Simplified method for trace anomaly calculations in $d \leq 6$

Fiorenzo Bastianelli*
Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna via Irnerio 46, I-40126 Bologna, Italy

N. D. Hari Dass†
Institute of Mathematical Sciences, C.I.T. Campus, Chennai 600 113, India
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We discuss a simplified method for computing trace anomalies in dimensions $d \leq 6$. It is known that in the quantum mechanical approach trace anomalies in d dimensions are given by a $(d/2+1)$-loop computation in an auxiliary 1D sigma model with arbitrary geometry. We show how one can obtain the same information using a simpler $(d/2)$-loop calculation on an arbitrary geometry supplemented by a $(d/2+1)$-loop calculation on the simplified geometry of a maximally symmetric space.

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Conformal anomalies, also called trace anomalies, have a variety of applications and are of central importance in string and quantum field theories [1]. They arise due to the fact that the regularization procedure brings in a scale dependence even though the classical theory was scale invariant. In particular, they appear whenever conformal field theories in even dimensions are considered in a curved background.[1] However their computation gets more involved as the spacetime dimension d gets bigger. Already for $d=6$ the computation is quite complicated and only recently the anomalies for free conformal fields have been completely identified [2] [we recall also that the trace anomalies for a class of supersymmetric (0,2) interacting 6-dimensional conformal field theory (CFT_6) in the large N limit was obtained in [3] using the supergravity dual as dictated by the AdS-CFT correspondence [4]]. Trace anomalies can be computed efficiently using the quantum mechanical approach of [5,6], which requires a $(n+1)$-loop calculation in an auxiliary 1D nonlinear sigma model to obtain the trace anomalies in $d=2n$ dimensions. The complications for $d=6$ trace anomalies are seen in this method as the need for calculating up to 4 loops in the 1D sigma model. The latter is laborious even in the newly developed dimensional regularization scheme [7] which requires finite covariant counterterms only.[2] Such a lengthy calculation was indeed performed recently in [12], confirming the correctness of the dimensional regularization scheme of the quantum mechanical path integral [7] as well as the correct value of the trace anomalies identified in [2].

In this paper we wish to discuss a simplified approach to obtain trace anomalies in $d \leq 6$ (hopefully it may be extended to higher dimensions as well in the future). The strategy we propose is to take advantage of some recent results concerning trace anomalies. These results guarantee that one may obtain all but one terms in the anomaly by a simpler n-loop calculation in the 1D nonlinear sigma model with arbitrary geometry. Then, the missing part of the anomaly can be identified by a $(n+1)$-loop calculation performed in the simpler geometry of a maximally symmetric space. It is this geometry which renders the higher loop calculation much easier.

Concretely, we first make use of the classification of Deser and Schwimmer [13] that divides trace anomalies into: *type A*, which are unique and always proportional to the Euler topological density, *type B*, whose number increases with the spacetime dimensions and are made up by local Weyl invariants, and *trivial anomalies*, which can be canceled by the variation of local counterterms and can be expressed as total derivatives. This classification makes it evident that the simplified geometry of a maximally symmetric space annihilates type B and trivial anomalies, and allows a simpler calculation of the type A anomaly, as done indeed in [14]. In the path integral method the type A anomaly can be obtained by a $(n+1)$-loop calculation for the sigma model on the maximally symmetric geometry. The latter simplifies drastically the calculation.[3] Then, inspecting the cohomological analysis for trace anomalies in $d=4$ [15] and $d=6$ [16] one notices that the remaining non-trivial part of the trace anomalies (i.e. type B) can be identified by certain terms in the curvature that are not affected by adding trivial anomalies, and at the same time are given by disconnected diagrams in the path integral approach. The latter are identified by a lower loop calculation (i.e. at n-loops).

Let us consider the case of a conformal scalar field in d dimensions

$$I=\int d^d x \sqrt{g}\, \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\xi R\phi^2) \tag{1}$$

where $\xi=(d-2)/[4(d-1)]$ and R is the curvature scalar.

*Email address: bastianelli@bo.infn.it
†Email address: dass@imsc.ernet.in

[1]More general background field configurations are also usefully considered sometimes.

[2]Mode regularization [8,5] and time slicing [9] need instead non-covariant counterterms which render higher loop calculations even more laborious. See Ref. [10] for an attempt to computerize the time slicing procedure and Refs. [11] for the original use of dimensional regularization in the infinite propagation time limit.

[3]It is conceivable that one may devise a way of computing this path integral exactly, thus deriving a compact formula for a generating functional for type A anomalies.

Our conventions for the curvature tensors follow from $[\nabla_\mu,\nabla_\nu]V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma$ and $R_{\mu\nu} = R_{\mu\sigma}{}^\sigma{}_\nu$. We employ a Euclidean signature.

As described in [5] one-loop trace anomalies can be obtained by computing a certain Fujikawa Jacobian suitably regulated and represented as a quantum mechanical path integral with periodic boundary conditions

$$\int d^d x\sqrt{g}\,\sigma(x)\langle T^\mu{}_\mu(x)\rangle = \lim_{\beta\to 0}\mathrm{Tr}[\sigma\,\mathrm{e}^{-\beta H}] = \lim_{\beta\to 0}\int_{PBC}\mathcal{D}x\;\sigma(x)\mathrm{e}^{-S[x]} \tag{2}$$

where on the left hand side $T^\mu{}_\mu$ denotes the trace of the stress tensor $T_{\mu\nu} = (2/\sqrt{g})(\delta I/\delta g^{\mu\nu})$ of the conformal scalar and $\sigma(x)$ is an arbitrary function describing an infinitesimal Weyl variation. In the first equality the infinitesimal part of the Fujikawa Jacobian has been regulated with the conformal scalar field kinetic operator $H = -\frac{1}{2}\nabla^2 - (\xi/2)R$. The limit $\beta\to 0$ should be taken after removing divergent terms in β (which is what the renormalization in QFT does). Thus, it picks up just the β independent term. Finally, on the right hand side the trace is given a representation as a quantum mechanical path integral corresponding to a model with Hamiltonian H and with periodic boundary conditions (PBC). Using the dimensional regularization scheme [7] the path integral requires the action

$$S[x] = \frac{1}{\beta}\int_{-1}^{0} dt\left[\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \beta^2[V(x) + V_{DR}(x)]\right] \tag{3}$$

with a scalar potential $V = -(\xi/2)R$ and the counterterm $V_{DR} = \frac{1}{8}R$, both needed to reproduce the correct Hamiltonian H. As in [7] a ghost action will be used to exponentiate the nontrivial part of the path integral measure

$$S_{gh} = \frac{1}{\beta}\int_{-1}^{0} dt\left[\frac{1}{2}g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu)\right] \tag{4}$$

where a^μ are bosonic and $b^\mu,\;c^\mu$ fermionic ghosts.

One can easily eliminate the arbitrary function $\sigma(x)$ from Eq. (2) to obtain the local formula

$$\langle T^\mu{}_\mu(x)\rangle = \lim_{\beta\to 0}\int_{x(-1)=x(0)=x}\mathcal{D}x\;\mathrm{e}^{-S[x]} \equiv \lim_{\beta\to 0}Z(\beta) \tag{5}$$

where again the limit has to be understood only as the indication of extracting the β independent part of the subsequent expression, while the boundary conditions on the path integral identify the initial and final points and keep them fixed.

Aiming at exemplifying the proposed method for $d\leqslant 6$ we need the transition amplitude $Z(\beta)$ on an arbitrary geometry up to three loops. This has already been computed and can be read off from various papers (see e.g. [8,12])

$$Z(\beta) = \frac{1}{(2\pi\beta)^{d/2}}\exp\left[\frac{\beta}{12}(6\xi-1)R + \beta^2\left(\frac{1}{720}(R^2_{mnab} - R^2_{mn}) + \frac{1}{120}(5\xi-1)\nabla^2 R\right) + O(\beta^3)\right]. \tag{6}$$

Now we need to compute at 4-loops on the simplified geometry of a maximally symmetric (MS) space, where the Riemann tensor is expressed by

$$R_{mnab} = b(g_{ma}g_{nb} - g_{mb}g_{na}) \tag{7}$$

with

$$b = \frac{R}{d(1-d)}. \tag{8}$$

We find it easier to use Riemann normal coordinates, so that cubic vertices are absent. The expansion of the metric in Riemann normal coordinates around a point (to be called the origin) is easily obtained by the method explained in [5] and reads

$$g_{mn}(x)dx^m dx^n = \left[\delta_{mn} + 2(x_m x_n - \delta_{mn}\vec{x}^2)\left(\frac{b}{6} - \frac{16}{6!}b^2\vec{x}^2 + \frac{8}{7!}b^3(\vec{x}^2)^2 + \cdots\right)\right]dx^m dx^n. \tag{9}$$

As an aside, we note that it is easy to evaluate recursively all the terms in the expansion and sum them up in a compact form

$$g_{mn}(x)dx^m dx^n = \vec{dx}^2 + 2\sum_{n=1}^{\infty}\frac{(-4b)^n(\vec{x}^2)^{n-1}}{(2n+2)!}\times[\vec{x}^2\vec{dx}^2 - (\vec{x}\cdot\vec{dx})^2] \tag{10}$$

$$= \vec{dx}^2 + \frac{1 - 2b\vec{x}^2 - \cos(2\sqrt{b\vec{x}^2})}{2b(\vec{x}^2)^2}\times[\vec{x}^2\vec{dx}^2 - (\vec{x}\cdot\vec{dx})^2]. \tag{11}$$

Using Eq. (9) into Eqs. (3) and (4) produces the required sigma model action. As usual, after getting the free propagators from the quadratic part of the action, one is left to compute perturbatively using Wick contractions

$$\hat{Z}(\beta) = \frac{1}{(2\pi\beta)^{d/2}}\exp\left[-\beta(1-4\xi)\frac{R}{8}\right]\langle \mathrm{e}^{-\hat{S}_{int}}\rangle \tag{12}$$

with

$$\hat{S}_{int} = \frac{1}{\beta}\int_{-1}^{0} dt\left(\frac{b}{6} - \frac{16}{6!}b^2\vec{x}^2 + \frac{8}{7!}b^3(\vec{x}^2)^2 + \cdots\right)\times(x_m x_n - \delta_{mn}\vec{x}^2)(\dot{x}^m\dot{x}^n + a^m a^n + b^m c^n) = S_4 + S_6 + S_8 + \cdots \tag{13}$$

where the subscripts indicate the power of the quantum fields appearing in the given vertex. Up to the order β^3 we only need to compute the following connected diagrams:

$$\langle e^{-S_{int}}\rangle = \exp\left[-\langle S_4\rangle - \langle S_6\rangle - \langle S_8\rangle + \frac{1}{2}\langle S_4^2\rangle_c + \langle S_4 S_6\rangle_c - \frac{1}{6}\langle S_4^3\rangle_c + O(\beta^4)\right]. \tag{14}$$

Using Wick contractions and dimensional regularization we find

$$\langle S_4\rangle = -\frac{\beta}{4!}R \qquad \langle S_6\rangle = -\frac{\beta^2}{5!}\frac{(d+2)}{9d(d-1)}R^2$$

$$\langle S_8\rangle = -\frac{\beta^3}{7!}\frac{(d+2)(d+4)}{15d^2(d-1)^2}R^3$$

$$\langle S_4^2\rangle_c = -\frac{\beta^2}{4!}\frac{1}{9d}R^2 \tag{15}$$

$$\langle S_4 S_6\rangle_c = -\frac{\beta^3}{6!}\frac{4(d+2)}{45d^2(d-1)}R^3$$

$$\langle S_4^3\rangle_c = -\frac{\beta^3}{6!}\frac{2(d^2-4)}{3d^2(d-1)^2}R^3$$

so that the full answer reads

$$\hat{Z}(\beta) = \frac{1}{(2\pi\beta)^{d/2}}\exp\left[\frac{\beta}{4!}(12\xi-2)R - \frac{\beta^2}{6!}\frac{(d-3)}{d(d-1)}R^2 + \frac{\beta^3}{8!}\frac{16(d+2)(d-3)}{9d^2(d-1)^2}R^3 + \cdots\right]. \tag{16}$$

We are now ready to describe concretely our method. In $d=2$ there is only the A type of anomaly. It can most easily be computed at two loops [i.e. order β in Eq. (14)] in the simplified MS geometry. Thus, using the two loop part of Eq. (16) and setting $d=2$ and $\xi=0$ gives

$$\langle T^\mu{}_\mu(x)\rangle_2 = -\frac{1}{24\pi}R \tag{17}$$

which is the correct anomaly for a scalar.

In $d=4$ we have instead $\xi=\frac{1}{6}$, and the 3-loop calculation on the MS geometry [i.e. using terms up to order β^2 in the exponent of Eq. (16)] gives

$$\langle T^\mu{}_\mu(x)\rangle_4\big|_{\text{type A}} = -\frac{1}{(2\pi)^2}\frac{1}{6!}\frac{1}{12}R^2 = -\frac{1}{(2\pi)^2}\frac{1}{6!}\frac{1}{2}E_4 \tag{18}$$

where in the second equality we have used the topological Euler density evaluated on the MS geometry

$$E_4 \equiv R^2_{mnab} - 4R^2_{mn} + R^2 = \frac{1}{6}R^2. \tag{19}$$

Now, the most general expression for $4d$ trace anomalies was obtained through a cohomological analysis in [15] and reads

$$\langle T^\mu{}_\mu(x)\rangle_4 = \frac{1}{(2\pi)^2}\frac{1}{6!}(aE_4 + cC + d\Box R) \tag{20}$$

where C is the square of the Weyl tensor representing the type B anomaly

$$C \equiv R^2_{mnab} - 2R^2_{mn} + \frac{1}{3}R^2 \tag{21}$$

and $\Box R$ is the unique trivial anomaly. Clearly a modification of the trivial anomaly cannot change the coefficient of the R^2 term appearing implicitly in Eq. (20). At the same time this coefficient can be produced by disconnected diagrams only. In fact, no index contraction signaling a connection through a propagator appears between the two factors of the curvature tensors belonging each to a separate vertex. Thus, the coefficient of the R^2 term is directly identified by a lower loop calculation, namely a 2-loop calculation on an arbitrary geometry (order β). Said differently, it is clear from Eq. (6) that for generic manifolds an explicit R^2 dependence can only come from the linear R terms in the exponent. In $d=4$ this term is absent. Since this cannot come from trivial anomalies one can conclude that

$$c = 3(r-a) \tag{22}$$

and hence $c=\frac{3}{2}$. Thus, the complete trace anomaly for a $d=4$ conformal scalar reads

$$\langle T^\mu{}_\mu(x)\rangle_4 = \frac{1}{(2\pi)^2}\frac{1}{6!}\left(-\frac{1}{2}E_4 + \frac{3}{2}C\right). \tag{23}$$

Now, let us address the $d=6$ case. In this spacetime dimensions $\xi=\frac{1}{5}$ and the 4-loop computation (order β^3) on the MS geometry presented above produces

$$\langle T^\mu{}_\mu(x)\rangle_6\big|_{\text{type A}} = -\frac{1}{(2\pi)^3}\frac{1}{8!}\frac{2}{135}R^3 = -\frac{1}{(2\pi)^3}\frac{1}{8!}\frac{5}{72}E_6 \tag{24}$$

where in the last equality we have used the following topological density evaluated on the MS geometry:

$$E_6 \equiv -\epsilon_{m_1n_1m_2n_2m_3n_3}\epsilon^{a_1b_1a_2b_2a_3b_3}R^{m_1n_1}{}_{a_1b_1} \times R^{m_2n_2}{}_{a_2b_2}R^{m_3n_3}{}_{a_3b_3} = \frac{16}{75}R^3. \tag{25}$$

The general expression of $6d$ trace anomalies can be obtained from a cohomological analysis [17,16] and is of the form

$$\langle T^a{}_a\rangle = \frac{1}{(2\pi)^3}\frac{1}{8!}(aE_6 + c_1I_1 + c_2I_2 + c_3I_3 + \text{trivial anomalies}) \tag{26}$$

with the three Weyl invariants given by

$$I_1 = C_{amnb} C^{mijn} C_i{}^{ab}{}_j$$
$$I_2 = C_{ab}{}^{mn} C_{mn}{}^{ij} C_{ij}{}^{ab} \tag{27}$$
$$I_3 = C_{mabc}\left(\nabla^2 \delta^m_n - 4R^m_n + \frac{6}{5} R\,\delta^m_n\right) C^{nabc} + \text{trivial anomalies,}$$

where C_{abmn} is the Weyl tensor in 6 dimensions and the coefficients a, c_1, c_2, c_3 will depend on the particular model considered. Now, it is important to recall that the specific expressions of the trivial anomalies have been recently found in [16]. Consulting those results (see, in particular, Table I) one notices that the coefficients of the three structures $R^3, RR^2_{mn}, RR^2_{mnab}$ can never be modified by adding trivial anomalies. At the same time those structures can be obtained by disconnected diagrams only. Thus, it suffices to use the results at the 3rd-loop order on an arbitrary geometry to fix them.

For the case of the conformal scalar field we use Eq. (6) and obtain

$$\langle T^\mu{}_\mu(x)\rangle_6 = \frac{1}{(2\pi)^3}\frac{1}{8!}(r_1 R^3 + r_2 RR^2_{mn} + r_3 RR^2_{mnab} + \text{other structures}) \tag{28}$$

with $r_1 = 7/225$ and $r_3 = -r_2 = 14/15$. Since these coefficients must correspond to the sum of the type A and B anomalies only, one finds with simple linear algebra

$$\langle T^a{}_a\rangle = \frac{1}{(2\pi)^3}\frac{1}{8!}\left(-\frac{5}{72}E_6 - \frac{28}{3}I_1 + \frac{5}{3}I_2 + 2I_3\right). \tag{29}$$

The general formulas relating the various coefficients are

$$c_1 = \frac{4}{3}(168a + 4r_2 + 9r_3)$$
$$c_2 = \frac{1}{3}(-408a + 300r_1 + 16r_2 - 19r_3) \tag{30}$$
$$c_3 = -\frac{5}{3}(24a - 15r_1 - r_2).$$

Before closing, it is amusing to note that on the three sphere S^3 [i.e. on the group manifold of $SU(2)$] the higher order corrections in Eq. (16) vanish. In fact, the transition amplitude on S^3 is known exactly [18]. From our path integral perspective this amplitude is saturated by the two loop correction given in Eq. (16).

We have discussed a simplified method for computing trace anomalies in $d \leq 6$. Apart from some amusing path integral computations on maximally symmetric spaces, our main results are Eqs. (22) and (30). It would be interesting to extend this method to higher dimensions. The main problem is to analyze how trivial anomalies may affect the structure of type B anomalies in $d \geq 8$. This is at present unknown. On the other hand, it is fortunate that most recent applications, such as the search for a C-theorem in higher dimensions, concern mostly the type A anomalies.

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