Exact solution to the Seiberg-Witten equation of noncommutative gauge theory

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(Received 13 April 2001; published 26 July 2001)

We derive an exact expression for the Seiberg-Witten map of noncommutative gauge theory. It is found by studying the coupling of the gauge field to the Ramond-Ramond potentials in string theory. Our result also proves the earlier conjecture by Liu.

I. INTRODUCTION

A noncommutative gauge theory can be realized by considering branes in string theory with a constant Neveu-Schwarz–Neveu-Schwarz (NS-NS) two-form field [1]. In [2], it was shown that there are two equivalent descriptions of the theory: one in terms of ordinary gauge fields A_i on a commutative space and another in terms of noncommutative gauge fields $\hat{A}_i$ on a noncommutative space whose coordinates obey the commutation relation

$$[x^i, x^j] = -i\theta^{ij}. \tag{1.1}$$

The map between A_i and $\hat{A}_i$, called the Seiberg-Witten map, is characterized[1] by the differential equation with respect to θ,

$$\delta\hat{A}_i(\theta) = -\frac{1}{4}\delta\theta^{jk}[\hat{A}_j*(\partial_k\hat{A}_i+\hat{F}_{ki})+(\partial_k\hat{A}_i+\hat{F}_{ki})*\hat{A}_j], \tag{1.2}$$

with the initial condition

$$\hat{A}_i(\theta=0) = A_i. \tag{1.3}$$

Here $*$ is the standard star product,

$$f(x)*g(x) = \lim_{y\to x}\exp\left[-i\theta^{ij}\frac{\partial^2}{\partial x^i\partial y^j}\right]f(x)g(y), \tag{1.4}$$

and the field strength $\hat{F}_{ij}$ is defined as

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i + i\hat{A}_i*\hat{A}_j - i\hat{A}_j*\hat{A}_i. \tag{1.5}$$

The differential equation (1.2) is known as the Seiberg-Witten equation.

There have been several attempts to solve the Seiberg-Witten equation. In [4], it was pointed out that the map can

be expressed in terms of a functional integral which quantizes the Poisson structure $\tilde{\theta}^{ij}$ related to θ^{ij} by

$$(\tilde{\theta}^{-1})_{ij} = (\theta^{-1})_{ij} + \partial_i A_j - \partial_j A_i. \tag{1.6}$$

By perturbatively evaluating the functional integral, one can obtain the Seiberg-Witten map order by order in a formal power series expansion in θ. In [5], the Seiberg-Witten map is expressed in terms of the Kontsevich map [6] which relates the star product associated with θ^{ij} to the one associated with $\tilde{\theta}^{ij}$ given by Eq. (1.6).[2] There is a procedure to compute the Kontsevich map as a formal power series expansion. The two approaches are related to each other since the Kontsevich map can be expressed in terms of a functional integral [7] which is similar to the one used in [4].

One can also try to solve Eq. (1.2) directly order by order in a power series expansion. The structure of the power series is examined in [8,9]. It was shown that it involves the so-called generalized star products, which also appear in the expansion of the open Wilson line,

$$\int dx*\left[e^{ikx}P\exp\left(i\int_0^1\hat{A}_i(x+l\tau)l^i d\tau\right)\right], \tag{1.7}$$

where

$$l^i = k_j\theta^{ji}, \tag{1.8}$$

and $*[\cdots]$ means that we take the standard star product (1.4) in the expansion of the expression in $[\cdots]$ in powers of $\hat{A}_i$. This suggests that the Seiberg-Witten map can be expressed in terms of the open Wilson line. Based on this observation and the earlier papers [4,5] mentioned in the above paragraph, it was conjectured in [10] that the (inverse of) Seiberg-Witten map is given in the momentum space by

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[1]As pointed out in [3], there is a possibility to modify the equation by performing field redefinition and gauge transformation.

[2]The method developed in [5] is also applicable to the case when θ^{ij} is not constant.

$$F_{ij}(k) \equiv \int dx\, e^{ikx}(\partial_i A_j(x) - \partial_j A_i(x))$$

$$= \int dx * \left[e^{ikx} \sqrt{\det(1 - \hat{f}\theta)} \left(\frac{1}{1 - \hat{f}\theta} \hat{f} \right)_{ij} \right.$$

$$\left. \times P \exp\left(i \int_0^1 \hat{A}_i(x + l\tau) l^i d\tau \right) \right], \qquad (1.9)$$

where

$$\hat{f}_{ij} = \int_0^1 \hat{F}_{ij}(x + l\tau) d\tau. \qquad (1.10)$$

Here we are using the same symbol x to denote both the commutative (in the first line) and the noncommutative coordinates (in the second line). The path-ordering with respect to τ is implicit in this expression and throughout the rest of the paper. It is clear that Eq. (1.9) obeys the initial condition (1.3). To the quadratic order in the power series expansion in θ, it was also checked in [10] that Eq. (1.9) satisfies the Seiberg-Witten equation.

In this paper, we derive an exact expression for the Seiberg-Witten map. We will discuss the case where the gauge group is $U(1)$. Solving the Seiberg-Witten equation is equivalent to finding a two-form $F_{ij} = F_{ij}(\hat{A}_i; \theta)$ which

(a) is gauge invariant,

$$F_{ij}(\hat{A}_i + \partial_i \hat{\lambda} + i\hat{A}_i * \hat{\lambda} - i\hat{\lambda} * \hat{A}_i; \theta) = F_{ij}(\hat{A}_i; \theta), \quad (1.11)$$

(b) obeys the Bianchi identity for the ordinary gauge theory:

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \qquad (1.12)$$

(c) satisfies the initial condition,

$$F_{ij}(\hat{A}_i; \theta = 0) = \partial_i \hat{A}_j - \partial_j \hat{A}_i. \qquad (1.13)$$

Modulo freedom of field redefinition and gauge transformation, the conditions (a) and (b) are equivalent to the Seiberg-Witten equation since the Bianchi identity (b) means that F_{ij} can be expressed as $F_{ij} = \partial_i A_j - \partial_j A_i$ for some A_i and the gauge invariance (a) guarantees that, under the noncommutative gauge transformation,

$$\hat{A}_i \rightarrow \hat{A}_i + \partial_i \hat{\lambda} + i\hat{A}_i * \hat{\lambda} - i\hat{\lambda} * \hat{A}_i, \qquad (1.14)$$

A_i transforms as an ordinary gauge field,

$$A_i \rightarrow A_i + \partial_i \lambda, \qquad (1.15)$$

for some λ which depends on $\hat{\lambda}$ and $\hat{A}_i$. These are exactly the conditions from which the Seiberg-Witten equation was derived [2]. The importance of the condition (b) in this context was stressed in [9].

If we realize the noncommutative gauge theory on p-branes in string theory, the two-form F_{ij} obeying the three conditions (a)–(c) can be found by identifying the current coupled to the Ramond-Ramond potential $C^{(p-1)}$. The gauge invariance (a) is manifest if we use the point-splitting regularization on the string worldsheet, and the Bianchi identity (b) is the consequence of the gauge invariance of the Ramond-Ramond potential,

$$C^{(p-1)} \rightarrow C^{(p-1)} + d\epsilon, \qquad (1.16)$$

where ϵ is an arbitrary $(p-2)$ form in the bulk. From the resulting expression for F_{ij}, it is straightforward to verify that the initial condition (c) is satisfied. The fact that the initial condition is satisfied is presumably related to the topological nature of the Ramond-Ramond coupling and the lack of α' corrections to it.[3]

When the noncommutative space is $2n$ dimensional, namely, when the rank of θ is $2n$, the Seiberg-Witten map[4] we find from the Ramond-Ramond current is

$$F_{ij}(k) + \theta_{ij}^{-1} \delta(k) = \frac{1}{\text{Pf}(\theta)} \int dx * \left[e^{ikx} (\theta - \theta \hat{f}\theta)_{ij}^{n-1} \right.$$

$$\left. \times P \exp\left(i \int_0^1 \hat{A}_i(x + l\tau) l^i d\tau \right) \right].$$
$$\qquad (1.17)$$

Here the integral $\int dx$ is over the space coordinates on the brane and is normalized as

$$\int dx = \int \frac{dx^1 \cdots dx^{2n}}{(2\pi)^{2n}}, \qquad (1.18)$$

the two-form $(\theta - \theta \hat{f}\theta)_{ij}^{n-1}$ in the integrand is defined as

$$(\theta - \theta \hat{f}\theta)_{ij}^{n-1} \equiv -\frac{1}{2^{n-1}(n-1)!} \epsilon_{ij i_1 i_2 \cdots i_{2n-2}}$$

$$\times \int_0^1 d\tau_1 (\theta - \theta \hat{F}(x + l\tau_1)\theta)^{i_1 i_2} \cdots$$

$$\times \int_0^1 d\tau_{n-1} (\theta - \theta \hat{F}(x + l\tau_{n-1})\theta)^{i_{2n-3} i_{2n-2}},$$
$$\qquad (1.19)$$

and the Pfaffian is normalized as

$$\text{Pf}(\theta) = \frac{1}{2^n n!} \epsilon_{i_1 \cdots i_{2n}} \theta^{i_1 i_2} \cdots \theta^{i_{2n-1} i_{2n}}. \qquad (1.20)$$

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[3]This result is in contrast with the case of the energy-momentum tensor studied in our earlier paper [11]. There it was shown that the energy-momentum tensor of the noncommutative theory derived from the coupling to the bulk graviton does not reduce to the one in the ordinary gauge theory in the limit $\theta \rightarrow 0$.

[4]It is known that a solution to the Seiberg-Witten equation is not unique. For example, there is the field redefinition ambiguity we mentioned in the above. It would be interesting to find out if this solution, which naturally comes from the string theory computation, has a special status among all possible solutions.

Note that the right-hand side of Eq. (1.17) depends only on $\hat{A}_i(x)$, θ^{ij} and k_i. In particular, the combination $(\theta - \theta \hat{f} \theta)_{ij}^{n-1}/\text{Pf}(\theta)$ does not depend on the normalization of the ϵ-symbol.

In order to make the logical structure of this paper transparent, we will first prove that Eq. (1.17) satisfies the three conditions (a)–(c) independently of the string theory origin of the formula. In particular, the proof holds for any n even though the string theory computation only works for $n \leq 4$. After the proof is completed, we will explain how the solution is found from the string theory computation of the Ramond-Ramond coupling.

It turns out that the map (1.17) can be re-expressed in the form (1.9). Thus we have also proven the conjecture in [10]. Since we now have the exact expression for the Seiberg-Witten map, it may also be possible to find an expression for the Kontsevich map in the case of Eq. (1.6).

This paper is organized as follows. In Sec. II, we prove that Eq. (1.17) satisfies the three conditions (a)–(c) and therefore gives the Seiberg-Witten map. We also show that it is equivalent to Eq. (1.9) conjectured in [10]. In Sec. III, we discuss its relation to the coupling of the noncommutative gauge field to the Ramond-Ramond potentials in string theory.[5] In Sec. IV, we discuss applications and extensions of our result.

After the first version of this paper appeared, we received two papers [12,13], whose contents overlap with Sec. III of this paper.

II. PROOF

In this section, we will prove that Eq. (1.17) obeys the three conditions (a)–(c) for the Seiberg-Witten map. The gauge invariance (a) is manifest because of the use of the open Wilson line [14–16]. We will show that it also satisfies the Bianchi identity (b) and the initial condition (c).

A. Bianchi identity

In order to prove the Bianchi identity, it is useful to introduce the following currents of rank $2s$:

$$J^{i_1 \cdots i_{2s}}(k) = \frac{1}{\text{Pf}(\theta)} \int dx$$

$$* \left[e^{ikx} \int_0^1 d\tau_1 (\theta - \theta \hat{F}(x + l\tau_1)\theta)^{[i_1, i_2} \cdots \right.$$

$$\times \int_0^1 d\tau_n (\theta - \theta \hat{F}(x + l\tau_n)\theta)^{i_{2s-1}, i_{2s}]}$$

$$\left. \times P \exp\left(i \int_0^1 \hat{A}_i(x + l\tau) l^i d\tau \right) \right]. \quad (2.1)$$

Here the indices $i_1, \ldots, i_{2s}$ are totally antisymmetrized with a factor of $1/(2s)!$ for each term. For noncommutative gauge theory in $2n$ dimensions, the Seiberg-Witten map (1.17) can be written as

$$F_{ij}(k) + \theta_{ij}^{-1}\delta(k) = -\frac{1}{2^{n-1}(n-1)!} \epsilon_{iji_1 \cdots i_{2n-2}}$$

$$\times J^{i_1 \cdots i_{2n-2}}(k). \quad (2.2)$$

Therefore, to prove that the left-hand side of Eq. (2.2) obeys the Bianchi identity, it is sufficient to show that these currents are conserved,

$$k_{i_1} J^{i_1 \cdots i_{2s}}(k) = 0. \quad (2.3)$$

The conservation law can be proven by performing integration by parts in the τ-integrals in Eq. (2.1). Before describing a proof for general s, it would be instructive to show how it works for $s = 1$ and $s = 2$. When $s = 1$,

$$k_i \int dx * \left[e^{ikx} \int_0^1 d\tau' (\theta - \theta \hat{F}(x + l\tau')\theta)^{ij} P \exp\left(i \int_0^1 \hat{A}_i(x + l\tau) l^i d\tau \right) \right]$$

$$= \int dx * \left[e^{ikx} \int_0^1 d\tau' (l^j - \theta^{ji} \hat{F}_{ii'} l^{i'}) P \exp\left(i \int_0^1 \hat{A}_i(x + l\tau) l^i d\tau \right) \right]$$

$$= i\theta^{jj'} \int dx \partial_{j'} * \left[e^{ikx} P \exp\left(i \int_0^1 \hat{A}_i(x + l\tau) l^i d\tau \right) \right]$$

$$= 0. \quad (2.4)$$

Here we decomposed the factor in the second line as follows:

$$l^j - \theta^{ji} \hat{F}_{ii'} l^{i'} = i\theta^{jj'}(ik_{j'} + i\partial_{j'}\hat{A}_{i'} l^{i'}) + \theta^{jj'} l^{i'} D_{i'}\hat{A}_{j'}, \quad (2.5)$$

and used the identity that

$$\int dx * \left[e^{ikx} \int_0^1 d\tau' \, l^{i'} D_{i'} \hat{A}_{j'}(x+l\tau') P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau) l^i d\tau \right) \right] = 0, \tag{2.6}$$

which was shown in (B.4) in [11].

To prove the current conservation for $s=2$, we use the following identity:

$$\int dx * \left[e^{ikx} \int_0^1 d\tau_1 l^i (\hat{F}_{ij}(x+l\tau_1) - \theta_{ij}^{-1}) \int_0^1 d\tau_2 \mathcal{O}(x+l\tau_2) \, P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau) l^i d\tau \right) \right]$$

$$= -i \int dx * \left[e^{ikx} \int_0^1 d\tau' D_j \mathcal{O}(x+l\tau') P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau) l^i d\tau \right) \right]. \tag{2.7}$$

The conservation law for $s=2$,

$$k_i \int dx * \left[e^{ikx} \int_0^1 d\tau_1 (\theta - \theta\hat{F}(x+l\tau_1)\theta)^{i[j} \int_0^1 d\tau_2 (\theta - \theta\hat{F}(x+l\tau_2)\theta)^{k,l]} P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau)l^i d\tau \right) \right] = 0, \tag{2.8}$$

follows from this by setting $\mathcal{O} = \hat{F}_{kl} - \theta_{kl}^{-1}$ and using the Bianchi identity

$$D_j \hat{F}_{kl} + D_k \hat{F}_{lj} + D_l \hat{F}_{jk} = 0 \tag{2.9}$$

for $\hat{F}$. What remains is to show Eq. (2.7). This follows from the following two identities. The first one is

$$\int dx * \left[e^{ikx} \int_0^1 d\tau_1 l^i D_i \hat{A}_j(x+l\tau_1) \int_0^1 d\tau_2 \, \mathcal{O}(x+l\tau_2) P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau)l^i d\tau \right) \right]$$

$$= \int dx * \left[e^{ikx} \int_0^1 d\tau' [\hat{A}_j, \mathcal{O}](x+l\tau') P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau)l^i d\tau \right) \right], \tag{2.10}$$

which can be derived from (B.5) in [11]. The second one is

$$\int dx * \left[e^{ikx} \int_0^1 d\tau_1 \{ -k_j - l^i \partial_j \hat{A}_i(x+l\tau_1) \} \int_0^1 d\tau_2 \mathcal{O}(x+l\tau_2) P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau)l^i d\tau \right) \right]$$

$$= -i \int dx * \left[e^{ikx} \int_0^1 d\tau' \partial_j \mathcal{O}(x+l\tau') P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau)l^i d\tau \right) \right], \tag{2.11}$$

where we performed integration by parts on $\hat{A}_i$. By combining Eqs. (2.10) and (2.11) using

$$l^i D_i \hat{A}_j - l^i \partial_j \hat{A}_i - k_j = l^i (\hat{F}_{ij} - \theta_{ij}^{-1}), \tag{2.12}$$

we obtain the identity (2.7).

To give a proof of the conservation law (2.3) for general s, it is most convenient to use the matrix theory language [17–19]. The noncommutative gauge theory with a commutative time coordinate t and $2n$ noncommutative space coordinates x^i ($i = 1, \ldots, 2n$) can be constructed from matrix theory by setting the matrix variables X^i in the form

$$X^i = x^i + \theta^{ij} \hat{A}_j(x), \tag{2.13}$$

where x^i obeys the commutation relation,

$$[x^i, x^j] = -i\theta^{ij}. \tag{2.14}$$

Formulas in noncommutative gauge theory can then be expressed in the matrix theory language according to the map [20]

$$[X^i, X^j] = -i(\theta^{ij} - \theta^{ii'} \hat{F}_{i'j'} \theta^{j'j}), \tag{2.15}$$

$$e^{ikX} = * \left[e^{ikx} P\exp\left(i \int_0^1 \hat{A}_i(x+l\tau) l^i \right) \right],$$

$$\text{tr}(\cdots)=\frac{1}{\text{Pf}(\theta)}\int dx*[\cdots],$$

with $l^i=k_j\theta^{ji}$. For a more precise description of the map between gauge invariant operators of matrix theory and the noncommutative gauge theory, see [21].[6]

Following the rule (2.15), we can express the currents (2.1) in noncommutative gauge theory using the matrix theory variables as

$$J(k)=\text{tr}(e^{ikX}),$$

$$J^{ij}(k)=i\,\text{tr}([X^i,X^j]e^{ikX}),$$

$$J^{ijlm}(k)=\frac{i^2}{3}\int_0^1 d\tau\,\text{tr}([X^i,X^j]e^{i\tau kX}[X^l,X^m]e^{i(1-\tau)kX})+\frac{i^2}{3}\int_0^1 d\tau\,\text{tr}([X^i,X^l]e^{i\tau kX}[X^m,X^j]e^{i(1-\tau)kX})$$

$$+\frac{i^2}{3}\int_0^1 d\tau\,\text{tr}([X^i,X^m]e^{i\tau kX}[X^j,X^l]e^{i(1-\tau)kX}),$$

$$\vdots$$

$$J^{i_1\cdots i_{2n}}(k)=\frac{i^n(n-1)!}{(2n)!}\int_0^1 d\tau_1\int_{\tau_1}^1 d\tau_2\cdots\int_{\tau_{n-2}}^1 d\tau_{n-1}\text{tr}([X^{i_1},X^{i_2}]e^{i\tau_1 kX}[X^{i_3},X^{i_4}]$$

$$\times e^{i(\tau_2-\tau_1)kX}\cdots[X^{i_{2n-1}},X^{i_{2n}}]e^{i(1-\tau_{n-1})kX})+([(2n)!-1]\ \text{more terms to antisymmetrize the indices}).$$

(2.16)

This facilitates our proof of the conservation law:

$$k_{i_1}J^{i_1\cdots i_{2s}}(k)=0.$$

(2.17)

In order to prove the conservation law in the matrix theory language, we will make use of the cyclicity of the trace, $\text{tr}(AB)=\text{tr}(BA)$. A care is needed here since this does not necessarily hold for infinite dimensional matrices. For example, in the background $X^i=x^i$ which gives rise to a noncommutative gauge theory from matrix theory, we have

$$[x^i,x^j]=-i\theta^{ij}.$$

(2.18)

Therefore $\text{tr}(x^ix^j)=\text{tr}(x^jx^i)$ is obviously untrue here. Fortunately, the conservation law can be proven under the weaker assumption about the cyclicity of the trace as

$$\text{tr}([X^i,X^j]\mathcal{O})=\text{tr}(\mathcal{O}[X^i,X^j]),\quad \text{tr}(e^{ikX}\mathcal{O})=\text{tr}(\mathcal{O}e^{ikX}),$$

(2.19)

for any $\mathcal{O}$ generated by any number of commutators $[X^i,X^j]$ and exponentials $e^{ik'X}$ with a possibility of a single insertion of X^i. This holds for X^i considered in this paper. $[X^i=x^i+\theta^{ij}\hat{A}_j(x)$ and we are allowed to perform integration by parts on $\hat{A}_j.]$

As a warmup, let us repeat the proof for $s=1$ and $s=2$ using the matrix theory language. For $s=1$, we can show the conservation for matrices X^i satisfying Eq. (2.19) as follows:

$$k_iJ^{ij}(k)=\text{tr}([ikX,X^j]e^{ikX})=\int_0^1 d\tau\,\text{tr}(e^{i\tau kX}[ikX,X^j]e^{i(1-\tau)kX})=\text{tr}([e^{ikX},X^j])=0.$$

(2.20)

For $s=2$, we need to perform the integration by parts in τ as

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[6]See also the formula (A6) given in the Appendix.

$$k_i J^{ijlm}(k) = \frac{i}{3} \int_0^1 d\tau \, \mathrm{tr}(e^{i(1-\tau)kX}[ikX, X^j] e^{i\tau kX}[X^l, X^m]) + (2 \text{ more terms})$$

$$= -\frac{i}{3} \int_0^1 d\tau \frac{d}{d\tau} \mathrm{tr}(e^{i(1-\tau)kX} X^j e^{i\tau kX}[X^l, X^m]) + (2 \text{ more terms})$$

$$= -\frac{i}{3} \mathrm{tr}(X^j e^{ikX}[X^l, X^m] - e^{ikX} X^j [X^l, X^m]) + (2 \text{ more terms})$$

$$= -\frac{i}{3} \mathrm{tr}([[X^l, X^m], X^j] e^{ikX}) + (2 \text{ more terms})$$

$$= 0. \tag{2.21}$$

To go from the fourth to the fifth line, we used the cyclicity of the trace. The last line follows from the Jacobi identity.

One can easily see that each step in Eqs. (2.20) and (2.21) has a corresponding step in the proof (2.4)–(2.12) using the gauge theory variables. If one wishes, one can also re-express the proof for arbitrary s in the following using the gauge theory variables, although the use of matrix theory variables substantially simplifies the proof.

Now we are ready to prove the conservation law for arbitrary s.[7] In the original form of the current in (2.16), the indices $i_1, i_2, \ldots, i_{2s}$ are totally antisymmetrized. However, we can always bring one of them i_1 to the first using the cyclic symmetry of the τ-integral form, while the rest of the indices $i_2, i_3, \ldots i_{2n}$ are still totally antisymmetrized. One of the terms appeared in Eq. (2.17) is then

$$-i \int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{n-2}}^1 d\tau_{n-1} \mathrm{tr}\, e^{i(1-\tau_{n-1})kX}[ikX, X^{i_2}] e^{i\tau_1 kX}[X^{i_3}, X^{i_4}] e^{i(\tau_2-\tau_1)kX}[X^{i_5}, X^{i_6}] e^{i(\tau_3-\tau_2)kX} \cdots [X^{i_{2n-1}}, X^{i_{2n}}]$$

$$= i \int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{n-2}}^1 d\tau_{n-1} \left(\frac{d}{d\tau_1} + \frac{d}{d\tau_2} + \cdots + \frac{d}{d\tau_{n-1}} \right) \mathrm{tr}\, e^{i(1-\tau_{n-1})kX} X^{i_2} e^{i\tau_1 kX}$$

$$\times [X^{i_3}, X^{i_4}] e^{i(\tau_2-\tau_1)kX}[X^{i_5}, X^{i_6}] e^{i(\tau_3-\tau_2)kX} \cdots [X^{i_{2n-1}}, X^{i_{2n}}]$$

$$= -i \int_0^1 d\tau_2 \int_{\tau_2}^1 d\tau_3 \cdots \int_{\tau_{n-2}}^1 d\tau_{n-1} \mathrm{tr}\, e^{i(1-\tau_{n-1})kX} X^{i_2}[X^{i_3}, X^{i_4}] e^{i\tau_2 kX}[X^{i_5}, X^{i_6}] e^{i(\tau_3-\tau_2)kX} \cdots [X^{i_{2n-1}}, X^{i_{2n}}]$$

$$+ i \int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{n-3}}^1 d\tau_{n-2} \mathrm{tr}\, X^{i_2} e^{i\tau_1 kX}[X^{i_3}, X^{i_4}] e^{i(\tau_2-\tau_1)kX} \cdots e^{(1-\tau_{n-2})}[X^{i_{2n-1}}, X^{i_{2n}}]. \tag{2.22}$$

In the last step, we used the formula derived using the integration by parts:

$$\int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{n-2}}^1 d\tau_{n-1} \left(\frac{d}{d\tau_1} + \frac{d}{d\tau_2} + \cdots + \frac{d}{d\tau_{n-1}} \right) f(\tau_1, \tau_2, \ldots, \tau_{n-1})$$

$$= -\int_0^1 d\tau_2 \int_{\tau_2}^1 d\tau_3 \cdots \int_{\tau_{n-2}}^1 d\tau_{n-1} f(0, \tau_2, \ldots, \tau_{n-1}) + \int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{n-3}}^1 d\tau_{n-2} f(\tau_1, \tau_2, \ldots, \tau_{n-2}, 1), \tag{2.23}$$

where f is an arbitrary function of $\tau_1, \tau_2, \ldots, \tau_{n-1}$. Using the antisymmetry in the indices, $i_2, i_3, \ldots, i_{2n}$, we can rewrite Eq. (2.22) as follows:

$$-i \int_0^1 d\tau_2 \int_{\tau_2}^1 d\tau_3 \cdots \int_{\tau_{n-2}}^1 d\tau_{n-1} \mathrm{tr}[X^{i_2}, [X^{i_3}, X^{i_4}]] e^{i\tau_2 kX}[X^{i_5}, X^{i_6}] e^{i(\tau_3-\tau_2)kX} \cdots [X^{i_{2n-1}}, X^{i_{2n}}] e^{i(1-\tau_{n-1})kX}. \tag{2.24}$$

This vanishes because of the Jacobi identity.

We have proven the conservation of the current (2.1). Thus Eq. (2.2) satisfies the Bianchi identity.

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[7]The following proof also resolves the question raised in [22] regarding the gauge invariance of the Ramond-Ramond couplings and extends the earlier work [23] on conservation of currents in matrix theory.

B. Initial condition

Since the conditions (a) and (b) are equivalent to the Seiberg-Witten equation (1.2), we now have a solution to the equation, modulo field redefinition and gauge transformation. What remains to verify is the initial condition (c). Although we can check this directly by expanding the map (1.17) in powers of θ, it is more useful to rewrite Eq. (1.17) in such a way that the initial condition is manifest. In this process, we find that Eq. (1.17) is equivalent to Eq. (1.9), therefore proving the conjecture in [10].

To see the relation between Eqs. (1.17) and (1.9), let us first show the identity

$$\frac{1}{\mathrm{Pf}(\theta)}(\theta-\theta\hat{f}\theta)^{n-1}_{ij}-\sqrt{\det(1-\hat{f}\theta)}\left(\frac{1}{1-\hat{f}\theta}\hat{f}\right)_{ij}=\frac{1}{\mathrm{Pf}(\theta)}\theta^{-1}_{ij}\mathrm{Pf}(\theta-\theta\hat{f}\theta). \tag{2.25}$$

This can be shown by writing the two terms on the left-hand side of the equation as

$$\frac{1}{\mathrm{Pf}(\theta)}(\theta-\theta\hat{f}\theta)^{n-1}_{ij}=\frac{1}{\mathrm{Pf}(\theta)}\mathrm{Pf}(\theta-\theta\hat{f}\theta)\left(\frac{1}{\theta-\theta\hat{f}\theta}\right)_{ij}, \tag{2.26}$$

and[8]

$$\sqrt{\det(1-\hat{f}\theta)}\left(\frac{1}{1-\hat{f}\theta}\hat{f}\right)_{ij}=\frac{1}{\mathrm{Pf}(\theta)}\mathrm{Pf}(\theta-\theta\hat{f}\theta)\left(\frac{1}{\theta-\theta\hat{f}\theta}\theta\hat{f}\right)_{ij}, \tag{2.27}$$

and taking the difference of the two. Therefore we find

$$\frac{1}{\mathrm{Pf}(\theta)}\int dx*\left[e^{ikx}(\theta-\theta\hat{f}\theta)^{n-1}_{ij}P\exp\left(i\int^1_0\hat{A}_i(x+l\tau)l^id\tau\right)\right]$$

$$=\int dx*\left[e^{ikx}\sqrt{\det(1-\hat{f}\theta)}\left(\frac{1}{1-\hat{f}\theta}\hat{f}\right)_{ij}P\exp\left(i\int^1_0\hat{A}_i(x+l\tau)l^id\tau\right)\right]$$

$$+\theta^{-1}_{ij}\frac{1}{\mathrm{Pf}(\theta)}\int dx*\left[e^{ikx}\mathrm{Pf}(\theta-\theta\hat{f}\theta)P\exp\left(i\int^1_0\hat{A}_i(x+l\tau)l^id\tau\right)\right]. \tag{2.28}$$

Next we show

$$\frac{1}{\mathrm{Pf}(\theta)}\int dx*\left[e^{ikx}\mathrm{Pf}(\theta-\theta\hat{f}\theta)P\exp\left(i\int^1_0\hat{A}_i(x+l\tau)l^id\tau\right)\right]=\delta(k). \tag{2.29}$$

Note that the left-hand side is the Ramond-Ramond current of the maximum rank $2n$,

$$\frac{1}{\mathrm{Pf}(\theta)}\int dx*\left[e^{ikx}\mathrm{Pf}(\theta-\theta\hat{f}\theta)P\exp\left(i\int^1_0\hat{A}_i(x+l\tau)l^id\tau\right)\right]=\frac{1}{2^nn!}\epsilon_{i_1\cdots i_{2n}}J^{i_1\cdots i_{2n}}(k). \tag{2.30}$$

To prove Eq. (2.29), it is simplest to use the matrix theory representation (2.16). We will show the current $J^{i_1\cdots i_{2n}}(k)$ of the maximum rank[9] is invariant under an arbitrary infinitesimal variation of the matrix variable near the background $X^i=x^i$ with $[x^i,x^j]=-i\theta^{ij}$, namely, it is topological. Once it is shown, we can evaluate the left-hand side of Eq. (2.29) at the background $X^i=x^i$ which corresponds to $\hat{A}_i(x)=0$ and find

$$\frac{1}{\mathrm{Pf}(\theta)}\int dx*\left[e^{ikx}\mathrm{Pf}(\theta-\theta\hat{f}\theta)P\exp\left(i\int^1_0\hat{A}_i(x+l\tau)l^id\tau\right)\right]=\frac{1}{\mathrm{Pf}(\theta)}\int dx*[e^{ikx}\mathrm{Pf}(\theta)]=\delta(k). \tag{2.31}$$

Now let us prove that the right-hand side of Eq. (2.30) is indeed topological. It is instructive to consider the simplest case of $n=1$ first,

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[8]We define the sign of the square root, $\sqrt{\det(1-\hat{f}\theta)}$, so that it agrees with that of $\mathrm{Pf}(\theta-\theta\hat{f}\theta)/\mathrm{Pf}(\theta)$.

[9]In this paper, we are setting all the scalar fields to be zero. Thus $J^{i_1\cdots i_{2n}}(k)$ is the current of the maximum rank for the noncommutative gauge theory in $(2n+1)$ dimensions.

$$\delta \,\mathrm{tr}(\epsilon_{ij}[X^i,X^j]e^{ikX}) = \epsilon_{ij}\mathrm{tr}\left(2[\,\delta X^i,X^j]e^{ikX} + \int_0^1 d\tau[X^i,X^j]e^{i\tau kX}ik_m\delta X^m e^{i(1-\tau)kX}\right)$$

$$= (2\epsilon_{ij}ik_m + ik_i\epsilon_{jm})\mathrm{tr}\left(\delta X^i\int_0^1 d\tau e^{i\tau kX}[X^j,X^m]e^{i(1-\tau)kX}\right)$$

$$= 0. \tag{2.32}$$

To go from the second to the third line, we used the cyclicity of the trace. In the last line, we used the identity in two dimensions,

$$2\epsilon_{ij}\epsilon^{jl} + \epsilon_{jm}\epsilon^{jm}\delta_i^l = 0. \tag{2.33}$$

In general, we have

$$\epsilon_{i_1\cdots i_{2n}}\int_0^1 d\tau_1\int_{\tau_1}^1 d\tau_2\cdots\int_{\tau_{n-2}}^1 d\tau_{n-1}\,\delta\,\mathrm{tr}([X^{i_1},X^{i_2}]e^{i\tau_1 kX}\cdots[X^{i_{2n-1}},X^{i_{2n}}]e^{i(1-\tau_{n-1})kX})$$

$$= (2nk_{i_{2n}}\epsilon_{ii_1\cdots i_{2n-1}} + k_i\epsilon_{i_1 i_2\cdots i_{2n}})\epsilon^{i_1\cdots i_{2n}}\int_0^1 d\tau_0\int_{\tau_0}^1 d\tau_1\cdots\int_{\tau_{n-2}}^1 d\tau_{n-1}\frac{n}{(2n)!}\epsilon_{i'_1\cdots i'_{2n}}$$

$$\times\mathrm{tr}(\delta X^i e^{i\tau_0 kX}[X^{i'_1},X^{i'_2}]e^{i(\tau_0-\tau_1)kX}\cdots[X^{i'_{2n-1}},X^{i'_{2n}}]e^{i(1-\tau_{n-1})kX})$$

$$= 0. \tag{2.34}$$

In the last line, we used the identity in $2n$ dimensions:

$$2n\epsilon_{ii_1\cdots i_{2n-1}}\epsilon^{i_1\cdots i_{2n-1}j} + \epsilon_{i_1\cdots i_{2n}}\epsilon^{i_1\cdots i_{2n}}\delta_i^j = 0. \tag{2.35}$$

Thus we have proven that the right-hand side of Eq. (2.30) is topological. Combining Eqs. (2.28) and (2.31), we find

$$\frac{1}{\mathrm{Pf}(\theta)}\int dx*\left[e^{ikx}(\theta-\theta\hat{f}\theta)_{ij}^{n-1}P\exp\left(i\int_0^1\hat{A}_i(x+l\tau)l^i d\tau\right)\right]$$

$$-\int dx*\left[e^{ikx}\sqrt{\det(1-\hat{f}\theta)}\left(\frac{1}{1-\hat{f}\theta}\hat{f}\right)_{ij}P\exp\left(i\int_0^1\hat{A}_i(x+l\tau)l^i d\tau\right)\right] = \theta_{ij}^{-1}\delta(k). \tag{2.36}$$

Therefore, the conjectured expression (1.9) agrees with Eq. (1.17). This completes the proof that Eq. (1.17) gives an exact Seiberg-Witten map.

III. RELATION TO THE RAMOND-RAMOND COUPLING

In Sec. II, we have proven that Eq. (1.17) satisfies the conditions (a)–(c). Now we would like to explain the string theoretical origin of the formula. As we mentioned in the Introduction, we found the expression for the Seiberg-Witten map (1.17) by studying the coupling of noncommutative gauge theory realized on p-branes to the Ramond-Ramond $(p-1)$-form in the bulk. The dual of the Ramond-Ramond current $J^{i_1\cdots i_{p-1}}$ on the $(p+1)$-dimensional world volume is a two-form. It is clear that this two-form must be invariant under the noncommutative gauge transformation, and thus it obeys the condition (a). The condition (b) is satisfied since the coupling should also be invariant under the Ramond-Ramond gauge transformation $C^{(p-1)}\rightarrow C^{(p-1)}+d\epsilon$. If we assume that there is no α' correction to the Ramond-Ramond coupling, we can expect that the two-form is related to the field strength F_{ij} of the commutative variable as follows [24–27]:

$$\int C^{(p-1)}\wedge(F+\theta^{-1}). \tag{3.1}$$

If that is the case, the condition (c) should also hold. This was our motivation for Eq. (1.17).

The couplings of noncommutative gauge theory to closed string states in the bulk can be derived in various different ways. One approach is to evaluate disk amplitudes on a Dp brane with a background of NS-NS two-form field and take the Seiberg-Witten limit. In [11], the energy-momentum tensor of the noncommutative theory was derived in this way. Alternatively, one can start with matrix theory [17] (i.e., many D0 branes instead of a Dp brane), compute the coupling of the bulk fields to the matrix variables, and evaluate it in the background which gives rise to the noncommutative gauge theory on a Dp brane [17–19]. This approach was

suggested in [28] and was carried out explicitly in [21] in the case of the coupling to the bulk graviton, where it was found to give the same result as that obtained using the first approach [11].

Here we will adopt the second approach since the currents coupled to the Ramond-Ramond potentials have already been studied in matrix theory [29–31].[10] For our purpose, it is sufficient to have the currents coupled to the space-time components of the Ramond-Ramond potential, $C_{0i_1 \cdots i_p}$, where the index 0 is for the timelike coordinate and $i = 1, \ldots, 9$ are for the spacelike coordinates in the type IIA string theory. The relevant couplings deduced in [29–31] are of the form

$$\int dt \, \mathrm{Str}(C_0(t,X) + C_{0ij}(t,X)[X^i,X^j]$$
$$+ C_{0ijkl}(t,X)[X^i,X^j][X^k,X^l] + \cdots). \quad (3.2)$$

Here X^i are matrix coordinates, and the symmetrized trace Str is defined by expanding $C_{0i_1 \cdots i_p}(t,X)$ in powers of X's and totally symmetrize them together with $[X^i,X^j]$, each of which is treated as one unit in the symmetrization. In the momentum basis, the currents coupled to the Ramond-Ramond potentials can be read off from Eq. (3.2) as

$$J(k) = \mathrm{Str}(e^{ikX}),$$

$$J^{ij}(k) = i \, \mathrm{Str}([X^i,X^j]e^{ikX}),$$

$$\vdots$$

$$J^{i_1 \cdots i_{2s}}(k) = \frac{i^s}{(2s)!} \mathrm{Str}([X^{i_1},X^{i_2}] \cdots [X^{i_{2s-1}},X^{i_{2s}}]e^{ikX})$$
$$+ ([(2s)!-1] \text{ more terms to antisymmetrize}$$
$$\text{the indices}). \quad (3.3)$$

We should point out that the symmetrized traces in Eqs. (3.2) and (3.3) make sense only when X's are trace class operators since they are defined by expanding $C_{0i_1 \cdots i_p}(X)$ in powers of X's before taking the trace. If X's are infinite dimensional, a trace of powers of X's may not be well-defined, though a trace of e^{ikX} may still exist. In fact, this is the case when $X^i = x^i + \theta^{ij}\hat{A}_j(x)$ with $[x^i,x^j] = -i\theta^{ij}$.

On the other hand, the currents (2.16) we used in Sec. II make sense even when X's are of the form $X^i = x^i + \theta^{ij}\hat{A}_j(x)$. Moreover, they agree with Eq. (3.3) when X's are finite dimensional. In fact, if $\mathcal{O}_1 \cdots \mathcal{O}_n$ and X are trace class operators, one can prove

$$\mathrm{Str}(\mathcal{O}_1 \cdots \mathcal{O}_n e^{ikX}) = \int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{n-2}}^1 d\tau_{n-1}$$
$$\times \mathrm{tr}(\mathcal{O}_1 e^{i\tau_1 kX}\mathcal{O}_2 \cdots \mathcal{O}_n e^{i(1-\tau_{n-1})kX})$$
$$+ ([(n-1)!-1] \text{ more terms to}$$
$$\text{symmetrize } \mathcal{O}_i\text{'s}). \quad (3.4)$$

Here the symmetrized trace Str on the left-hand side is defined by expanding e^{ikX} in powers of X's and symmetrizing them with $\mathcal{O}_1 \cdots \mathcal{O}_n$. On the other hand, the symmetrization on the right-hand side exchanges $\mathcal{O}_1 \cdots \mathcal{O}_n$ only. The equivalence (3.4) for $n=2$ has been proven in our previous paper [21]. A general proof for arbitrary n is given in the appendix of this paper. As shown in [21], the τ-integral expressions such as Eq. (2.16) naturally arise from disk amplitudes of a single closed string state and an arbitrary number of open string states. In this case, $\tau_1, \ldots, \tau_n$ are identified as locations of the open string vertex operators on the boundary of the worldsheet disk. In [21], this is shown explicitly for the coupling of X's to the graviton in the bulk. We expect the situation is the same for the coupling to the Ramond-Ramond potentials. This is the string theory origin of the formula (2.1).

IV. DISCUSSION

In this paper, we proved that Eq. (1.17) satisfies the conditions (a)–(c) for the Seiberg-Witten map. We also showed that it is equivalent to Eq. (1.9) and therefore proved the conjecture in [10].

The exact Seiberg-Witten map can be used to understand the relation between the commutative and noncommutative descriptions of D-branes with a strong NS-NS two-form field. For example, it may be possible to study the noncommutative solitons [33] in the language of the commutative variables.

In this paper, we set all the scalar fields to be zero and focused on the Seiberg-Witten map between $A_i(x)$ and $\hat{A}_i(x)$. It is straightforward to include these in the analysis. We can also add commutative dimensions by starting from many Dp branes with $p>0$ rather than D0 branes and by using the results in [30] and [31] about the Ramond-Ramond coupling of these branes.

ACKNOWLEDGMENTS

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[10]In the case of constant Ramond-Ramond potentials, the coupling to noncommutative gauge fields was studied in [32].

APPENDIX: τ-ORDERED INTEGRAL=SYMMETRIZED TRACE

In this appendix, we will prove the equivalence (3.4) of the symmetrized trace and the τ-ordered trace.[11]
First let us perform the τ integrals in Eq. (3.4) explicitly. For any operators $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$,

$$\int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{m-2}}^1 d\tau_{m-1} \mathrm{tr}\, \mathcal{O}_1 e^{i\tau_1 kX} \mathcal{O}_2 e^{i(\tau_2-\tau_1)kX} \cdots \mathcal{O}_{m-1} e^{i(\tau_{m-1}-\tau_{m-2})kX} \mathcal{O}_m e^{i(1-\tau_{m-1})kX}$$

$$= \int_0^1 d\tau_1' \int_0^{1-\tau_1'} d\tau_2' \int_0^{1-\tau_1'-\tau_2'} d\tau_3' \cdots \int_0^{1-\tau_1'-\cdots-\tau_{m-2}'} d\tau_{m-1}' \mathrm{tr}\, \mathcal{O}_1 e^{i\tau_1' kX}$$

$$\times \mathcal{O}_2 e^{i\tau_2' kX} \cdots \mathcal{O}_{m-1} e^{i\tau_{m-1}' kX} \mathcal{O}_m e^{i(1-\tau_1'-\tau_2'-\cdots-\tau_{m-1}')kX}$$

$$= \sum_{a_1=0}^\infty \sum_{a_2=0}^\infty \cdots \sum_{a_m=0}^\infty \int_0^1 d\tau_1' \int_0^{1-\tau_1'} d\tau_2' \cdots \int_0^{1-\tau_1'-\cdots-\tau_{m-2}'} d\tau_{m-1}' \frac{1}{a_1! a_2! \cdots a_m!}$$

$$\times \mathrm{tr}\, \mathcal{O}_1 (i\tau_1' kX)^{a_1} \mathcal{O}_2 (i\tau_2' kX)^{a_2} \cdots \mathcal{O}_{m-1} (i\tau_{m-1}' kX)^{a_{m-1}} \mathcal{O}_m (i(1-\tau_1'-\tau_2'-\cdots-\tau_{m-1}')kX)^{a_m}$$

$$= \sum_{a_1=0}^\infty \sum_{a_2=0}^\infty \cdots \sum_{a_m=0}^\infty \frac{1}{(a_1+a_2+\cdots+a_m+m-1)!} \mathrm{tr}\, \mathcal{O}_1 (ikX)^{a_1} \mathcal{O}_2 (ikX)^{a_2} \cdots \mathcal{O}_{m-1} (ikX)^{a_{m-1}} \mathcal{O}_m (ikX)^{a_m}$$

$$= \sum_{n=0}^\infty \frac{1}{(n+m-1)!} \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} \cdots \sum_{p_{m-1}=0}^{n-p_1-p_2-\cdots-p_{m-2}} \mathrm{tr}\, \mathcal{O}_1 (ikX)^{p_1}$$

$$\times \mathcal{O}_2 (ikX)^{p_2} \cdots \mathcal{O}_{m-1} (ikX)^{p_{m-1}} \mathcal{O}_m (ikX)^{n-p_1-p_2-\cdots-p_{m-1}}. \tag{A1}$$

Here we changed the integration variables in the first step as

$$\tau_1' = \tau_1, \quad \tau_2' = \tau_2 - \tau_1, \quad \cdots \quad \tau_{m-1}' = \tau_{m-1} - \tau_{m-2}, \tag{A2}$$

and used the following formula in performing the τ' integrals:

$$\int_0^1 d\tau_1' \int_0^{1-\tau_1'} d\tau_2' \cdots \int_0^{1-\tau_1'-\cdots-\tau_{m-2}'} d\tau_{m-1}' \tau_1'^{\alpha_1-1} \tau_2'^{\alpha_2-2} \cdots \tau_{m-1}'^{\alpha_{m-1}-1} (1-\tau_1'-\tau_2'-\cdots-\tau_{m-1}')^{\beta-1}$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_{m-1})\Gamma(\beta)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_{m-1}+\beta)} \quad \text{for } \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \beta > 0. \tag{A3}$$

On the other hand, since

$$\mathrm{Str}[(ikX)^n \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_m]$$

$$= \frac{1}{(n+m-1)!} \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} \cdots \sum_{p_{m-1}=0}^{n-p_1-p_2-\cdots-p_{m-2}} n!\, \mathrm{tr}\, \mathcal{O}_1 (ikX)^{p_1} \mathcal{O}_2 (ikX)^{p_2} \cdots \mathcal{O}_{m-1} (ikX)^{p_{m-1}}$$

$$\times \mathcal{O}_m (ikX)^{n-p_1-p_2-\cdots-p_{m-1}} + ([(m-1)!-1] \text{ more terms to symmetrize in } \mathcal{O}_2, \mathcal{O}_3, \ldots, \mathcal{O}_m), \tag{A4}$$

where Str is the symmetrized trace with respect to $X, \mathcal{O}_1, \ldots, \mathcal{O}_m$, we can write

$$\mathrm{Str}[e^{ikX} \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_m] = \sum_{n=0}^\infty \frac{1}{n!} \mathrm{Str}[(ikX)^n \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_m]$$

$$= \int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{m-2}}^1 d\tau_{m-1} \mathrm{tr}\, \mathcal{O}_1 e^{i\tau_1 kX} \mathcal{O}_2 e^{i(\tau_2-\tau_1)kX} \cdots \mathcal{O}_{m-1} e^{i(\tau_{m-1}-\tau_{m-2})kX} \mathcal{O}_m e^{i(1-\tau_{m-1})kX}$$

$$+ ([(m-1)!-1] \text{ more terms to symmetrize}). \tag{A5}$$

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[11]We assume that the symmetrized trace is well-defined. This means that, if we define the symmetrized trace in terms of a power series expansion in X's, X must be trace class operators.

Here we made use of the cyclicity of the trace. Therefore, we have shown the equivalence of the two expressions (3.3) and (2.16). Furthermore, we can show that

$$\text{tr}\left[\exp\left(i\int_0^1 d\tau kX\right)\int_0^1 d\tau_1 \mathcal{O}_1 \cdots \int_0^1 d\tau_m \mathcal{O}_m\right] = \int_0^1 d\tau_1 \int_{\tau_1}^1 d\tau_2 \cdots \int_{\tau_{m-2}}^1 d\tau_{m-1} \text{tr}\mathcal{O}_1 e^{i\tau_1 kX} \mathcal{O}_2 e^{i(\tau_2 - \tau_1)kX} \cdots$$

$$\times \mathcal{O}_{m-1} e^{i(\tau_{m-1} - \tau_{m-2})kX} \mathcal{O}_m e^{i(1 - \tau_{m-1})kX}$$

$$+ ([(m-1)! - 1] \text{ more terms to symmetrize})$$

$$= \text{Str}[e^{ikX}\mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_m], \tag{A6}$$

where the operators are τ-ordered in the first line. This formula is the generalization of (27) in [21] to the case where more than two operators are inserted and useful when we transform the current $J^{i_1 i_2 \cdots i_{2n}}$ (2.16) to the form (2.1) used in the noncommutative gauge theory.

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[1] A. Connes, M.R. Douglas, and A. Schwarz, J. High Energy Phys. **02**, 003 (1998).

[2] N. Seiberg and E. Witten, J. High Energy Phys. **09**, 032 (1999).

[3] T. Asakawa and I. Kishimoto, J. High Energy Phys. **11**, 024 (1999)

[4] K. Okuyama, J. High Energy Phys. **03**, 016 (2000).

[5] B. Jurco and P. Schupp, Eur. Phys. J. C **14**, 367 (2000); B. Jurco, P. Schupp, and J. Wess, Nucl. Phys. **B584**, 784 (2000); "Nonabelian noncommutative gauge theory via noncommutative extra dimensions," hep-th/0102129

[6] M. Kontsevich, "Deformation quantization of Poisson manifolds, I," q-alg/9709040.

[7] A.S. Cattaneo and G. Felder, Commun. Math. Phys. **212**, 591 (2000).

[8] M.R. Garousi, Nucl. Phys. **B579**, 209 (2000).

[9] T. Mehen and M.B. Wise, J. High Energy Phys. **12**, 008 (2000).

[10] H. Liu, "*-Trek II: *ₙ operations, open Wilson lines and the Seiberg-Witten map," hep-th/0011125.

[11] Y. Okawa and H. Ooguri, Nucl. Phys. **B599**, 55 (2001).

[12] S. Mukhi and N.V. Suryanarayana, "Gauge-invariant couplings of noncommutative branes to Ramond-Ramond backgrounds," hep-th/0104045.

[13] H. Liu and J. Michelson, "Ramond-Ramond couplings of noncommutative D-branes," hep-th/0104139.

[14] N. Ishibashi, S. Iso, H. Kawai, and Y. Kitazawa, Nucl. Phys. **B573**, 573 (2000).

[15] S.R. Das and S. Rey, Nucl. Phys. **B590**, 453 (2000).

[16] D.J. Gross, A. Hashimoto, and N. Itzhaki, "Observables of non-commutative gauge theories," hep-th/0008075.

[17] T. Banks, W. Fischler, S.H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).

[18] B. de Wit, J. Hoppe, and H. Nicolai, Nucl. Phys. **B305**, 545 (1988).

[19] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, Nucl. Phys. **B498**, 467 (1997).

[20] N. Seiberg, J. High Energy Phys. **09**, 003 (2000).

[21] Y. Okawa and H. Ooguri, "Energy-momentum tensors in Matrix theory and in noncommutative gauge theories," to appear in the Proceedings of the conference, *Strings 2001*, Mumbai, India, hep-th/0103124.

[22] J.H. Schwarz, "Comments on Born-Infeld theory," hep-th/0103165.

[23] M. Van Raamsdonk, Nucl. Phys. **B542**, 262 (1999)

[24] J. Polchinski and Y. Cai, Nucl. Phys. **B296**, 91 (1988).

[25] C.G. Callan, C. Lovelace, C.R. Nappi, and S.A. Yost, Nucl. Phys. **B308**, 221 (1988).

[26] M. Li, Nucl. Phys. **B460**, 351 (1996).

[27] M.R. Douglas, "Branes within branes," hep-th/9512077.

[28] S.R. Das and S.P. Trivedi, J. High Energy Phys. **02**, 046 (2001).

[29] W.I. Taylor and M. Van Raamsdonk, Nucl. Phys. **B558**, 63 (1999)

[30] W.I. Taylor and M. Van Raamsdonk, Nucl. Phys. **B573**, 703 (2000).

[31] R.C. Myers, J. High Energy Phys. **12**, 022 (1999).

[32] S. Mukhi and N.V. Suryanarayana, J. High Energy Phys. **11**, 006 (2000).

[33] For a recent review of this subject, see J.A. Harvey, "Komaba lectures on noncommutative solitons and D-branes," hep-th/0102076.