ and $m=406$ MeV [2]. Although for p below m all the calculations are reasonably consistent, the high-momentum tail is dominated by the asymptotic wave function $\varphi_\pi^0(p)$, which overestimates $\varphi_\pi(p)$, as it should be for an attractive Coulomb-type interaction, and for small p it approaches the nonperturbative eigenfunction, as well as for very high momentum. The analytical approximation works quite good for momentum up to about $1.5m$; however, it does not have the asymptotic tail for high momentum. The relativistic phase space diminishes the magnitude of $\varphi(p)$ as clearly seen in Eq. (60).

The calculation of the pion charge radius was performed in the renormalized effective QCD-inspired Hamiltonian theory with Eqs. (36) and (40), and although we have simplified the spin dependence in the dynamical equation, it is important in the evaluation of the radius. For this purpose we have used an effective pseudoscalar Lagrangian to construct the spin part of the pion wave function; it gives the absolute normalization of the lowest Fock component of the light-cone wave function in terms of the weak decay constant f_π, the constituent quark mass, and the number of colors. Turning off the Coulomb-like interaction, we retrieved the well-known result valid in the soft-pion limit [14]. We also compared the values for r_π obtained with expressions where the quark spin is neglected and the wave function normalized to 1. We pointed out the necessity of the correct normalization of the wave function of the $\uparrow\downarrow$ model according to the f_π value to be able to fit r_π, which is reminiscent of the fact that f_π and r_π are closely related in the light-front phenomenology [12].

The experimental pion radius of 0.67 ± 0.02 fm was fitted with $m_u=m_{\bar{u}}=386$ MeV and $\alpha=0.18$, resulting in $r_\pi=0.67$ fm and the mass of the singlet-$2s$ state of 768 MeV. We remind the reader that the pion mass is input in the renormalized model calculation. The singlet-$2s$ excited-state mass of the $\bar{u}u$ system was identified with the ρ-meson mass in the present effective QCD model. We stress that in the singlet channel the hyperfine interaction is attractive, which is not valid for the spin-1 channel of the vector mesons. Once the hyperfine part is substituted by a potential that does not distinguish between spin-1 and spin-0 states, the identification with the physics of chiral symmetry breaking is strictly lost. However, we believe that the Dirac-delta interaction parametrizes the short-range physics, which is brought to the model by the empirical value of the pion mass. In essence, without being too naive, in order to fit the pion charge radius with success and with reasonable parameters, it was essential that (i) the Coulomb-like interaction be *attractive* and (ii) the *normalization* of the form factor be consistent with the empirical f_π value.

With respect to the masses of the ground state of the pseudoscalar and vector mesons with one up antiquark together with strange, charm, or bottom quarks, which were calculated as a function of the constituent quark mass, the results were in qualitative agreement with the data for $\alpha=0.18$. We tried a better fit of this data, since it was clear that the split between the heavy-meson masses would not be adjusted in the present calculation for α small. This, in fact, seems reasonable, since these mesons are weakly bound and the quarks can be found in a region where the confinement potential, not present in our model, is important. The difference between the vector and pseudoscalar mesons masses for the same $\bar{q}q$ pair is underestimated for $\alpha=0.18$ above the kaon mass. We found that for $\alpha=0.5$ the experimental data are reproduced. This gives us some hope that it is possible to refine the $\uparrow\downarrow$ model to include more physics than initially thought. In regard to this extension, it is reasonable to think that α should on average increase with the size scale, indicating a confining behavior, which we found stronger for the heavy mesons since they are less bound than the pion and the kaon. In short, a reasonable description of the physics of the pion and other scalar and vector mesons was found, taking into account the simplicity of the renormalized effective light-cone QCD-inspired theory.

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APPENDIX: DERIVATION OF THE RENORMALIZED *T* MATRIX

In this appendix, the solution of Eq. (17) to find the renormalized T matrix, Eq. (19), is performed in detail. We want to solve Eq. (17):

$$T_{\mathcal{R}}(M^2)=V+V^{\delta}_{\mathcal{R}}+(V+V^{\delta}_{\mathcal{R}})G_0^{(+)}(M^2)T_{\mathcal{R}}(M^2). \quad \text{(A1)}$$

The regular part of the potential V is defined by Eq. (8) and the renormalized singular interaction is given by Eq. (13), rewritten as a matrix equation

$$V^{\delta}_{\mathcal{R}}=|\chi\rangle\lambda_{\mathcal{R}}(\mu^2)\langle\chi|-|\chi\rangle\lambda_{\mathcal{R}}(\mu^2)\langle\chi|G_0^{(+)}(\mu^2)V^{\delta}_{\mathcal{R}}, \quad \text{(A2)}$$

which has the solution

$$V^{\delta}_{\mathcal{R}}=|\chi\rangle v_{\mathcal{R}}(\mu^2)\langle\chi|, \quad \text{(A3)}$$

with the function

$$v_{\mathcal{R}}(\mu^2)=[\lambda_{\mathcal{R}}(\mu^2)+\langle\chi|G_0^{(+)}(\mu^2)|\chi\rangle]^{-1}. \quad \text{(A4)}$$

The function $v_{\mathcal{R}}(\mu^2)$ contains the divergences in the momentum integrals which exactly cancel such infinities in Eq. (A1). It is enough for the formal manipulations that will come. However, one could equally well introduce a cutoff in Eqs. (A1) and (A4), and perform the limit of the cutoff going to infinity just after the solution of Eq. (A1), in which all the necessary cancellations happen and the limit is finite.

Next, Eq. (A1) is rewritten as

$$[1-VG_0^{(+)}(M^2)]T_{\mathcal{R}}(M^2) =V+V^{\delta}_{\mathcal{R}}+V^{\delta}_{\mathcal{R}}G_0^{(+)}(M^2)T_{\mathcal{R}}(M^2), \quad \text{(A5)}$$

and inverting the operator in the left-hand using the regular T matrix, $T^V(M^2)$, solution of Eq. (20), one has

$$T_{\mathcal{R}}(M^2)=T^V(M^2)+[1+T^V(M^2)G_0^{(+)}(M^2)]|\chi\rangle v_{\mathcal{R}}(\mu^2) \times\langle\chi|[G_0^{(+)}(M^2)T_{\mathcal{R}}(M^2)+1]. \quad \text{(A6)}$$

The ''bra'' function $\langle\chi|G_0^{(+)}(M^2)T_{\mathcal{R}}(M^2)$ has to be calculated in order to find the renormalized T matrix. We multiply Eq. (A6) by $\langle\chi|G_0^{(+)}(M^2)$ on both sides, and solving it we get

$$\langle\chi|G_0^{(+)}(M^2)T_{\mathcal{R}}(M^2)=\frac{\langle\chi|G_0^{(+)}(M^2)T^V(M^2)+\langle\chi|G^{V(+)}(M^2)|\chi\rangle v_{\mathcal{R}}(\mu^2)\langle\chi|}{1-\langle\chi|G^{V(+)}(M^2)|\chi\rangle v_{\mathcal{R}}(\mu^2)}, \quad \text{(A7)}$$

where the regular potential resolvent is

$$G^{V(+)}(M^2)=G_0^{(+)}(M^2)+G_0^{(+)}(M^2)T^V(M^2)G_0^{(+)}(M^2). \quad \text{(A8)}$$

The ''bra'' function of Eq. (A7) is introduced back into Eq. (A6), and with a little algebra one finds

$$T_{\mathcal{R}}(M^2)=T^V(M^2)+\frac{[1+T^V(M^2)G_0^{(+)}(M^2)]|\chi\rangle\langle\chi|[G_0^{(+)}(M^2)T^V(M^2)+1]}{v_{\mathcal{R}}^{-1}(\mu^2)-\langle\chi|G^{V(+)}(M^2)|\chi\rangle}, \quad \text{(A9)}$$

which, after introducing Eq. (A4) and the explicit form of the resolvent of Eq. (A8), results in the renormalized T matrix of Eq. (19).

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