Perturbative test of exact vacuum expectation values of local fields in affine Toda theories

Changrim Ahn,[1,2] P. Baseilhac,[3] Chanju Kim,[4,*] and Chaiho Rim[5,2]
[1]*Department of Physics, Ewha Womans University Seoul 120-750, Korea*
[2]*Asia Pacific Center for Theoretical Physics Yoksam-dong 678-39, Seoul, 135-080, Korea*
[3]*Department of Mathematics, University of York Heslington, York YO105DD, United Kingdom*
[4]*School of Physics, Korea Institute for Advanced Study Seoul, 130-012, Korea*
[5]*Department of Physics, Chonbuk National University Chonju 561-756, Korea*
(Received 15 March 2001; published 5 July 2001)

Vacuum expectation values of local fields for all dual pairs of nonsimply laced affine Toda field theories recently proposed are checked against perturbative analysis. The computations based on Feynman diagram expansion are performed up to the two-loop level. We obtain, good agreement.

DOI: 10.1103/PhysRevD.64.046002 PACS number(s): 11.25.Hf, 11.55.Ds

I. INTRODUCTION

The vacuum expectation values (VEVs) of local fields play an important role in quantum field theory (QFT) and statistical mechanics [1,2]. In QFT defined as perturbed conformal field theory (CFT), they constitute the basic ingredients for multipoint correlation functions, using short-distance expansions [2,3]. Recently, important progress has been made in the calculations of the VEVs in two-dimensional integrable QFT. In Ref. [4], an explicit expression for the VEVs of the exponential field in the sine-Gordon and sinh-Gordon models—$A_1^{(1)}$ affine Toda field theory (ATFT)—was proposed. Moreover, it was shown in Ref. [5] that this expression can be obtained as the minimal solution of certain ‘‘reflection relations’’ which involve the Liouville ‘‘reflection amplitude’’ [6], where the sinh-Gordon QFT was considered as a perturbed Liouville conformal field theory. Subsequently, this ‘‘reflection relations’’ method was successfully generalized to other models, for which the VEVs were calculated. We refer the reader to Refs. [7–11] for details.

It is thus natural to study the case of dual pairs of nonsimply laced ATFTs, as well as the simply laced one which had been previously considered [12]. In addition to the technical aspect, such VEVs can provide interesting information as this class of models appears in various physics contexts [13–16]. These include special case of 3D $U(1)$ or XY model [12], VEV of the spin field σ in the $\mathbb{Z}_n$-Ising models [17] perturbed by the leading thermal operator, asymptotics of the cylindrically symmetric solutions of the classical Toda equations. More recently, exact off-shell results for coupled minimal models were considered in Ref. [18].

ATFTs can be considered as perturbed Toda field theories (TFTs). In Ref. [19] the ‘‘reflection amplitudes’’ for all nonsimply laced TFTs were proposed as well as the exact relation between the masses of the particles and the parameters in the ATFT action. On the one hand, reflection amplitudes are the main objects which can be used for studying the UV asymptotics of the ground state energy $E(R)$ [or effective central charge $c_{\rm eff}(R)$] for the system on the circle of size R [6,20,21,19]. This result agrees well with the TBA result at small R [22,23], which can provide a nontrivial test for the S matrix in Refs. [24,25]. On the other hand, reflection amplitudes were also used to calculate the exact VEVs for all dual pairs of nonsimply laced ATFTs [19].

However, to support the exact VEVs, it is desirable to check with various methods since the ‘‘reflection relations’’ method has *no* rigorous mathematical proof. For simplest cases such as sinh-Gordon and Bullough-Dodd, the exact VEVs have been checked both nonperturbatively and perturbatively [4,7,26,27]. The purpose of this paper is to check the conjectured VEVs of ATFTs using perturbation theory up to two-loop level. In Sec. II, we review some basic facts about ATFTs and the axiomatic equations satisfied by the VEVs which lead to the exact solutions given in Ref. [19]. Perturbative analysis follows in Sec. III where we compute VEVs of scalar fields up to one-loop and some composite operators up to two loops.

II. EXACT VACUUM EXPECTATION VALUES IN AFFINE TODA FIELD THEORIES

Let us first recall some known results about ATFTs which are relevant in further analysis. The ATFT with real coupling b corresponding to the affine Lie algebra[1] $\hat{\mathcal{G}}$ is generally described by the action in Euclidean space:

$$\mathcal{A}=\int d^2x\left[\frac{1}{8\pi}(\partial_\mu\boldsymbol{\varphi})^2+\sum_{i=0}^{r}\mu_{e_i}e^{b\mathbf{e}_i\cdot\boldsymbol{\varphi}}\right], \tag{2.1}$$

where $\{\mathbf{e}_i\}\in\Phi_{\mathbf{s}}(i=1,...,r)$ is the set of simple roots of $\hat{\mathcal{G}}$ of rank r and $-\mathbf{e}_0$ is a maximal root satisfying

$$\mathbf{e}_0+\sum_{i=1}^{r}n_i\mathbf{e}_i=0. \tag{2.2}$$

*Present address: Department of Physics, Seoul National University, 151-742, Seoul, Korea.

[1]Throughout the paper, we denote an untwisted algebra as $\hat{\mathcal{G}}$, while $\hat{\mathcal{G}}^\vee$ refers to a twisted one. Furthermore, $\mathcal{G}$ denotes a finite Lie algebra.

The fields in Eq. (2.1) are normalized such that

$$\langle \varphi_a(x)\varphi_b(y)\rangle = -\delta_{ab}\log|x-y|^2. \tag{2.3}$$

Since all potential terms in simply laced case have the same conformal dimensions, they all renormalize in the same way. It is then sufficient to introduce one scale parameter[2] μ in action (2.1). However, for the nonsimply laced case (except $BC_r \equiv A_{2r}^{(2)} - r \geq 2$— affine Lie algebra in which case three different parameters are necessary) we have to introduce two different parameters:[3] one is associated with the set of standard roots of length $|\mathbf{e}_i|^2 = 2$ and is denoted by $\mu_{\mathbf{e}_i} = \mu$ whereas the other, denoted by $\mu_{\mathbf{e}_i} = \mu'$, is associated with the set of nonstandard roots of length $|\mathbf{e}_i|^2 = l^2 \neq 2$.

In the presence of background charge, the ATFTs can be considered s CFTs perturbed by $e^{b\mathbf{e}_0\varphi}$. The background charge is given by

$$\mathbf{Q} = b\boldsymbol{\rho} + \frac{1}{b}\boldsymbol{\rho}^\vee, \tag{2.4}$$

where $\boldsymbol{\rho} = \frac{1}{2}\Sigma_{\alpha>0}\boldsymbol{\alpha}$ and $\boldsymbol{\rho}^\vee = \frac{1}{2}\Sigma_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}^\vee$ are, respectively, the Weyl and dual Weyl vector of $\mathcal{G}$. The sums in their definitions run over all positive roots $\{\boldsymbol{\alpha}\} \in \Phi_+$, dual roots $\{\boldsymbol{\alpha}^\vee\} \in \Phi_+^\vee$. Then, the stress-energy tensor $T(z)$, where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$ are complex coordinates of $\mathbb{R}^2$,

$$T(z) = -\frac{1}{2}(\partial_z\boldsymbol{\varphi})^2 + \mathbf{Q}\cdot\partial_z^2\boldsymbol{\varphi} \tag{2.5}$$

ensures the local conformal invariance of the TFT. The corresponding central charges were calculated in Ref. [28]. Defining $\mathbf{a} = (a_1, \ldots, a_r)$, the exponential fields

$$V_{\mathbf{a}}(x) = \exp(\mathbf{a}\cdot\varphi)(x) \tag{2.6}$$

are spinless conformal primary fields with dimensions

$$\Delta(\mathbf{a}) = \frac{\mathbf{Q}^2}{2} - \frac{(\mathbf{a}-\mathbf{Q})^2}{2}. \tag{2.7}$$

By analogy with the Liouville field theory [29,30,6] the physical space of states $\mathcal{H}$ in the TFTs consists of the continuum variety of primary states associated with the exponential fields (2.6) and their conformal descendents with

$$\mathbf{a} = i\mathbf{P} + \mathbf{Q} \quad \text{and} \quad \mathbf{P} \in \mathbb{R}^r. \tag{2.8}$$

In addition to the conformal invariance TFTs possess an extended symmetry generated by $W(\mathcal{G})$ algebra [31,32]. Indeed, for any arbitrary Weyl group element $\hat{s} \in \mathcal{W}$ the fields $V_{\mathbf{Q}+\hat{s}(\mathbf{a}-\mathbf{Q})}(x)$ are reflection images of each other and are related by the linear transformation:

$$V_{\mathbf{a}}(x) = R_{\hat{s}}(\mathbf{a}) V_{\mathbf{Q}+\hat{s}(\mathbf{a}-\mathbf{Q})}(x), \tag{2.9}$$

where $R_{\hat{s}}(\mathbf{a})$ is called the "reflection amplitude," an important object in CFT which defines the two-point functions of the operator $V_{\mathbf{a}}$. In Ref. [19] the following expression for the reflection amplitude $R_{\hat{s}}(\mathbf{a})$ for nonsimply laced TFT was obtained:

$$R_{\hat{s}}(\mathbf{a}) = \frac{A_{\hat{s}i\mathbf{P}}}{A_{i\mathbf{P}}}, \tag{2.10}$$

where

$$A_{i\mathbf{P}} \equiv A(\mathbf{P}) = \prod_{i=1}^{r}[\pi\mu_{\mathbf{e}_i}\gamma(\mathbf{e}_i^2 b^2/2)]^{i\boldsymbol{\omega}_i^\vee\cdot\mathbf{P}/b} \times \prod_{\mathbf{a}>0}\Gamma(1-i\mathbf{P}\cdot\boldsymbol{\alpha}b)\Gamma(1-i\mathbf{P}\cdot\boldsymbol{\alpha}^\vee/b)$$

with Eq. (2.8), the fundamental coweights $\boldsymbol{\omega}_i^\vee$ and we denote $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ as usual. We accept Eq. (2.10) as the proper analytical continuation of the function $R_{\hat{s}}(\mathbf{a})$ for all $\mathbf{a}$. For $\hat{s}_i \in \mathcal{W}_{\mathbf{s}}$, the subset of Weyl group elements associated with the simple roots $\mathbf{e}_i$, notice that the ratio $A(\hat{s}_i\mathbf{P})/A(\mathbf{P})$ reduce to the reflection amplitude $S_L(\mathbf{e}_i, \mathbf{P})$ of the Liouville field theory [6]:

$$\begin{aligned}\frac{A(\hat{s}_i\mathbf{P})}{A(\mathbf{P})} &= S_L(\mathbf{e}_i, \mathbf{P}) \\ &= [\pi\mu_{e_i}\gamma(\mathbf{e}_i^2 b^2/2)]^{-i\mathbf{P}\cdot\mathbf{e}_i^\vee/b} \\ &\quad \times \frac{\Gamma(1+i\mathbf{P}\cdot\mathbf{e}_i b)\Gamma(1+i\mathbf{P}\cdot\mathbf{e}_i^\vee/b)}{\Gamma(1-i\mathbf{P}\cdot\mathbf{e}_i b)\Gamma(1-i\mathbf{P}\cdot\mathbf{e}_i^\vee/b)}.\end{aligned} \tag{2.11}$$

Then, as ATFTs can be realized as CFTs perturbed by some relevant operators [33], in the conformal perturbation theory (CPT) approach one can formally rewrite any N-point correlation functions of local operators $\mathcal{O}_a(x)$ as

$$\begin{aligned}&\langle \mathcal{O}_{a_1}(x_1)\cdots\mathcal{O}_{a_N}(x_N)\rangle_{\text{ATFT}} \\ &\quad = Z^{-1}(\lambda)\langle \mathcal{O}_{a_1}(x_1)\cdots\mathcal{O}_{a_N}(x_N) e^{-\lambda\int d^2x\Phi_{\text{pert}}(x)}\rangle_{\text{TFT}}\end{aligned}$$

where

$$Z(\lambda) = \langle e^{-\lambda\int d^2x\Phi_{\text{pert}}(x)}\rangle_{\text{TFT}},$$

Φ_{pert} is the perturbing local field, λ is the CPT expansion parameter which characterizes the strength of the perturbation and $\langle\cdots\rangle_{\text{TFT}}$ denotes the expectation value in the TFT. Whereas vertex operators (2.6) satisfy reflection relations (2.9) in the CFT, the CPT framework provides[4] similar relations among their expectation values in the perturbed case. In other words, if dots stands for any local insertion one has

$$\langle V_{\mathbf{a}}(x)(\cdots)\rangle_{\text{TFT}} = R_{\hat{s}}(\mathbf{a})\langle V_{\mathbf{Q}+\hat{s}(\mathbf{a}-\mathbf{Q})}(x)(\cdots)\rangle_{\text{TFT}}. \tag{2.12}$$

[2]For the sinh-Gordon model ($A_1^{(1)}$ ATFT) μ is generally called the cosmological constant.

[3]We choose the convention that the length squared of the long roots are 4 for $C_r^{(1)}$ and 2 for the other untwisted algebras.

[4]At the moment, there is no rigorous proof of this assumption.

Then, if we define the one-point function $G(\mathbf{a})$ as the VEV of the vertex operator $V_{\mathbf{a}}(x)$ for nonsimply laced ATFT by

$$G(\mathbf{a})=\langle \exp(\mathbf{a}\cdot\boldsymbol{\varphi})(x)\rangle_{\mathrm{TFT}}. \tag{2.13}$$

one can formally rewrite this expression[5] as

$$\langle e^{\mathbf{a}\cdot\varphi}(x)\rangle_{\mathrm{ATFT}}=Z^{-1}(\lambda)\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int\prod_{j=1}^{n}d^2y_j\langle e^{\mathbf{a}\cdot\varphi}(x)\times e^{b\mathbf{e}_0\cdot\varphi}(y_1)\cdots e^{b\mathbf{e}_0\cdot\varphi}(y_n)\rangle_{\mathrm{TFT}}. \tag{2.14}$$

Indeed, using Eq. (2.12) one expects that similar relations hold for $G(\mathbf{a})$. If this VEV satisfies the system of functional equations associated with $\mathcal{W}_s$ then it also automatically satisfies more complicated reflection relations. Furthermore, as was shown in previous works [14,15], ATFTs can be understood as different perturbation of TFTs. The simplest case (beyond the sinh-Gordon model) is the Bullough-Dodd model which can be understood alternatively [7] as a perturbed Liouville CFT with coupling constant b or a perturbed Liouville CFT with coupling constant $-b/2$. Here one can proceed similarly. We denote $\Phi_{\mathbf{s}}(\mathcal{G})$ as the set of simple roots of the finite Lie algebra $\mathcal{G}$, $\boldsymbol{\eta}$ the extra root associated with the perturbation and $\{\boldsymbol{\epsilon}_i\}$ an orthogonal basis ($\boldsymbol{\epsilon}_i\cdot\boldsymbol{\epsilon}_j=\delta_{ij}$) in $\mathbb{R}^r$. Each one of the ATFT Lagrangian representation, denoted $\mathcal{L}_b[\Phi_{\mathbf{s}}(\mathcal{G})]$, associated with $\Phi_{\mathbf{s}}(\mathcal{G})$ and the coupling constant b can be rewritten in two different ways:

$$\mathcal{L}_b[\Phi_{\mathbf{s}}(B_r^{(1)})]\equiv\mathcal{L}_b[\Phi_{\mathbf{s}}(B_r)\oplus\boldsymbol{\eta}\equiv\mathbf{e}_0=-(\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2)],$$

$$\equiv\mathcal{L}_{-b}[\overline{\Phi}_{\mathbf{s}}(D_r)\oplus\boldsymbol{\eta}=-\boldsymbol{\epsilon}_r];$$

$$\mathcal{L}_b[\Phi_{\mathbf{s}}(C_r^{(1)})]\equiv\mathcal{L}_b[\Phi_{\mathbf{s}}(C_r)\oplus\boldsymbol{\eta}\equiv\mathbf{e}_0=-2\boldsymbol{\epsilon}_1],$$

$$\equiv\mathcal{L}_{-b}[\overline{\Phi}_{\mathbf{s}}(C_r)\oplus\boldsymbol{\eta}\equiv-2\boldsymbol{\epsilon}_r];$$

$$\mathcal{L}_b[\Phi_{\mathbf{s}}(F_4^{(1)})]\equiv\mathcal{L}_b[\Phi_{\mathbf{s}}(F_4)\oplus\boldsymbol{\eta}\equiv\mathbf{e}_0=-\boldsymbol{\epsilon}_1-\boldsymbol{\epsilon}_2],$$

$$\equiv\mathcal{L}_{-b}\left[\overline{\Phi}_{\mathbf{s}}(B_4)\oplus\boldsymbol{\eta}\equiv-\frac{1}{2}(\boldsymbol{\epsilon}_1-\boldsymbol{\epsilon}_2-\boldsymbol{\epsilon}_3-\boldsymbol{\epsilon}_4)\right];$$

$$\mathcal{L}_b[\Phi_{\mathbf{s}}(G_2^{(1)})]\equiv\mathcal{L}_b[\Phi_{\mathbf{s}}(G_2)\oplus\boldsymbol{\eta}\equiv\mathbf{e}_0=-\sqrt{2}\boldsymbol{\epsilon}_1],$$

$$\equiv\mathcal{L}_{-b}[\overline{\Phi}'_{\mathbf{s}}(A_2)\oplus\boldsymbol{\eta}\equiv-\sqrt{2/3}\boldsymbol{\epsilon}_2],$$

where the different sets of simple roots are reported in Appendix B. Using Eq. (2.12) implies that the VEV (2.13) must satisfy *simultaneously* two irreducible systems of functional equations corresponding to two different sets $\mathcal{W}_s$. It results that $G(\mathbf{a})$ obeys the functional equations

$$G(\tau\mathbf{a})=S_L(\mathbf{e}_j,\mathbf{P})G\{\tau[\mathbf{Q}+\hat{s}_j(\mathbf{a}-\mathbf{Q})]\}\quad\text{for all}\quad\hat{s}_j\in\mathcal{W}_{\mathbf{s}}, \tag{2.15}$$

where

$$B_r^{(1)}:(\tau)_{ij}=\delta_{ij}\ \text{ for }\ \mathcal{G}\equiv B_r\ \text{ and }\ (\tau)_{ij}=-\delta_{ir+1-j}\ \text{ for }\ \mathcal{G}\equiv D_r;$$

$$C_r^{(1)}:(\tau)_{ij}=\delta_{ij}\ \text{ and }\ (\tau)_{ij}=-\delta_{ir+1-j}\ \text{ for }\ \mathcal{G}\equiv C_r;$$

$$F_4^{(1)}:(\tau)_{ij}=\delta_{ij}\ \text{ for }\ \mathcal{G}\equiv F_4\ \text{ and }\ (\tau)_{ij}=\delta_{ij}(\delta_{2j}+\delta_{3j}+\delta_{4j}-\delta_{1j})\ \text{ for }\ \mathcal{G}\equiv B_4;$$

$$G_2^{(1)}:(\tau)_{ij}=\delta_{ij}\ \text{ for }\ \mathcal{G}\equiv G_2\ \text{ and }\ (\tau)_{ij}=-\delta_{i\,3-j}\ \text{ for }\ \mathcal{G}\equiv A_2,$$

with coupling constant b. Notice that by simply looking at the Dynkin diagram symmetry of $B_r^{(1)}$ and $C_r^{(1)}$ (see Fig. 1) one can also differently deduce

[5]In fact, the integrals in Eq. (2.14) are highly infrared divergent. By analogy with the situation appearing in the perturbed Liouville QFT [7], one can get around this infrared problem by considering a 2D world sheet Σ_g topologically equivalent to a sphere equipped by a background metric $g_{\nu\sigma}(x)=\rho(x)\delta_{\nu\sigma}$. Then the terms $\rho(y_k)$ which appear in the integrals analogous to those in Eq. (2.14) provide an efficient infrared cutoff. We report the reader to Ref. [7] for details.

$$G(a_1,a_2,\ldots,a_{r-1},a_r)=G(-a_1,a_2,\ldots,a_{r-1},a_r)\quad \text{for}\ B_r^{(1)};$$

$$G(a_1,a_2,\ldots,a_{r-1},a_r)=G(-a_r,-a_{r-1},\ldots,-a_2,-a_1)\quad \text{for}\ \ C_r^{(1)}. \tag{2.16}$$

The reflection relations (2.15) (or, equivalently the relations (2.16) for $B_r^{(1)}$ and $C_r^{(1)}$) constituted the starting point in deriving the expectation values $G(\mathbf{a})$. Following previous works, we also assumed that $G(\mathbf{a})$ is a meromorphic function of $\mathbf{a}$.

Furthermore, for real coupling constant b, the spectrum for any dual pair of nonsimply laced ATFT consists of r particles with the masses M_a ($a=1,\ldots,r$) expressed in terms of the mass parameter $\overline{m}$. These spectra are reported in Appendix A. The exact relation between the parameters of the action μ and μ' and the masses associated with the spectrum of the physical particles was obtained in Ref. [19] using the Bethe ansatz method (see, for example, Refs. [34,35]). We report the reader to [19] for details. By replacing these mass-μ relations in the ''minimal'' solution[6] of the functional equations (2.15), the following exact expression for the VEVs (2.13) was proposed [19]:

$$G(\mathbf{a})=[\overline{m}k(\mathcal{G})\kappa(\mathcal{G})]^{-\mathbf{a}^2}\left[\frac{\mu\gamma(1+b^2)}{\mu'\gamma(1+b^2l^2/2)}\right]^{\mathbf{d}\cdot\mathbf{a}(1-B)/Hb}$$
$$\times\left[\frac{(-\pi\mu\gamma(1+b^2))^{l^2/2}}{-\pi\mu'\gamma(1+b^2l^2/2)}\right]^{\mathbf{d}\cdot\mathbf{a}B/Hb}$$
$$\times\exp\int_0^\infty\frac{dt}{t}[\mathbf{a}^2e^{-2t}-\mathcal{F}(\mathbf{a},t)]. \tag{2.17}$$

with

$$\mathcal{F}(\mathbf{a},t)=\sum_{\alpha>0}\left[\frac{\sinh(a_\alpha bt)\sinh\{[a_\alpha b-2Q_\alpha b+H(1+b^2)]t\}\sinh\left[\left(\frac{b^2|\alpha|^2}{2}+1\right)t\right]}{\sinh(t)\sinh\left(\frac{b^2|\alpha|^2}{2}t\right)\sinh[H(1+b^2)t]}\right]$$

where we denote $a_{\boldsymbol{\alpha}}=\mathbf{a}\cdot\boldsymbol{\alpha}$ and

$$\mathbf{d}=\frac{\boldsymbol{\rho}^\vee h^\vee-\boldsymbol{\rho}h}{1-l^2/2}.$$

The expressions $k(\mathcal{G})$ and $\kappa(\mathcal{G})$ can be found in Ref. [19]. Here, it is convenient to introduce the ''deformed'' Coxeter number [24,25]

$$H=h(1-B)+h^\vee B\quad \text{with}\quad B=\frac{b^2}{1+b^2}, \tag{2.18}$$

where $h(h^\vee)$ is the Coxeter (dual Coxeter) number of $\mathcal{G}(\mathcal{G}^\vee)$. The integral in Eq. (2.17) is convergent if

$$\boldsymbol{\alpha}\cdot\mathbf{Q}-H(b+1/b)<\mathrm{Re}(\boldsymbol{\alpha}\cdot\mathbf{a})<\boldsymbol{\alpha}\cdot\mathbf{Q}\quad \text{for all}\quad \boldsymbol{\alpha}\in\Phi \tag{2.19}$$

and is defined through analytic continuation outside this domain. The particular case of Eq. (2.17) corresponds to the simply laced one for which the result is in perfect agreement with Ref. [12].

Similarly, it is straightforward to obtain the VEVs of an ATFT based on a twisted affine Lie algebra $\hat{\mathcal{G}}^\vee$. The reflection amplitudes corresponding to the TFT, i.e., the conformal part were easily obtained from Eq. (2.10) by using the duality relation for the parameters $\mu_{\mathbf{e}_i}$ and $\mu_{\mathbf{e}_i}^\vee$ associated with the dual pairs of ATFTs [19]:

$$\pi\mu_{\mathbf{e}_i}\gamma\left(\frac{b^2\mathbf{e}_i^2}{2}\right)=\left[\pi\mu_{\mathbf{e}_i}^\vee\gamma\left(\frac{\mathbf{e}_i^{\vee 2}}{2b^2}\right)\right]^{b^2\mathbf{e}_i^2/2} \tag{2.20}$$

and the change $b\to 1/b$. Each one of the Lagrangian associated with $\hat{\mathcal{G}}^\vee$ can be written in two different ways. In any case, the resulting system of functional equations which has to be satisfied by the VEV is nothing else than the dual of Eq. (2.15). To express the corresponding solution in terms of the mass of the physical particles, the mass-μ relations in the twisted case [19] are used. Finally, the result for the VEV $G(\mathbf{a})$ for all twisted affine Lie algebras is obtained from Eq. (2.17) with the change $b\to 1/b$.

[6]Notice that the prefactor which was given in Ref. [19] was presented in a slightly different, but equivalent, form.

It is similarly straightforward to study the $BC_r \equiv A_{2r}^{(2)}$ (self-dual) remaining case which was considered in Ref. [36]. Notice that the expectation values (2.17) can be used to derive the bulk free energy of the ATFT:

$$f_{\hat{\mathcal{G}}} = -\lim_{V\to\infty} \frac{1}{V} \ln Z, \tag{2.21}$$

where V is the volume of the 2D space and Z is the singular part of the partition function associated with action (2.1). For specific values $\mathbf{a} \in b\{\mathbf{e}_i\}$, with $\{\mathbf{e}_i\} \in \Phi_{\mathbf{s}}$ $(i=1,...,r)$ or $\mathbf{e}_0$, the integral in Eq. (2.17) can be evaluated explicitly. Using the exact mass-μ relations and the obvious relations

$$\partial_\mu f(\mu) = \sum_{\{i\}} \langle e^{b\mathbf{e}_i \cdot \varphi} \rangle \quad \text{or} \quad \partial_{\mu'} f(\mu') = \sum_{\{i'\}} \langle e^{\mathbf{b}\mathbf{e}_{i'} \cdot \varphi} \rangle, \tag{2.22}$$

where $\{i\}$ and $\{i'\}$ denote, respectively, the whole set of long and short roots, the following bulk free energy was obtained [19]:

$$f_{\hat{\mathcal{G}}} = \frac{\overline{m}^2 \sin(\pi/H)}{8\sin(\pi B/H)\sin[\pi(1-B)/H]},$$

$$\hat{\mathcal{G}} = B_r^{(1)} \text{ and } C_r^{(1)},$$

$$f_{\hat{\mathcal{G}}} = \frac{\overline{m}^2 \cos[\pi(1/3 - 1/H)]}{16\cos(\pi/6)\sin(\pi B/H)\sin[\pi(1-B)/H]},$$

$$\hat{\mathcal{G}} = G_2^{(1)} \text{ and } F_4^{(1)}$$

and similarly with the change $B \to (1-B)$ for $(B_r^{(1)})^\vee$, $(C_r^{(1)})^\vee$, $(G_2^{(1)})^\vee$, and $(F_4^{(1)})^\vee$. In particular, these results were in perfect agreement with the values obtained using the Bethe ansatz approach [19].

III. PERTURBATIVE CHECKS

To support the result (2.17) of Ref. [19] beyond the nonperturbative check (provided by the bulk free energy calculation), we present here a perturbative check. We expand the vacuum expectation value (2.17) in power series in b and compare each coefficient with the one obtained from standard Feynman perturbation theory associated with Eq. (2.1). In the first part of this section, we consider the VEV of the field $\langle \varphi \rangle$ which is given by

$$\langle \boldsymbol{\varphi} \rangle = \frac{\delta}{\delta \mathbf{a}} G(\mathbf{a}) \bigg|_{\mathbf{a}=0}. \tag{3.1}$$

Since the result renders the same conclusion for all ATFTs, we choose $D_r^{(1)}$ series as illustrative examples and omit the details for other simply laced cases ($A_r^{(1)}$ case is trivial as seen shortly). It also provides a useful step to the calculation of $B_r^{(1)}$ series which is obtained from $D_r^{(1)}$ through folding procedure. Finally we present the result of an exceptional algebra $G_2^{(1)}$.

In a second part, as an additional check we also consider the "fully connected" composite operator expectation value of $\langle \boldsymbol{\varphi}^a \boldsymbol{\varphi}^b \rangle$ defined by

$$\langle\langle \boldsymbol{\varphi}^\alpha \boldsymbol{\varphi}^b \rangle\rangle \equiv \langle \boldsymbol{\varphi}^a \boldsymbol{\varphi}^b \rangle - \langle \boldsymbol{\varphi}^a \rangle \langle \boldsymbol{\varphi}^b \rangle = \frac{1}{2} \frac{\delta^2 \ln G(\mathbf{a})}{d\mathbf{a}^a \delta \mathbf{a}^b} \bigg|_{\mathbf{a}=0}. \tag{3.2}$$

Since these quantities are quite complicated to calculate perturbatively, we will content ourselves with considering only some simple combinations of them up to two loops for $B_3^{(1)}$, $C_2^{(1)}$, and $G_2^{(1)}$ cases.

A. Perturbative checks of $\langle \varphi \rangle$

Using Eqs. (2.17) and (3.1) one finds the result

$$\langle \varphi \rangle = \frac{\mathbf{d}}{Hb} \ln\left[\frac{\mu\gamma(1+b^2)}{\mu'\gamma(1+b^2 l^2/2)}\right] + B\frac{\mathbf{d}}{Hb}(l^2/2-1)\ln[-\pi\mu\gamma(1+b^2)] + b\int_0^\infty dt \sum_{\alpha>0}\left[\alpha \frac{\sinh\{[2\boldsymbol{\alpha}\cdot\mathbf{Q}b - H(1+b^2)]t\}\sinh\left[\left(\frac{b^2|\boldsymbol{\alpha}|^2}{2}+1\right)t\right]}{\sinh(t)\sinh[H(1+b^2)t]\sinh\left(\frac{b^2|\boldsymbol{\alpha}|^2}{2}t\right)}\right]. \tag{3.3}$$

To proceed further,[7] we expand $\langle \varphi \rangle$ order by order in b and write the result as

$$\langle \varphi \rangle = \frac{1}{b}\mathcal{K} + b\mathcal{L} + \mathcal{O}(b^2). \tag{3.4}$$

For the simply laced case, this expression is drastically simplified:

$$\mathcal{K} = -\sum_{\mathbf{a}>0} \boldsymbol{\alpha} \ln \gamma\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\rho}}{h}\right);$$

$$\mathcal{L} = -\int_0^\infty dt \frac{\coth(t)}{\sinh(ht)} \left\{ \sum_{\alpha>0} \alpha \sinh[(h - 2\boldsymbol{\alpha}\cdot\boldsymbol{\rho})t] \right\}. \tag{3.5}$$

[7]However, notice that $\langle \varphi \rangle = 0$ identically for $A_r^{(1)}$ series since $\mu' = \mu$, $l^2 = 2$ and $\Sigma_{\boldsymbol{\alpha}>0} \boldsymbol{\alpha} \sinh[(2\boldsymbol{\alpha}\cdot\mathbf{Q} - hQ)bt] = 0$ [37,12].

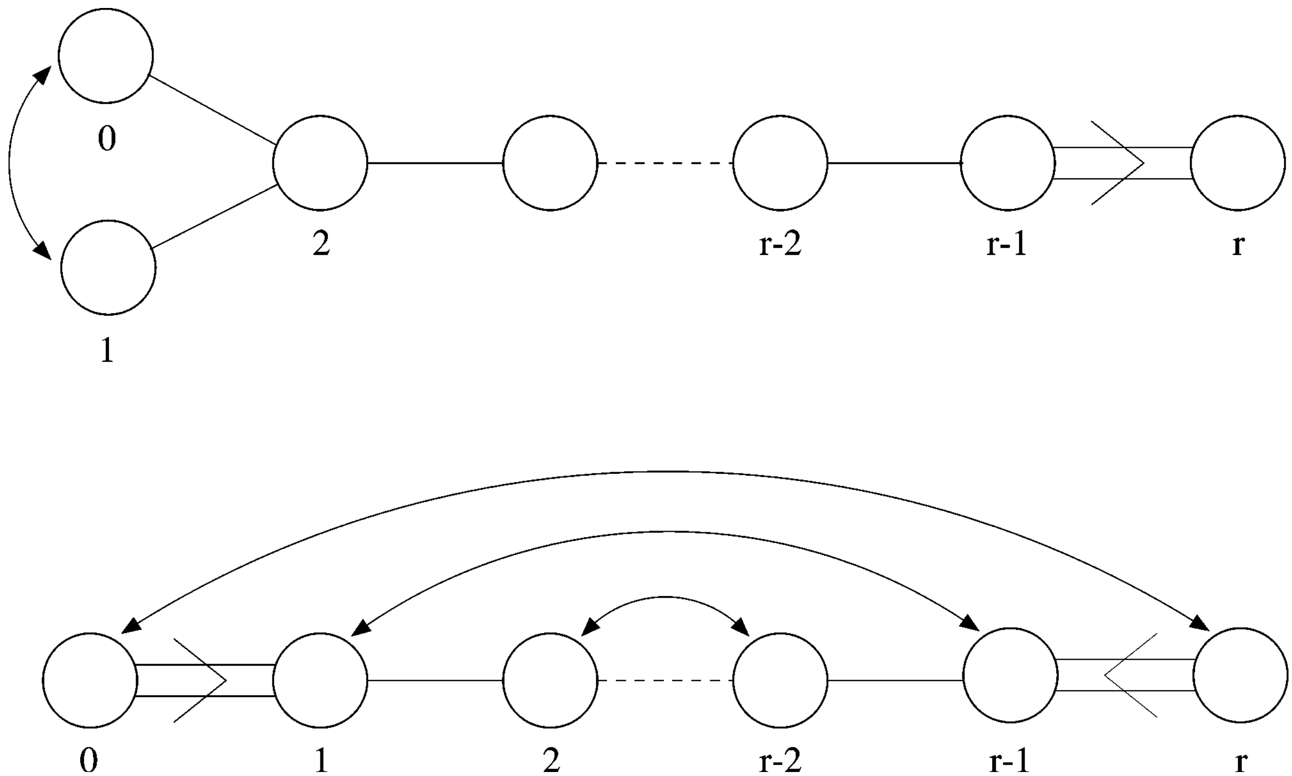


FIG. 1. Automorphisms associated with the Dynkin diagram $B_r^{(1)}$ and $C_r^{(1)}$ corresponding to Eq. (2.16).

Let us introduce the component notation $\mathcal{K}_i = \mathbf{e}_i \cdot \mathcal{K}$ and $\mathcal{L}_i = \mathbf{e}_i \cdot \mathcal{L}$.

For the $D_r^{(1)}$ series, the nonperturbative results are given as

$$\mathcal{K}_1 = \mathcal{K}_{r-1} = \mathcal{K}_r = -\left(1 - \frac{4}{h}\right)\ln 2; \tag{3.6}$$

$$\mathcal{L}_1 = \mathcal{L}_{r-1} = \mathcal{L}_r = \frac{1}{2h}\left[\xi\left(\frac{1}{h}\right) + \xi\left(\frac{2}{h}\right) - \xi\left(\frac{1}{2} + \frac{1}{h}\right) - \xi\left(\frac{1}{2}\right)\right],$$

where $h = 2r-2$ and for $k = 2,3,...,r-2$ we have

$$\mathcal{K}_k = \ln 2 - \left(1 - \frac{4}{h}\right)\ln 2; \tag{3.7}$$

$$\mathcal{L}_k = \frac{1}{2h}\left[-\xi\left(\frac{1}{2} + \frac{k}{h}\right) - \xi\left(\frac{1}{2} + \frac{k-1}{h}\right) - \xi\left(\frac{k}{h}\right) - \xi\left(\frac{k-1}{h}\right) + \xi\left(\frac{2k-2}{h}\right) + 2\xi\left(\frac{2k-1}{h}\right) + \xi\left(\frac{2k}{h}\right)\right],$$

where we define $\xi(x) = \Psi(x) + \Psi(1-x)$ in terms of the digamma function $\Psi(x) = d \ln \Gamma(x)/dx$.

Perturbative analysis of the action (2.1) begins with shifting $\boldsymbol{\varphi} \to \boldsymbol{\varphi}_{cl} + \boldsymbol{\varphi}$ such that $\boldsymbol{\varphi}_{cl}$ satisfies the minimum of the ATFT potential. This classical solution reproduces exactly the leading term in Eq. (3.4):

$$\boldsymbol{\varphi}_{cl} \cdot \mathbf{e}_i = \mathcal{K}_i . \tag{3.8}$$

This identity provides the amusing relations among the $\gamma(x)$ functions, when x is related to Lie algebra quantity, which is observed for general case in Refs. [12,19]

One-loop perturbative calculation is conveniently done using the classical mass eigenstate representation [25]. $D_r^{(1)}$ series representation is given by

$$\mathbf{e}_1 = (-l_1^1, -l_1^2, \ldots, -l_1^{r-2}, 1, 0), \tag{3.9}$$

$$\mathbf{e}_k = (l_{k-1}^1 - l_k^1, l_{k-1}^2 - l_k^2, \ldots, l_{k-1}^{r-2} - l_k^{r-2}, 0, 0)$$

for

$$k = 2,...,r-2,$$

$$\mathbf{e}_{r-1} = (l_{r-2}^1, l_{r-2}^2, \ldots, l_{r-2}^{r-2}, 0, -1),$$

$$\mathbf{e}_r = (l_{r-2}^1, l_{r-2}^2, \ldots, l_{r-2}^{r-2}, 0, 1)$$

where $l_k^a = (2/\sqrt{h})\sin(2ak\pi/h)$.

The next-to leading order term, i.e., the field expectation value to the one-loop order $\langle\varphi\rangle_b$ is given by tadpole diagrams which in general needs to be appropriately regularized. The perturbative result is, however, finite for the mass eigenstate representation, and does not depend on the regularization scheme for $D_r^{(1)}$ series (and in general for simply laced cases).

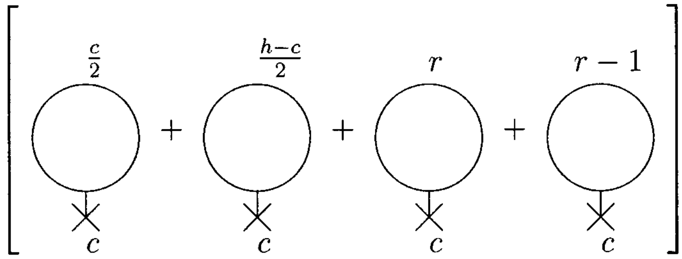
To distinguish from the component notation, φ_j, which is obtained from the proposed VEV, the perturbative mass eigenstate component is denoted as Φ^c. If the values are correct, then the relation between these two quantities should be $\varphi_j=\Sigma_c\Phi^c\mathbf{e}_j^c$ where $\mathbf{e}_j^c$ is the cth component of the mass eigenstate representation $\mathbf{e}_j$.

Φ^c vanishes when $c=r-1$, r and $c=\text{odd}\leqslant r-2$,

$$\langle\Phi^c\rangle_b=0 \tag{3.10}$$

and otherwise

$$\langle\Phi^c\rangle_b=\frac{1}{2}\Big[\ \cdots\ \Big]$$



$$=-\frac{b}{\sqrt{h}Z_c}\left(Z_{\frac{c}{2}}^2\ln(4Z_{\frac{c}{2}}^2)-Z_{\frac{h-c}{2}}^2\ln(4Z_{\frac{h-c}{2}}^2)\right), \tag{3.11}$$

where $Z_a=\sin(a\pi/h)$. The divergent terms cancel each other and the total contribution remains finite.

With the help of various relations of the digamma function and trigonometric function one can prove that $\mathcal{L}$'s in Eqs. (3.6) and (3.7) coincide with the ones in Eq. (3.11). Considering this as a nontrivial check, one can view this as a useful identity between digamma functions and trigonometric functions

$$b\mathcal{L}_i=\sum_{c=\text{even}}^{r-2}\langle\Phi^c\rangle_b\mathbf{e}_i^c. \tag{3.12}$$

For example, we have for $i=1$,

$$\xi\left(\frac{1}{h}\right)+\xi\left(\frac{2}{h}\right)-\xi\left(\frac{1}{2}+\frac{1}{h}\right)-\xi\left(\frac{1}{2}\right)$$
$$=\sum_{c=\text{even}}^{r-2}8\cos\left(\frac{c\pi}{h}\right)\left\{\sin^2\left(\frac{c\pi}{2h}\right)\ln\left[4\sin^2\left(\frac{c\pi}{2h}\right)\right]\right.$$
$$\left.-\cos^2\left(\frac{c\pi}{2h}\right)\ln\left[4\cos^2\left(\frac{c\pi}{2h}\right)\right]\right\}. \tag{3.13}$$

For the nonsimply laced case, the situation becomes more involved. By expanding Eq. (3.3), one finds the following coefficients:

$$\mathcal{K}=\frac{\mathbf{d}}{h}\ln\left(\frac{2\mu}{l^2\mu'}\right)-\sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}^\vee\ln\gamma\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\rho}^\vee}{h}\right),$$

$$\mathcal{L}=\frac{\mathbf{d}}{h}\left\{\left(\frac{l^2}{2}-1\right)[2\gamma_E+\ln(\pi\mu b^2)]+(h-h^\vee)\ln\left(\frac{2\mu}{l^2\mu'}\right)\right\}$$
$$-\int_0^\infty dt\frac{1}{\sinh(ht)}\left\{\coth(t)\sum_{\alpha>0}\alpha\sinh[(h-2\boldsymbol{\alpha}\cdot\boldsymbol{\rho}^\vee)t]\right.$$
$$\left.-\frac{2}{h}\sum_{\alpha>0}\boldsymbol{\alpha}^\vee\boldsymbol{\alpha}\cdot(h\boldsymbol{\rho}-h^\vee\boldsymbol{\rho}^\vee)\cosh[(h-2\boldsymbol{\alpha}\cdot\boldsymbol{\rho}^\vee)t]\right\}$$

where $\gamma_E=0.5772...$ is Euler's number.

The explicit value for $B_r^{(1)}$ series takes the form

$$\mathcal{K}_1=\frac{2}{h}\ln\left(\frac{2\mu'}{\mu}\right)-\ln 2; \tag{3.14}$$

$$\mathcal{K}_k=\frac{2}{h}\ln\left(\frac{2\mu'}{\mu}\right),\quad k=2,3,...,r-1;$$

$$\mathcal{K}_r=-\left(1-\frac{2}{h}\right)\ln\left(\frac{2\mu'}{\mu}\right)+\ln 2;$$

and

$$\mathcal{L}_1=\frac{1}{h}\mathcal{J}+\mathcal{I}_1+\Delta\mathcal{I}_1; \tag{3.15}$$

$$\mathcal{L}_r=\left(\frac{1}{h}-\frac{1}{2}\right)\mathcal{J}+\mathcal{I}_r+\Delta\mathcal{I}_r;$$

$$\mathcal{K}_k=\frac{1}{h}\mathcal{J}+\mathcal{I}_k+\Delta\mathcal{I}_k,\quad k=2,3,...,r-1,$$

where

$$\mathcal{J}=\left[2\gamma_E+\ln(\pi\mu b^2)+\frac{2}{h}\ln\left(\frac{\mu'}{2\mu}\right)\right] \tag{3.16}$$

and

$$\mathcal{I}_1=\frac{1}{2h}\left\{\xi\left(\frac{1}{h}\right)+\xi\left(\frac{2}{h}\right)-\xi\left(\frac{1}{2}+\frac{1}{h}\right)-\xi\left(\frac{1}{2}\right)\right\};$$

$$\mathcal{I}_r=\frac{1}{2h}\left\{2\xi\left(\frac{1}{h}\right)-\xi\left(\frac{2}{h}\right)-\xi\left(\frac{1}{2}\right)\right\};$$

$$\mathcal{I}_k=\frac{1}{2h}\left\{\xi\left(\frac{2k-2}{h}\right)+2\xi\left(\frac{2k-1}{h}\right)+\xi\left(\frac{2k}{h}\right)-\xi\left(\frac{k}{h}\right)\right.$$
$$\left.-\xi\left(\frac{k-1}{h}\right)-\xi\left(\frac{1}{2}+\frac{k-1}{h}\right)-\xi\left(\frac{1}{2}+\frac{k}{h}\right)\right\}. \tag{3.17}$$

Note that $\mathcal{I}_k$'s $(k=1,...,r-1)$ are identical to $\mathcal{L}_k$'s in Eq. (3.7) for $D_{r+1}^{(1)}$ series. $\Delta\mathcal{I}$'s are given by

$$\Delta\mathcal{I}_1=\Delta\mathcal{I}_r=\frac{1}{2h^2}\left\{-2\xi\left(\frac{1}{h}\right)+4\xi\left(\frac{2}{h}\right)-2\xi\left(\frac{1}{2}+\frac{1}{h}\right)\right\}; \tag{3.18}$$

$$\Delta\mathcal{I}_k=\frac{1}{2h^2}\left\{(4-4k)\xi\left(\frac{2k-2}{h}\right)+4k\xi\left(\frac{2k}{h}\right)-2k\xi\left(\frac{k}{h}\right)\right.$$
$$+(2k-2)\xi\left(\frac{k-1}{h}\right)+(2k-2)\xi\left(\frac{1}{2}+\frac{k-1}{h}\right)$$
$$\left.-2k\xi\left(\frac{1}{2}+\frac{k}{h}\right)\right\},$$

and turn out to be identical to each other:

$$\Delta\mathcal{I}_1=\Delta\mathcal{I}_k=\Delta\mathcal{I}_r=\frac{4}{h^2}\ln 2. \tag{3.19}$$

As noted for the simply laced case, $\mathcal{K}$ is identified with the classical value φ_{cl}. For $B_r^{(1)}$ series

$$b\mathbf{e}_i\cdot\boldsymbol{\varphi}_{cl}=\ln\left(\frac{\mu n_i}{\mu_{e_i}}\right)-\frac{1}{h}\sum_{j=0}^{r}n_j\ln\left(\frac{\mu n_j}{\mu_{\mathbf{e}_i}}\right), \tag{3.20}$$

which agrees with $\mathcal{K}$ in Eq. (3.14).

Beyond the classical result, however, renormalization should be carefully incorporated unlike in the simply laced case. The classical mass eigenstate representation of $B_r^{(1)}$ is obtained by folding the one of $D_{r+1}^{(1)}$ (3.9),

$$\mathbf{e}_1=(-l_1^1,-l_1^2,\ldots,-l_1^{r-1},1)$$

$$\mathbf{e}_k=(l_{k-1}^1-l_k^1,l_{k-1}^2-l_k^2,\ldots,l_{k-1}^{r-1}-l_k^{r-1},0)$$

for

$$k=2,...,r-1,$$

$$\mathbf{e}_r=(l_{r-1}^1,l_{r-1}^2,\ldots,l_{r-1}^{r-1},0) \tag{3.21}$$

from which we obtain the one-loop contribution $\langle\boldsymbol{\varphi}\rangle_b$:

$$\langle\Phi_r\rangle_b=0;$$

$$\langle\Phi^c\rangle_b=\frac{b}{4\sqrt{h}Z_c^2}g_rZ_{2c}\quad\text{when}\quad c=\text{odd}\leqslant r-1.0;$$

$$\langle\Phi^c\rangle_b=-\frac{b}{4\sqrt{h}Z_c^2}\{4Z_c(g_{c/2}Z_{c/2}^2-g_{(h-c)/2}Z_{(h-c)/2}^2)+g_rZ_{2c}\}$$
$$\text{when}\quad c=\text{even}\leqslant r-1, \tag{3.22}$$

where $Z_a=\sin(a\pi/h)$ as is given in Eq. (3.11). g_a is the Euclidean integration of the tadpole diagram

$$g_a\equiv\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+m_a^2}, \tag{3.23}$$

where m_a is the physical mass equivalent to M_a in Appendix A up to this order of b^2. Its explicit value is given by $m_a=2\overline{m}_0\sin(\pi a/h)$ for $a=1,2,...,r-1$ and $m_r=\overline{m}_0$ with $\overline{m}_0^2=2^{2+2/h}(\pi\mu b^2)(\mu'/\mu)^{2/h}$.

Here, to evaluate the one-loop diagram we are using the normal ordering with respect to the free field theory. In this scheme g_a is given by

$$g_a=\frac{1}{4\pi}\left[\ln\left(\frac{m_a}{2}\right)^2+2\gamma_E\right]=\left[\mathcal{J}+\ln\left(\frac{m_a}{m_0}\right)^2+\frac{2}{h}\ln 2\right]. \tag{3.24}$$

Then, using the identity

$$\left(\sum_{c=\text{odd}}^{r-1}-\sum_{c=\text{even}}^{r-1}\right)\csc^2\left(\frac{c\pi}{h}\right)\sin\left(\frac{2c\pi}{h}\right)\sin\left(\frac{2kc\pi}{h}\right)=\frac{k}{2h},$$
$$k=1,...,r-1, \tag{3.25}$$

we find that the $\mathcal{J}$ parts of Eq. (3.22) agree exactly with those of Eq. (3.15), i.e.,

$$b\mathcal{L}_i|_{\mathcal{J}\text{-part}}=\sum_c \mathbf{e}_i^c\langle\Phi^c\rangle_b|_{\mathcal{J}-\text{part}}.$$

Furthermore, since the term $\ln(m_a/\overline{m}_0)^2$ in Eq. (3.24) reproduces $\mathcal{I}_k$'s which are the same as $\mathcal{L}_k$'s in Eq. (3.7) of the $D_{r+1}^{(1)}$ series for $k=1,...,r-1$, the agreement (3.12) in the $D_r^{(1)}$ case immediately implies

$$b\mathcal{L}_i|_{\mathcal{I}.\text{part}}=\sum_c \mathbf{e}_i^c\langle\Phi^c\rangle_b|_{\mathcal{I}\text{-part}}.$$

Finally, $\Delta\mathcal{I}_k$ terms come from the last term $(2/h)\ln 2$ in Eq. (3.24). This establishes the exact agreement between the perturbative and nonperturbative results for the $B_r^{(1)}$ case.

For the exceptional algebra $G_2^{(1)}$, we have

$$\mathcal{K}_1=\frac{1}{2}\ln\left(\frac{\mu'}{3\mu}\right)+2\ln\gamma\left(\frac{1}{6}\right)-4\ln\gamma\left(\frac{1}{3}\right);$$

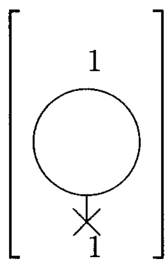
$$\mathcal{K}_2=-\frac{1}{2}\ln\left(\frac{\mu'}{3\mu}\right)-\ln\gamma\left(\frac{1}{6}\right)+2\ln\gamma\left(\frac{1}{3}\right);$$

$$\mathcal{L}_1=\frac{1}{6}\left\{4\gamma_E+\ln\left[(\pi\mu b^2)\left(\frac{\pi\mu' b^2}{3}\right)\right]\right\}+\frac{1}{2}\ln 3+\frac{2}{9}\ln 2;$$

$$\mathcal{L}_2=-\frac{1}{3}\left\{2\gamma_E+\frac{1}{2}\ln\left[(\pi\mu b^2)\left(\frac{\pi\mu' b^2}{3}\right)\right]\right\}-\frac{2}{9}\ln 2-\frac{1}{4}\ln 3. \tag{3.26}$$

On the other hand, the corresponding one-loop diagram is given by

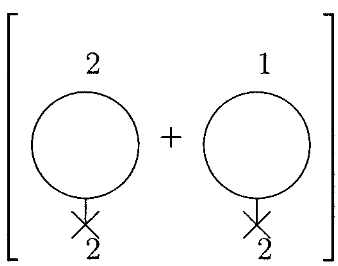
$$\langle \Phi^1 \rangle_b = \frac{1}{2}$$



(3.27)

and

$$\langle \Phi^2 \rangle_b = \frac{1}{2}$$



(3.28)

After explicit calculations as in the previous case, we find $\langle\Phi^2\rangle_b=b\mathcal{L}_2$ and $\langle\Phi^1\rangle_b=\frac{1}{2}\mathcal{L}_1+\mathcal{L}_2$ which completes the perturbative check for $G_2^{(1)}$.

B. Perturbative checks of the composite operators $\langle\varphi_a\varphi_b\rangle$

From the expression (2.17) and using Eq. (3.2) we have the VEV of composite operator

$$G^{ab}\equiv\langle\langle\varphi^a\varphi^b\rangle\rangle$$
$$=-\delta^{ab}\sum_{i=1}^{r}\frac{n_i}{H(1+b^2)}\ln[-\pi\mu_{\mathbf{e}_i}\gamma(1+b^2\mathbf{e}_i^2/2)]$$
$$+\int_0^\infty\frac{dt}{t}[\delta^{ab}e^{-2t}-\mathcal{F}^{ab}],$$

where

$$\mathcal{F}^{ab}=b^2t^2\sum_{\alpha>0}\alpha^a\alpha^b$$
$$\times\frac{\sinh[(1+b^2\alpha^2/2)t]\cosh\{[(1+b^2)H-2b\boldsymbol{\alpha}\cdot\mathbf{Q}]t\}}{\sinh(t)\sinh(b^2\boldsymbol{\alpha}^2t/2)\sinh[(1+b^2)Ht]}. \tag{3.29}$$

These are in general rather complicated quantities to calculate perturbatively due to various divergences to be taken care of up to some finite part. For some combinations such as the relative value of the composite operator $G^{aa}-G^{rr}(a=1,\dots,r-1)$, however, the propagators are renormalized with an overall renormalization constant and therefore, most of the complications due to the renormalization scheme disappears. Therefore, such quantities provides an additional independent check of the nonperturbative result in a simple way. Since the perturbative calculation is done in the classical mass eigenstate representation, in this section we will use the mass eigenstate representation for $\boldsymbol{\alpha}$ in Eq. (3.29).

For $C_2^{(1)}$, the composite operator value is given by

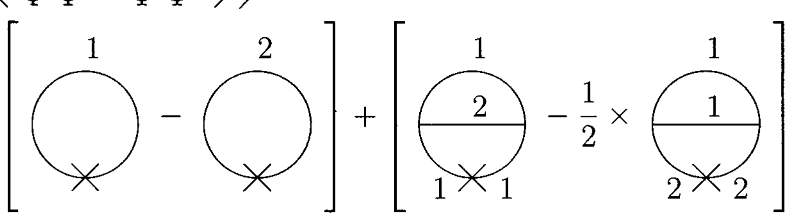
$$G^{12}=0;$$

$$G^{11}-G^{22}=\int_0^\infty dt\,\frac{b^2t\sinh(1+b^2)t[4\cosh(4+8b^2)t-4]}{\sinh t\sinh b^2t\sinh(4+6b^2)t}$$
$$=\ln 2+b^2(0.79221\ldots)+\mathcal{O}(b^4). \tag{3.30}$$

The corresponding value is confirmed perturbatively: $\langle\langle\Phi^1\Phi^2\rangle\rangle=0$ since there is no vertex at all for this case. The other one is given by

$$\langle\langle\Phi^1\Phi^1-\Phi^2\Phi^2\rangle\rangle$$

$$=$$



$$+\mathcal{O}(b^4)$$

$$=\ln 2+b^2(0.79221\ldots)+\mathcal{O}(b^4) \tag{3.31}$$

whose Feynman integration is done in Appendix C. This agrees with the nonperturbative results (3.30).

For the case $B_3^{(1)}$, the nonperturbative result gives

$$G^{12}=G^{23}=0;$$

$$G^{11}-G^{33}=-2\int_0^\infty \frac{dt}{t}(\mathcal{F}^{11}-\mathcal{F}^{33})$$
$$=b^2(-0.195326\ldots)+\mathcal{O}(b^4);$$

$$G^{22}-G^{33}=-2\int_0^\infty \frac{dt}{t}(\mathcal{F}^{22}-\mathcal{F}^{33})$$
$$=-\ln 3+b^2(-0.321552\ldots)+\mathcal{O}(b^4), \quad (3.32)$$

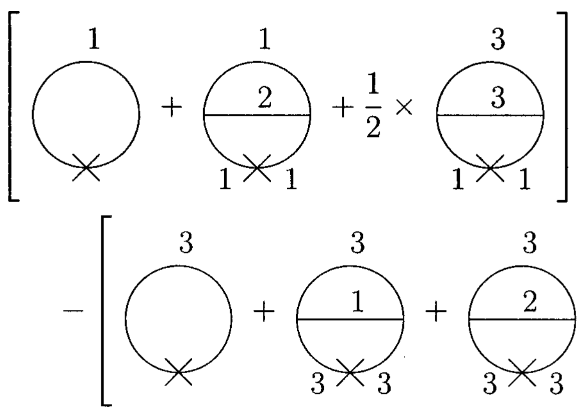
where

$$\mathcal{F}^{11}-\mathcal{F}^{33}=\frac{b^2t^2}{\sinh(t)\sinh[(6+5b^2)t]}\left\{\frac{\sinh[(1+b^2)t]}{\sinh(b^2t)}\times\{-\cosh[(4+3b^2)t]+2\cosh(b^2t)-\cosh[(2+b^2)t]\}+\frac{\sinh[(1+b^2/2)t]}{2\sinh(b^2t/2)}\times\{\cosh[(4+4b^2)t]+\cosh[(2+b^2)t]-2\}\right\};$$

$$\mathcal{F}^{22}-\mathcal{F}^{33}=\frac{b^2t^2}{\sinh(t)\sinh[(6+5b^2)t]}\left\{\frac{\sinh[(1+b^2)t]}{\sinh(b^2t)}\times\{\cosh[(4+3b^2)t]-\cosh[(2+b^2)t]\}+\frac{\sinh[(1+b^2/2)t]}{2\sinh(b^2t/2)}\{\cosh[(4+4b^2)t]+\cosh[(2+b^2)t]-2\}\right\}.$$

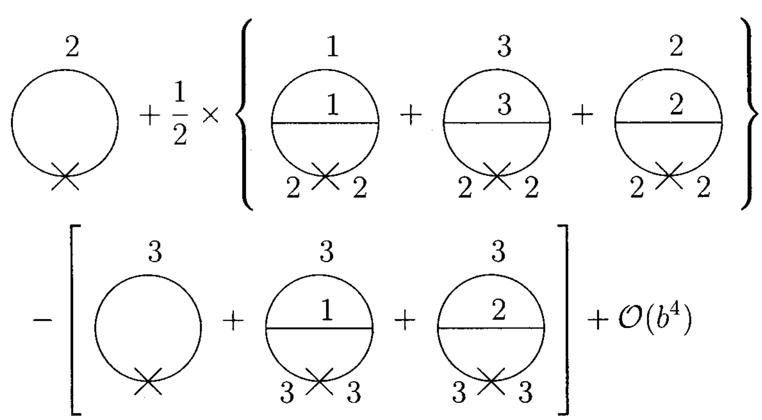
The perturbative calculation gives the result $\langle\langle\Phi^1\Phi^3\rangle\rangle=\langle\langle\Phi^2\Phi^3\rangle\rangle=0$ since there is no vertex at all in this case. The relative values of the composite operators are

$$\langle\langle\,\Phi^1\Phi^1-\Phi^3\Phi^3\,\rangle\rangle =$$



$$+\,\mathcal{O}(b^4)$$
$$=b^2(-0.195326\ldots)+\mathcal{O}(b^4)\;; \quad (3.33)$$

$$\langle\langle\,\Phi^2\Phi^2-\Phi^3\Phi^3\,\rangle\rangle =$$



$$+\,\mathcal{O}(b^4)$$
$$=-\ln 3+b^2(-0.321552\ldots)+\mathcal{O}(b^4) \quad (3.34)$$

which exactly reproduces Eq. (3.32).

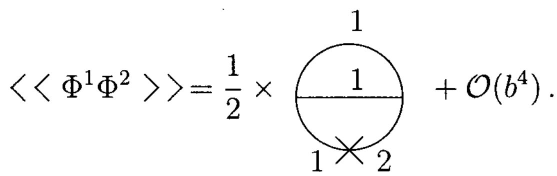
A similar check can be done for $G_2^{(1)}$. G^{12} is not vanishing but is given by

$$G^{12}=-\sqrt{3}b^2\sum_{\alpha>0}\left[\left(\frac{1}{2}\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2\right)\cdot\boldsymbol{\alpha}\right](\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha})$$

$$\times\int dt\,\frac{t\sinh[(1+b^2\alpha^2/2)t]\cosh[(1+b^2)H-2b\boldsymbol{\alpha}\cdot\mathbf{Q}]t}{\sinh(t)\sinh(b^2\boldsymbol{\alpha}^2t/2)\sinh[(1+b^2)Ht]}$$

$$=\frac{2}{\sqrt{3}}b^2(0.0488314\ldots)+\mathcal{O}(b^4) \tag{3.35}$$

whose value is also obtained from the perturbative diagram



(3.36)

Finally, the relative value of the composite operators

$$G^{11}-G^{22}=-\int_0^\infty\frac{dt}{\sinh(t)\sinh[(6+4b^2)t]}$$

$$\times\{\sinh[(1+b^2)t][\cosh(2t)$$

$$-\cosh(4t+2b^2t)]\}$$

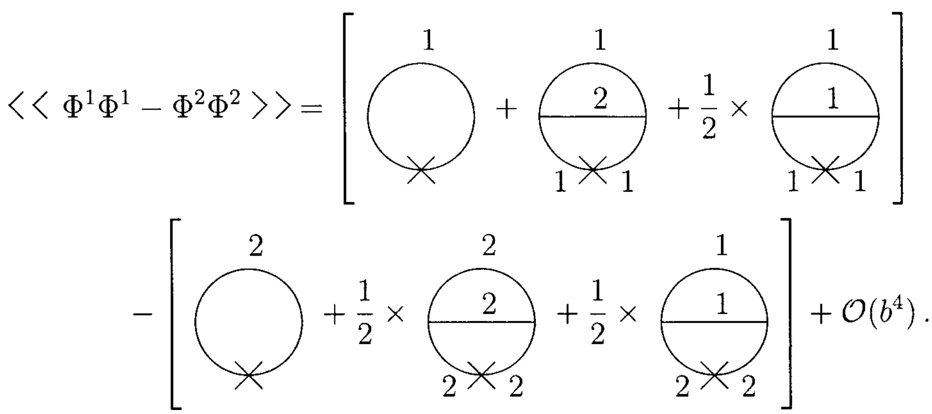
$$+\sinh\left(t+\frac{b^2t}{3}\right)\left[2\cosh\left(\frac{2b^2t}{3}\right)\right.$$

$$-\cosh\left(4t+\frac{10b^2t}{3}\right)-$$

$$\left.\cosh\left(2t+\frac{4b^2t}{3}\right)\right]$$

$$=\frac{1}{2}\ln 3+\frac{2}{3}b^2(0.183165\ldots)+\mathcal{O}(b^4) \tag{3.37}$$

is reproduced by the following perturbative diagrams:



(3.38)

It is straightforward to generalize the above perturbative calculation to other remaining cases and to confirm the proposed VEV.

ACKNOWLEDGMENTS

We thank V. A. Fateev for valuable discussions. P.B. thanks KIAS for hospitality where part of this work was done. This work is supported in part by Grant No. MOST 98-N6-01-01-A-05 (C.A.), and KOSEF 1999-2-112-001-5(C.A., C.R.). P.B.'s work is supported in part by the EU under Contract Nos. ERBFMRX CT960012 and HPMF-CT-1999-00094.

APPENDIX A

In contrast to the simply laced case for which the mass ratios correspond to the classical values, mass ratios for non-simply laced case get quantum corrections [24,25]. The mass spectrum for the dual cases remains the same with the change $b\to 1/b$, where the mass spectrum depends only on one parameter $\overline{m}$:

$$B_r^{(1)}:M_r=\overline{m},\quad M_a=2\overline{m}\sin(\pi a/H),\quad a=1,2,...,r-1,$$

$$C_r^{(1)}:M_a=2\overline{m}\sin(\pi a/H),\quad a=1,2,...,r,$$

$$G_2^{(1)}:M_1=\overline{m},\quad M_2=2\overline{m}\cos[\pi(1/3-1/H)],$$

$$F_4^{(1)}:M_1=\overline{m},\quad M_2=2\overline{m}\cos[\pi(1/3-1/H)],$$

$$M_3=2\overline{m}\cos[\pi(1/6-1/H)],\quad M_4=2M_2\cos(\pi/H). \tag{A1}$$

For nonsimply laced Lie algebras, the Coxeter and dual Coxeter numbers are

$$h_{B_r^{(1)}}=h_{(C_r^{(1)})}=2r, \quad h_{(B_r^{(1)})^\vee}=h_{A_{2r-1}^{(2)}}=2r-1,$$

$$h_{(C_r^{(1)})^\vee}=h_{D_{r+1}^{(2)}}=2(r+1),$$

$$h_{F_4^{(1)}}=12, \quad h_{(F_4^{(1)})^\vee}=9, \quad h_{G_2^{(1)}}=6, \quad h_{(G_2^{(1)})^\vee}=h_{D_4^{(3)}}=4.$$

APPENDIX B: NOTATIONS

$$\Phi_{\mathbf{s}}(A_2)=\sqrt{2}\epsilon_2, \quad \sqrt{3/2}\epsilon_1-1/\sqrt{2}\epsilon_2;$$

$$\Phi_{\mathbf{s}}(B_r)=\epsilon_i-\epsilon_{i+1}, \quad 1\leqslant i\leqslant r-1, \quad \epsilon_r;$$

$$\Phi_{\mathbf{s}}(C_r)=\epsilon_i-\epsilon_{i+1}, \quad 1\leqslant i\leqslant r-1, \quad 2\epsilon_r;$$

$$\Phi_{\mathbf{s}}(D_r)=\epsilon_i-\epsilon_{i+1}, \quad 1\leqslant i\leqslant r-1, \quad \epsilon_r+\epsilon_{r-1};$$

$$\Phi_{\mathbf{s}}(F_4)=\epsilon_i-\epsilon_{i+1}, \quad i\in\{2,3\},\epsilon_4, \quad \frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4);$$

$$\Phi_{\mathbf{s}}(G_2)=\sqrt{2/3}\epsilon_3, \quad 1/\sqrt{2}\epsilon_1-\sqrt{3/2}\epsilon_2;$$

and

$$\bar{\Phi}_{\mathbf{s}}(C_r)=\Phi_{\mathbf{s}}(C_r)\big|_{\epsilon_i\leftrightarrow\epsilon_{r+1-i}}; \quad \bar{\Phi}_{\mathbf{s}}(D_r)=\Phi_{\mathbf{s}}(D_r)\big|_{\epsilon_i\leftrightarrow\epsilon_{r+1-i}};$$

$$\bar{\Phi}_{\mathbf{s}}(A_2)=\Phi_{\mathbf{s}}(A_2)\big|_{\epsilon_1\leftrightarrow\epsilon_2};$$

$$\bar{\Phi}_{\mathbf{s}}(B_4)=\Phi_{\mathbf{s}}(B_4)\big|_{\epsilon_i\leftrightarrow-\epsilon_i, \quad i\in\{2,3,4\}}.$$

APPENDIX C: FEYNMAN INTEGRALS

The Feynman integration for $C_2^{(1)}$ in Eq. (3.31) is presented as the following. The lowest order diagrams (order of b^0) are represented as the Feynman integration

$$\langle\langle\Phi^1\Phi^1-\Phi^2\Phi^2\rangle\rangle_0=-4\pi\int\frac{d^2p_E}{(2\pi)^2}\left(\frac{1}{p_E^2+M_1^2}-\frac{1}{p_E^2+M_2^2}\right)$$
$$=\ln\frac{M_2^2}{M_1^2}=\ln 2, \tag{C1}$$

where p_E is the Euclidean momentum. M_i is the physical mass and its value at the integration is considered up to this appropriate perturbative order in b. Since the wave function and mass renormalization is already done, the next-to-leading order diagrams (order of b^2) are represented as

$$\langle\langle\Phi^1\Phi^1-\Phi^2\Phi^2\rangle\rangle_b$$
$$=(4\pi)^2\int\frac{d^2p_E}{(2\pi)^2}\left(\frac{32I_{12}}{(p_E^2+M_1^2)^2}-\frac{16I_{11}}{p_E^2+M_2^2}\right)$$
$$=0.79221\ldots, \tag{C2}$$

where

$$I_{ij}=\int\frac{d^2k_E}{(2\pi)^2}\frac{1}{(k_E^2+M_i)^2}\frac{1}{(k_E+p_E)^2+M_j{}^2}$$
$$=\int_0^1\frac{dx}{4\pi}\frac{1}{-x(1-x)p_E^2+(1-x)M_i^2+xM_j^2}. \tag{C3}$$

The Feynman integrations (3.33) and (3.34) of the next-to leading order for $B_3^{(1)}$ are given by

$$\langle\langle\Phi^1\Phi^1-\Phi^3\Phi^3\rangle\rangle_b$$
$$=(4\pi)^2\int\frac{d^2p_E}{(2\pi)^2}\left(\frac{(I_{33}+2I_{12})}{(p_E^2+M_1^2)^2}-\frac{(2I_{13}+2I_{33})}{(p_E^2+M_3^2)^2}\right)$$
$$=-0.195326\ldots,$$

$$\langle\langle\Phi^2\Phi^2-\Phi^3\Phi^3\rangle\rangle_b$$
$$=(4\pi)^2\int\frac{d^2p_E}{(2gp)^2}\left(\frac{(I_{33}+I_{11}+9I_{22})}{(p_E^2+M_2^2)^2}-\frac{(2I_{13}+2I_{23})}{(p_E^2+M_3^2)^2}\right)$$
$$=-0.321552\ldots. \tag{C4}$$

The Feynman integrations (3.36) and (3.38) of the next-to leading order for $G_2^{(1)}$ are evaluated respectively as

$$\langle\langle\Phi^1\Phi^2\rangle\rangle_b=(4\pi)^2\frac{2}{\sqrt{3}}\int\frac{d^2p_E}{(2\pi)^2}\left(\frac{I_{11}}{(p_E^2+M_1^2)(p_E^2+M_2^2)}\right)$$
$$=\frac{2}{\sqrt{3}}\times 0.0488314\ldots,$$

$$\langle\langle\Phi^1\Phi^1-\Phi^2\Phi^2\rangle\rangle_b$$
$$=(4\pi)^2\int\frac{d^2p_E}{(2\pi)^2}\left(\frac{\left(2I_{12}+\frac{4}{3}I_{11}\right)}{(p_E^2+M_1^2)^2}-\frac{(9I_{12}+I_{11})}{p_E^2+M_2^2}\right)$$
$$=\frac{2}{3}\times 0.183165\ldots. \tag{C5}$$

[1] A. Patashinskii and V. Pokrovskii, *Fluctuation Theory of Phase Transitions* (Pergamon, Oxford, 1979).
[2] M. Shifman, A. Vainstein, and V. Zakharov, Nucl. Phys. **B147**, 385 (1979).
[3] Al. Zamolodchikov, Nucl. Phys. **B348**, 619 (1991).
[4] S. Lukyanov and A. Zamolodchikov, Nucl. Phys. **B493**, 571 (1997).
[5] V. A. Fateev, S. Lukyanov, A. Zamolodchikov, and Al. B. Zamolodchikov, Phys. Lett. B **406**, 83 (1997).
[6] A. B. Zamolodchikov and Al. B. Zamolodchikov, Nucl. Phys.

B477, 577 (1996).
[7] V. A. Fateev, S. Lukyanov, A. Zamolodchikov, and Al. B. Zamolodchikov, Nucl. Phys. **B516**, 652 (1998).
[8] A. LeClair, Phys. Lett. B **230**, 423 (1989); N. Rechetikhin and F. Smirnov, Commun. Math. Phys. **131**, 157 (1990); D. Bernard and A. Leclair, Nucl. Phys. **B340**, 721 (1990); C. J. Efthimiou, *ibid.* **B398**, 697 (1993).
[9] F. Smirnov, Int. J. Mod. Phys. A **6**, 1407 (1991).
[10] R. Guida and N. Magnoli, Phys. Lett. B **411**, 127 (1997).
[11] P. Baseilhac and V. A. Fateev, Nucl. Phys. **B532**, 567 (1998).
[12] V. A. Fateev, Mod. Phys. Lett. A **15**, 259 (2000).
[13] C. Tracy and H. Widom, Commun. Math. Phys. **190**, 697 (1998).
[14] H. J. de Vega and V. A. Fateev, J. Phys. A **25**, 2693 (1992); F. A. Smirnov, Int. J. Mod. Phys. A **6**, 1407 (1991); V. A. Fateev, *ibid.* **6**, 2109 (1991).
[15] I. Vaysburd, Phys. Lett. B **335**, 161 (1994); Nucl. Phys. **B446**, 387 (1995).
[16] A. LeClair, A. W. W. Ludwig, and G. Mussardo, Nucl. Phys. **B512**, 523 (1998).
[17] V. A. Fateev and A. B. Zamolodchikov, Sov. Phys. JETP **63**, 215 (1985); Phys. Lett. A **92**, 37 (1992).
[18] P. Baseilhac, Nucl. Phys. **B594**, 607 (2001).
[19] C. Ahn, P. Baseilhac, V. A. Fateev, C. Kim, and C. Rim, Phys. Lett. B **481**, 114 (2000).
[20] C. Ahn, C. Kim, and C. Rim, Nucl. Phys. **B556**, 505 (1999).
[21] C. Ahn, V. A. Fateev, C. Kim, C. Rim, and B. Yang, Nucl. Phys. **B565**, 611 (2000).
[22] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).
[23] Al. B. Zamolodchikov, Nucl. Phys. **B342**, 695 (1990).
[24] G. W. Delius, M. T. Grisaru, and D. Zanon, Nucl. Phys. **B382**, 365 (1992).
[25] E. Corrigan, P. E. Dorey, and R. Sasaki, Nucl. Phys. **B408**, 579 (1993).
[26] V. V. Mkhitaryan, R. H. Poghossian, and T. A. Sedrakyan, ‘‘Perturbation theory in radial quantization approach and the expectation values of exponential fields in sine-Gordon model,’’ hep-th/9910128.
[27] R. H. Poghossian, Nucl. Phys. **B570**, 506 (2000).
[28] T. Hollowood and P. Mansfield, Phys. Lett. B **226**, 73 (1989).
[29] T. Curtright and C. Thorn, Phys. Rev. Lett. **48**, 1309 (1982); E. Braaten, T. Curtright, and C. Thorn, Phys. Lett. **118B**, 115 (1982); Ann. Phys. (N.Y.) **147**, 365 (1983).
[30] J. L. Gervais and A. Neveu, Nucl. Phys. **B238**, 125 (1984); **B238**, 396 (1984); **B257[FS14]**, 59 (1985).
[31] V. Fateev and S. Lukyanov, Sov. Sci. Rev., Sect. A **212**, 212 (1990).
[32] P. Bouwknegt and K. Schoutens, *W-symmetry* (World Scientific, Singapore, 1995).
[33] A. B. Zamolodchikov, Adv. Stud. Pure Math. **19**, 641 (1989).
[34] Al. B. Zamolodchikov, Int. J. Mod. Phys. A **10**, 1125 (1995).
[35] V. A. Fateev, Phys. Lett. B **324**, 45 (1994).
[36] P. Baseilhac, JHEP Proceedings of the 4th annual TMR conference ‘‘Nonperturbative quantum effects 2000,’’ hep-th/0009245.
[37] C. Destri and H. J. de Vega, Nucl. Phys. **B358**, 251 (1991).