Naked singularities in higher dimensional Vaidya space-times

S. G. Ghosh*
Department of Mathematics, Science College, Congress Nagar, Nagpur 440 012, India

Naresh Dadhich†
Inter-University Center for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India
(Received 30 November 2000; published 23 July 2001)

We investigate the end state of the gravitational collapse of a null fluid in higher-dimensional space-times. Both naked singularities and black holes are shown to be developing as the final outcome of the collapse. The naked singularity spectrum in a collapsing Vaidya region (4D) gets covered with the increase in dimensions and hence higher dimensions favor a black hole in comparison to a naked singularity. The cosmic censorship conjecture will be fully respected for a space of infinite dimension.

DOI: 10.1103/PhysRevD.64.047501 PACS number(s): 04.20.Dw, 04.20.Cv, 04.70.Bw

I. INTRODUCTION

Inspired by work in string theory and other field theories, there has been considerable interest in recent times to find solutions of the Einstein equation in dimensions greater than four. It is believed that the underlying space-time in the large energy limit of Planck energy may have higher dimensions than the usual four. At this level, all the basic forces of nature are supposed to unify and, hence, it would be pertinent in this context to consider solutions of the gravitational field equation in higher dimensions. Of course this consideration would be relevant when the usual four-dimensional manifold picture of space-time becomes inapplicable. This would perhaps happen as we approach a singularity, whether in cosmology or in gravitational collapse.

Gravitational collapse continues to occupy the center stage in gravitational research since the formulation of the singularity theorems [1] and cosmic censorship conjecture (CCC) [2]. The singularity theorems revealed that the occurrence of singularities is a generic property of space-times in classical general relativity (GR). However, these theorems say nothing about the detailed features of the singularities such as their visibility to an external observer as well as their strength. On the other hand, CCC states that GR contains a built-in feature that precludes formation of naked singularities (see Ref. [3] for reviews). The CCC remains as one of the most outstanding unresolved questions in GR. However, there are many known examples in literature showing that both naked singularities and black holes can form in gravitational collapse [4]. The central shell focusing singularity can be naked or covered depending upon the choice of initial data. There is a critical branch of solutions, where a transition from naked singularity to black hole occurs. In particular, gravitational collapse of spherical matter in the form of radiation (null fluid) described by the Vaidya metric [5] is well studied for investigating CCC [6–11].

The main aim of the paper is to examine what role the dimensionality of space-time plays in the context of CCC. Interestingly, it turns out that as dimension increases, the window for naked singularity shrinks. That is, gravity seems to get strengthened with an increase in dimensions of space. We shall first generalize previous space-times. The metric for this purpose is already known [12] and we shall call it the higher-dimensional (HD) Vaidya metric. It turns out that higher dimensions seem to favor a black hole. This would be discussed in Sec. II, which will be followed by concluding remarks.

II. SINGULARITIES IN HIGHER-DIMENSIONAL VAIDYA SPACE-TIMES

The metric of collapsing null fluid in the HD case is [12]

$$ds^2=-\left[1-\frac{2m(v)}{(n-1)r^{(n-1)}}\right]dv^2+2dv\,dr+r^2d\Omega_n^2, \quad (1)$$

where $v\in(-\infty,\infty)$ is the null coordinate that represents advanced Eddington time, $r\in[0,\infty)$ is the radial coordinate, and

$$d\Omega^2=d\theta_1^2\sin^2\theta_1(d\theta_2^2+\sin^2\theta_2(d\theta_3^2+\cdots+\sin^2\theta_{n-1}d\theta_n^2))$$

is a metric on the n sphere, and $n=D-2$, where D is the total number of dimensions. The arbitrary function $m(v)$ (which is restricted only by the energy conditions), represents the mass at advanced time v. The energy-momentum tensor can be written in the form

$$T_{ab}=\frac{n}{(n-1)r^n}\dot{m}(v)k_ak_b \quad (2)$$

with the null vector k_a satisfying $k_a=-\delta_a^v$ and $k_ak^a=0$. We have used the units that fix the speed of light and gravitational constant via $8\pi G=c=1$. Clearly, for the weak energy condition to be satisfied, we require that $\dot{m}(v)$ be non-negative, where an overdot represents a derivative with respect to v. Thus, mass function is a non-negative increasing function of v for imploding radiation.

*Author to whom all correspondence should be directed. Email address: sgghosh@hotmail.com
†Email address: nkd@iucaa.ernet.in

TABLE I. Variation of λ_c and X_0 with D.

Dimensions $(D=n+2)$	Critical value $\lambda_c=\frac{1}{n}\left(\frac{n-1}{2n}\right)^{n-1}$	Tangent $\left(X_0=\frac{2n}{n-1}\right)$
4	1/8	4
5	1/27	3
6	27/2048	2.6667
7	256/5000	2.5

A. Self-similar case

The physical situation here is that of a radial influx of null fluid in an initially empty region of the higher-dimensional Minkowskian space-time. The first shell arrives at $r=0$ at time $v=0$ and the final at $v=T$. A central singularity of growing mass is developed at $r=0$. For $v<0$ we have $m(v)=0$, i.e., higher-dimensional Minkowskian space-time, and for $v>T$, $\dot{m}(v)=0$, $m(v)$ is positive definite. The metric for $v=0$ to $v=T$ is HD Vaidya, and for $v>T$ we have the HD Schwarzschild solution. In order to get an analytical solution for our HD case, we choose

$$2m(v)=\begin{cases} 0, & v<0, \\ \lambda(n-1)v^{(n-1)}(\lambda>0), & 0\leqslant v\leqslant T, \\ m_0(>0), & v>T. \end{cases} \tag{3}$$

With this choice of $m(v)$ the space-time is self-similar [13], admitting a homothetic killing vector, which is given by the Lie derivative

$$Ł_{\xi}g_{ab}=\xi_{a;b}+\xi_{b;a}=2g_{ab}. \tag{4}$$

Let $K^a=dx^a/dk$ be the tangent vector to a null geodesic, where k is an affine parameter. Along with null geodesics, we have

$$\xi^a K_a=rK_r+vK_v=C \tag{5}$$

and writing $K^v=P/r$ as in Ref. [8], we obtain

$$K^r=\left[1-\frac{2m(v)}{(n-1)r^{(n-1)}}\right]\frac{P}{2r}. \tag{6}$$

In Eq. (5), because of Eqs. (3), K^v and K^r, yields

$$P=\frac{2C}{2-(v/r)+\lambda(v/r)^n} \tag{7}$$

and so the geodesics are completely determined. Radial null geodesics of the metric (1), by virtue of K^v and K^r, satisfy

$$\frac{dr}{dv}=\frac{1}{2}\left[1-\frac{2m(v)}{(n-1)r^{(n-1)}}\right]. \tag{8}$$

Clearly, the above differential equation has a singularity at $r=0$, $v=0$. The nature (a naked singularity or a black hole) of the collapsing solutions can be characterized by the existence of radial null geodesics coming out of the singularity. The motion near singularity is characterized by the roots of an algebraic equation that we derive next. Equation (8), upon using Eqs. (3), turns out to be

$$\frac{dr}{dv}=\frac{1}{2}[1-\lambda X^{(n-1)}], \tag{9}$$

where $X\equiv v/r$ is the tangent to a possible outgoing geodesic. The central shell focusing singularity is at least locally naked [14] (for brevity we have addressed it as naked throughout this paper), if and only if there exists $X_0\in(0,\infty)$, which satisfies

$$X_0=\lim_{r\to 0\, v\to 0}X=\lim_{r\to 0\, v\to 0}\frac{v}{r}=\lim_{r\to 0\, v\to 0}\frac{dv}{dr}=\frac{2}{1-\lambda X_0^{(n-1)}} \tag{10}$$

or

$$\lambda X_0^n-X_0+2=0. \tag{11}$$

Thus any solution $X=X_0>0$ of Eq. (11) would correspond to a naked singularity of the space-time, i.e., to future directed null geodesics emanating from the singularity $(v=0,r=0)$. The smallest such X_0 corresponds to the earliest ray emanating from the singularity and is called the Cauchy horizon of the space-time. If X_0 is the smallest positive root of Eq. (11), then there are no naked singularities in the region $X<X_0$. Hence in the absence of positive real roots, the central singularity is not naked (censored) because, in that case, there are no outgoing future directed null geodesics from the singularity. Thus, occurrence of positive real roots implies that the strong CCC is violated, though not necessarily the weak CCC. The global nakedness of singularity can then be seen by making a junction onto HD Schwarschild space-time.

We now examine the condition for occurrence of naked singularity. With a straight forward calculation it can be shown that Eq. (11) always admits two real positive roots for $\lambda\leqslant\lambda_c$, where λ_c is the critical value of the parameter λ deciding the existence of a naked singularity or a black hole. The values of λ_c and X_0 are summarized in Tables I and II for the various D.

Thus it follows that singularity will be naked if $\lambda\leqslant\lambda_c$. On the other hand, if the inequality is reversed, $\lambda>\lambda_c$, no naked singularity would form and gravitational collapse would result in a black hole. Note that X_0 is bounded below by the value 2, $X_0\to 2$ as $\lambda\to 0$ or $D\to\infty$.

TABLE II. Values of X_0 for $\lambda<\lambda_c$.

Dimensions (D)	$\lambda<\lambda_c$	Two tangents (X_0)
4	0.11	2.970 86, 6.120 05
5	0.035	2.655 12, 3.497 81
6	0.013	2.547 64, 2.806 51
7	0.0051	2.452 46, 2.551 46

It is interesting to note that λ_c decreases significantly as we increase the value of D. Thus the spectrum of the naked singularity gets covered with the introduction of extra dimensions (see Table I). The two roots in Table II indicate the naked singularity window in the slope of the tangent to geodesics emanating from the singularity, which pinches with an increase in dimension.

The degree of inhomogeneity of collapse is defined as $\mu \equiv 1/\lambda$ (see Ref. [10]). Thus, the inhomogeneity factor increases with D. From the physical point of view, an increase in inhomogeneity should favor naked singularity and hence should increase the spectrum. On the other hand, an increase in dimensions also strengthens gravity, which would go as $r^{(2-D)}$, as the collapse approaches the singularity $r=0$. Among these two trends, the latter seems ultimately to have the upper hand, which results in shrinking of the naked singularity window for initial data.

Strength of the singularity

The strength of singularity, which is the measure of its destructive capacity, is the most important feature. Following Clark and Królak [15], we consider the null geodesics affinely parametrized by k and terminating at shell focusing singularity $r=v=k=0$. Then it would be a strong curvature singularity as defined by Tipler [16] if

$$\lim_{k\to 0} k^2\psi = \lim_{k\to 0} k^2 R_{ab}K^a K^b > 0, \tag{12}$$

where R_{ab} is the Ricci tensor. It is widely believed that a space-time does not admit analytic extension through a singularity, if it is a strong curvature singularity in the above sense.

Equation (12), with the help of Eqs. (2) and (3), and the expression for K^v, can be expressed as

$$\lim_{k\to 0} k^2\psi = \lim_{k\to 0} n\lambda X^{n-2}\left[\frac{kP}{r^2}\right]^2. \tag{13}$$

Our purpose here is to investigate the above condition along future directed null geodesics coming out of the singularity. First, we note that

$$\frac{dX}{dk} = \frac{1}{r}K^v - \frac{X}{r}K^r = (2-X+\lambda X^n)\frac{P}{2r^2} = \frac{C}{r^2}. \tag{14}$$

Using the fact that as singularity is approached, $k\to 0$, $r\to 0$, and $X\to a_+$ [a root of Eq. (11)] and using L'Hôpital's rule, we observe

$$\lim_{k\to 0}\frac{kP}{r^2} = \frac{2}{1+(n-2)\lambda X_0^{(n-1)}}, \tag{15}$$

and hence Eq. (13) gives

$$\lim_{k\to 0} k^2\psi = \frac{4n\lambda X_0^{(n-2)}}{[1+(n-2)\lambda X_0^{(n-1)}]^2} > 0. \tag{16}$$

Thus along the radial null geodesics, a strong curvature condition is satisfied and hence it is a strong curvature singularity.

B. Non-self-similar case

In the previous section we showed the occurrence of strong curvature naked singularities for the self-similar HD Vaidya space-times. Self-similarity is a strong geometric condition on space-time. It may be argued that naked singularity could be an artifact of the self-similarity. It is therefore important to investigate the non-self-similar case as well. It has been shown that in 4D naked singularity does occur for non-self-similar space-times [3,7,8]. In this section we wish to study a similar situation in HD Vaidya space-times.

Here we examine the mass function given by

$$2m(v) = (n-1)\beta^{(n-1)}v^{\alpha(n-1)}[1-2\alpha\beta v^{(\alpha-1)}], \tag{17}$$

$\alpha>1$ and β are constants. This breaks the basic requirement for self-similarity [13]. This class of solutions for 4D space-time have been discussed in Refs. [7,9]. As mentioned above, the null radiation shells start imploding at $v=0$ and the final shell arrives at $v=T$. The weak energy condition would require

$$T^{\alpha-1} < \frac{n-1}{2\beta(n\alpha-1)}. \tag{18}$$

It is clear that $v=0$, $m(v)=0$, i.e., we shall have a HD Minkowskian and for $v=T$, $dm/dv=0$, and $m(v)=m_0(T)>0$ HD Schwarzschild. The radial null geodesics for the mass function (17) can be obtained from Eq. (8) and is given by $r=\beta v^{\alpha}$. This integral curve meets the singularity with a tangent at $r=0$, indicating the occurrence of naked singularity. The singularity is also globally naked as $dr/dv>0$, with v increasing. It is straightforward to see that $(n-1)r^{(n-1)}>2m(T)$ is satisfied along this curve. We finally come to the question of the strength of the singularity. It is seen that singularities are strong curvature only if $m(v)\sim v^{(n-1)}$ in the approach to singularity.

III. CONCLUDING REMARKS

In the absence of a rigorous formulation as well as proof for either version of CCC, considerations of various examples showing occurrence of naked singularities remains the only tool to study this important problem. In this context, one question that could naturally arise is, what happens in higher dimensions that are currently being considered in view of their relevance for string theory and other field theories? Would the examples of naked singularity in 4D go over to HD or not? Our investigation shows that qualitatively the situation remains similar with monotonic shrinkage of a naked singularity window with increase in dimensions. Increase in dimensions favors a black hole.

Our main aim was to study the effect of the increase in dimension of the space on the collapse. As D increases, two opposing effects set in; one is an increase in inhomogeneity and the other a strengthening of the gravitational field. The former would favor naked singularity while the latter a black hole. It turns out that in the final analysis, it is the latter that has an upper hand and leads to the shrinkage of the naked singularity window. We have employed the Vaidya null radiation collapse scenario to study this effect.

The motivation for higher dimensions clearly comes from the string theory in which the effective action involves the dilaton scalar field or antisymmetric tensor field. The dilaton field couples nonminimally to the Ricci curvature. It would, however, be trivial in our case, as the scalar curvature R vanishes for the Vaidya solution. The case for the antisymmetric tensor field would be similar as well. Thus, the results obtained here would also be relevant and valid for effective supergravity theories. It may also be noted that the higher curvature terms in the string theory become important only in the close vicinity of naked singularity when curvatures become divergingly high. That is when singularity is strong curvature singularity. That means our analysis of gravitational collapse would be relevant and meaningful for the effective supergravity theories following from the string theory. The Vaidya metric in the 4D case has been extensively used to study the formation of naked singularity in spherical gravitational collapse. In this paper, we have generalized previous studies to the case of HD Vaidya space-times. We have shown that results of gravitational collapse, obtained in 4D Vaidya space-time, also go over to HD Vaidya space-times and essentially retain their physical behavior, i.e., strong curvature naked singularity.

ACKNOWLEDGMENTS

S.G.G. would like to thank IUCAA, Pune for the hospitality and UGC, Pune for MRP F. No. 23-118/2000 (WRO).

[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University, Cambridge, England, 1973).

[2] R. Penrose, Riv. Nuovo Cimento **1**, 252 (1969); in *General Relativity, an Einstein Centenary Volume*, edited by S. W. Hawking and W. Israel (Cambridge University, Cambridge, England, 1979).

[3] P. S. Joshi, *Global Aspects in Gravitation and Cosmology* (Clarendon, Oxford, 1993); C. J. S. Clarke, Class. Quantum Grav. **10**, 1375 (1993); R. M. Wald, gr-qc/9710068; T. P. Singh, J. Astrophys. Astron. **20**, 221 (1999); S. Jhingan and G. Magli, gr-qc/9903103; P. S. Joshi, gr-qc/0006101.

[4] See, for example, Joshi [3].

[5] P. C. Vaidya, Proc.-Indian Acad. Sci., Sect. A **33**, 264 (1951); Gen. Relativ. Gravit. **31**, 119 (1999).

[6] A. Papapetrou, in *A Random Walk in Relativity and Cosmology*, edited by N. Dadhich, J. K. Rao, J. V. Narlikar, and C. V. Vishveshwara (Wiley, New York, 1985).

[7] K. Rajagopal and K. Lake, Phys. Rev. D **35**, 1531 (1987); K. Lake, *ibid.* **43**, 1416 (1991).

[8] I. H. Dwivedi and P. S. Joshi, Class. Quantum Grav. **6**, 1599 (1989); **8**, 1339 (1991).

[9] P. S. Joshi and I. H. Dwivedi, J. Math. Phys. **32**, 2167 (1991); Gen. Relativ. Gravit. **24**, 129 (1992).

[10] J. P. S. Lemos, Phys. Rev. Lett. **68**, 1447 (1992); Phys. Rev. D **59**, 044020 (1999).

[11] J. Jhingan, N. Dadhich, and P. S. Joshi, Phys. Rev. D **63**, 044010 (2001).

[12] B. R. Iyer and C. V. Vishveshwara, Pramana **32**, 749 (1989); L. K. Patel and N. Dadhich, gr-qc/9909068.

[13] A general spherically symmetric space-time is self-similar if there exists a radial coordinate r and orthogonal time coordinate t for which $g_{tt}(ct,cr)=g_{tt}(t,t)$ and $g_{rr}(ct,cr)=g_{rr}(t,r)$ for every $c>0$.

[14] Globally or locally naked singularities here means violating the weak or strong form of the CCC. Discussion of the two forms of CCC can be found in Ref. [2] (above).

[15] C. J. S. Clarke and A. Królak, J. Geom. Phys. **2**, 127 (1986).

[16] F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980).