Area spectrum in Lorentz covariant loop gravity

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We use the manifestly Lorentz covariant canonical formalism to evaluate eigenvalues of the area operator acting on Wilson lines. To this end we modify the standard definition of the loop states to make it applicable to the present case of noncommutative connections. The area operator is diagonalized by using the usual shift ambiguity in the definition of the connection. The eigenvalues are then expressed through quadratic Casimir operators. No dependence on the Immirzi parameter appears.

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I. INTRODUCTION

Quantization of gravity is an extremely hard and interesting problem that remains unsolved so far. During recent years, a number of approaches have achieved definite progress in treating various aspects of quantum gravity. The most elaborated and popular line of research is string theory, which includes perturbative gravity in its spectrum and unifies it with other interactions. An alternative (or, perhaps, complementary) approach is loop quantum gravity [1] (for review, see Ref. [2]). This program relies on the Dirac canonical quantization. It is explicitly nonperturbative and background independent and thus it realizes the basic principles of general relativity. During the previous decade this approach has got rigorous mathematical foundations [3] and has led to interesting qualitative predictions about quantum spacetime.

These predictions originate from remarkable results obtained in the framework of loop quantum gravity, which are calculations of the volume and area spectra [4–6]. It appeared, however, that the area spectrum depends on the so-called Immirzi parameter [7]. It parametrizes a canonical transformation [8], which introduces a new connection field. The reason for this dependence is that this transformation cannot be realized unitarily in the Hilbert space of quantum theory [9]. In the language of quantum-field theory this indicates the presence of a quantum anomaly. There exist two different types of the quantum anomalies. The first type of anomalies appear when a symmetry of the classical action cannot be preserved by quantization due to divergencies or other quantum effects. Chiral and conformal anomalies belong to this type. Their presence indicates emergence of a new physics. The most celebrated example is the chiral anomaly in QCD, which has been used for description of the low-energy hadron physics since late 1960s. Rather naturally, it has been suggested [9] that the anomaly in the mentioned canonical transformation belongs to this type and, consequently, the Immirzi parameter is a new fundamental constant.

One cannot, however, exclude the second possibility. An anomaly could appear if a symmetry is involuntarily broken by the choice of a particular quantization scheme. If this is the case, the remedy can be in applying another quantization scheme that explicitly preserves as many important symmetries as possible. This is the route we take in the present paper by applying the manifestly Lorentz covariant quantization of Ref. [10] to the calculation of the area spectrum.

There is already some evidence that the Immirzi parameter dependence may disappear in a more symmetric quantization scheme. In Ref. [10], the path-integral quantization scheme of Ref. [11] has been extended to arbitrary values of the Immirzi parameter. It has been demonstrated that the Immirzi parameter dependence does not appear in the path integral. We should stress that in principle, the path-integral formalism is capable of seeing nonperturbative effects (as, e.g., the virtual black-hole formation [12]). Another important result was obtained recently by Samuel [13] who demonstrated that the Barbero connection is not a Lorentz connection.

Recently, it was recognized that it was important for the theory to be Lorentz covariant in spin foam models [14], which represent the modern development of loop quantum gravity [15]. However, the Lorentz covariance has been introduced there without any reference to the canonical quantization. It is an important task to develop a Lorentz covariant formulation "from the first principles."

In this paper we apply the Lorentz covariant canonical quantization developed in Ref. [10] to loop quantum gravity. We rederive the spectrum of the area operator in the new framework. To this end we construct the Wilson line operator with true Lorentz connection. Since the Dirac brackets of the connections are nonzero, there is no connection representation. However, by choosing an appropriate vacuum state, we are able to construct the quantum states corresponding to the Wilson lines, which behave in a very similar way to the ordinary loop states. However, the area operator is not nec-

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essarily diagonal on these states. To diagonalize this operator, we use the usual ambiguity in the connection: any connection can be shifted by a vector and will still remain a proper connection. It appears, that the shift is uniquely defined by the requirement that it vanishes on the constraint surface and that the area operator is diagonal on the Wilson line states. This new connection obeys a remarkably simple bracket algebra. Eigenvalues of the area operator are then calculated. They *do not* depend on the Immirzi parameter.

The paper is organized as follows. In Sec. II we summarize the covariant canonical formulation of Ref. [10]. In Sec. III we discuss the choice of the connection variables to be used in the Wilson line states. The area spectrum is calculated in Sec. IV. Section V is devoted to discussion of the results, problems, and future perspectives. The appendices are intended to list various definitions and useful properties.

We use the following notations for indices. The indices $i,j,\ldots$ from the middle of the alphabet label the space coordinates. The latin indices $a,b,\ldots$ from the beginning of the alphabet are the so(3) indices, whereas the capital letters $X,Y,\ldots$ from the end of the alphabet are the so(3,1) indices.

II. so(3,1)-COVARIANT CANONICAL FORMULATION

In this section we review the covariant formalism developed in Ref. [10]. It is a canonical formulation of general relativity based on the generalized Hilbert-Palatini action suggested by Holst [16]:

$$S_{(\beta)}=\frac{1}{2}\int \varepsilon_{\alpha\beta\gamma\delta}e^{\alpha}\wedge e^{\beta}\wedge\left(\Omega^{\gamma\delta}+\frac{1}{\beta}\star\Omega^{\gamma\delta}\right). \qquad (1)$$

Here the star operator is defined as $\star\omega^{\alpha\beta}=\frac{1}{2}\varepsilon^{\alpha\beta}{}_{\gamma\delta}\omega^{\gamma\delta}$, and $\Omega^{\alpha\beta}$ is the curvature of the spin connection $\omega^{\alpha\beta}$. A $3+1$ decomposition of the fields reads

$$e^0=N\,dt+\chi_a E^a_i\,dx^i,\quad e^a=E^a_i\,dx^i+E^a_iN^i\,dt,$$
$$\tilde{E}^i_a=h^{1/2}E^i_a,\quad \underset{\sim}{N}=h^{-1/2}N,\quad \sqrt{h}=\det E^a_i,$$
$$N^i=\mathcal{N}^i_D+\tilde{E}^i_a\chi^a\underset{\sim}{\mathcal{N}},\quad \underset{\sim}{N}=\underset{\sim}{\mathcal{N}}+\underset{\sim}{E}{}^a_i\chi_a\mathcal{N}^i_D. \qquad (2)$$

Here E^i_a is the inverse of E^a_i. The field χ_a describes the deviation of the normal to the spacelike hypersurface $\{t=0\}$ from the time direction.

Let us introduce matrix fields carrying one Lorentz index

$$A^X=(\tfrac{1}{2}\omega^{0a},\tfrac{1}{2}\varepsilon^a{}_{bc}\omega^{bc})\text{—connection multiplet,}$$
$$\tilde{P}^i_X=(\tilde{E}^i_a,\varepsilon_a{}^{bc}\tilde{E}^i_b\chi_c)\text{—first triad multiplet,}$$
$$\tilde{Q}^i_X=(-\varepsilon_a{}^{bc}\tilde{E}^i_b\chi_c,\tilde{E}^i_a)\text{—second triad multiplet,}$$
$$\tilde{P}^i_{(\beta)X}=\tilde{P}^i_X-\frac{1}{\beta}\tilde{Q}^i_X\text{—canonical triad multiplet,} \qquad (3)$$

which form multiplets in the adjoint representation of so(3,1). In Appendix A we present the relations between the triad multiplets and introduce the numerical matrices Π and R [Eqs. (A2) and (A3)] appearing in the formulas below. In terms of these fields, the decomposed action can be represented in the form

$$S_{(\beta)}=\int dt\,d^3x(\tilde{P}^i_{(\beta)X}\partial_t A^X_i+\mathcal{N}^X_{\mathcal{G}}\mathcal{G}_X+\mathcal{N}^i_D H_i+\underset{\sim}{\mathcal{N}}H),$$
$$\mathcal{G}_X=\partial_i\tilde{P}^i_{(\beta)X}+f^Z_{XY}A^Y_i\tilde{P}^i_{(\beta)Z},$$
$$H_i=-\tilde{P}^j_{(\beta)X}F^X_{ij},$$
$$H=-\frac{1}{2[1+(1/\beta^2)]}\tilde{P}^i_{(\beta)X}\tilde{P}^j_{(\beta)Y}f^{XY}_Z R^Z_W F^W_{ij},$$
$$F^X_{ij}=\partial_iA^X_j-\partial_jA^X_i+f^X_{YZ}A^Y_iA^Z_j, \qquad (4)$$

where f^Z_{XY} are so(3,1) structure constants, $\mathcal{N}^X_{\mathcal{G}}=A^X_0$. The so(3,1) indices are raised and lowered with the help of the Killing form

$$g_{XY}=\frac{1}{4}f^{Z_2}_{XZ_1}f^{Z_1}_{YZ_2},\quad g^{XY}=(g^{-1})^{XY},\quad g_{XY}=\begin{pmatrix}\delta_{ab} & 0\\ 0 & -\delta_{ab}\end{pmatrix}. \qquad (5)$$

The limit $\beta\to i$ gives Ashtekar gravity. Even though the Hamiltonian constraint H in Eq. (4) has apparently a pole at $\beta=i$, one can demonstrate [10] that this limit is nonsingular.

The canonical variables of the model are A^X_i and $\tilde{P}^i_{(\beta)X}$. G_X, H_i, and H are first class constraints obeying the algebra presented in Appendix C. We call them the Gauss law, diffeomorphism, and Hamiltonian constraints, respectively. There are also two sets of the second class constraints. They are represented by 3×3 symmetric fields

$$\phi^{ij}=\Pi^{XY}\tilde{Q}^i_X\tilde{Q}^j_Y=0, \qquad (6)$$
$$\psi^{ij}=f^{XYZ}\tilde{Q}^{[l}_X\tilde{Q}^{\{j]}_Y\partial_l\tilde{Q}^{i\}}_Z-2(\tilde{Q}\tilde{Q})^{\{i[j\}}\tilde{Q}^{l]}_ZA^Z_l=0, \qquad (7)$$
$$(\tilde{Q}\tilde{Q})^{ij}=g^{XY}\tilde{Q}^i_X\tilde{Q}^j_Y. \qquad (8)$$

Symmetrization is taken with the weight 1/2. Antisymmetrization includes no weight.

The existence of the second class constraints gives rise to the Dirac bracket [17]

$$\{K,L\}_D=\{K,L\}-\{K,\varphi_r\}(\Delta^{-1})^{rr'}\{\varphi_{r'},L\}, \qquad (9)$$

where $\varphi_r=(\phi^{ij},\psi^{ij})$. The matrix of commutators of the second class constraints $\Delta^{rr'}$ can be found in Appendix B. Both Δ and Δ^{-1} are triangular. Due to this, when one of the functions in Eq. (9), K or L is a first class constraint, the Dirac bracket coincides with the ordinary one (except for the case when $K=H$ and L depends on the connection). In particular, this gives

$$\{\mathcal{G}_X,\mathcal{G}_Y\}_D=f^Z_{XY}\mathcal{G}_Z, \qquad (10)$$

$$\{\mathcal{G}_X, A_i^Y\}_D = \delta_X^Y \partial_i - f_{XZ}^Y A_i^Z,$$

$$\{\mathcal{G}_X, \tilde{P}_Y^i\}_D = f_{XY}^Z \tilde{P}_Z^i.$$

Finally, the Dirac brackets of the canonical variables have the form

$$\{\tilde{P}^i_{(\beta)X}, \tilde{P}^j_{(\beta)Y}\}_D = 0,$$

$$\{A_i^X, \tilde{P}^j_{(\beta)Y}\}_D = \delta_i^j \delta_Y^X - \tfrac{1}{2} R^{XZ} (\tilde{Q}_Z^j \tilde{Q}_i^W + \delta_i^j I^W_{(Q)Z}) g_{WY},$$

$$\{A_i^X, A_j^Y\}_D = -\{A_i^X, \phi^{kl}\}(D_1^{-1})_{(kl)(mn)}\{\psi^{mn}, A_r^Z\} \times \{\tilde{P}^r_{(\beta)Z}, A_j^Y\}_D - \{A_i^X, \tilde{P}^r_{(\beta)Z}\}_D \{A_r^Z, \psi^{mn}\} \times (D_1^{-1})_{(mn)(kl)} \{\phi^{kl}, A_j^Y\}. \tag{11}$$

Here $\underset{\sim}{Q}_i^X$ is the inverse triad multiplet and

$$I^Y_{(P)X} := \tilde{P}_X^i \underset{\sim}{P}_i^Y, \quad I^Y_{(Q)X} := \tilde{Q}_X^i \underset{\sim}{Q}_i^Y \tag{12}$$

are projectors on $\tilde{Q}$ and $\tilde{P}$ multiplets (see Appendix B for details).

Quantization may go along the usual way. We may replace the canonical variables by operators and define a commutator on them as $[.,.] := i\hbar\{.,.\}_D$. Of course, when we replace the canonical variables by operators, the right-hand side of Eq. (11) becomes ambiguous. In actual calculations of the area spectrum we will use a shifted connection $\mathcal{A}$. As we will see in Sec. III B, for this connection no ordering ambiguity appears.

III. AREA OPERATOR AND THE WILSON LINE

A. Wilson line with canonical connection

In Ref. [10] it was suggested to use the Lorentz covariant formulation described above as a basis for a modified loop approach. The key point is that A_i^X is a true Lorentz connection [Eq. (10)] and so one can construct the Wilson line operator

$$\hat{U}_\alpha(a,b) = \mathcal{P}\exp\left(\int_a^b dx^i A_i^X T_X\right), \tag{13}$$

where α is a path between two points a and b, and T_X is a gauge generator. However, we encounter a serious obstacle, since instead of simple standard canonical commutation relations, we now have a complicated algebra of the Dirac brackets (11). In particular, the operators such as Eq. (13) fail to form the loop algebra. Moreover, since the connection A_i^X is noncommutative, the connection representation does not exist.

Nevertheless, one might hope to obtain some results relying on the bracket algebra (11) only. Let us try to obtain the spectrum of the area operator extensively investigated in the framework of the standard loop approach [4,5]. Here we follow the line of reasonings suggested in Ref. [2]. In particular, we use the same regularization technique for the area operator. Namely, we define the operator of the triad smeared over a two-dimensional surface embedded in the three manifold:

$$\tilde{P}_X(\Sigma) = \int_\Sigma d^2\sigma\, n_i(\sigma) \tilde{P}_X^i(\sigma), \tag{14}$$

where the embedding is described by the coordinates $x^i(\vec{\sigma})$ and the normal to the surface is given by $n_i = \varepsilon_{ijk}(\partial x^j/\partial\sigma^1)(\partial x^k/\partial\sigma^2)$. Then the regularized area operator is defined as follows:

$$\mathcal{S} = \lim_{\rho\to\infty} \sum_n \sqrt{g(S_n)}, \tag{15}$$

where the sum is taken over a partition ρ of S into small surfaces S_n, $\cup_n S_n = S$, and[1]

$$g(\Sigma) = g^{XY} \tilde{P}_X(\Sigma) \tilde{P}_Y(\Sigma). \tag{16}$$

We define a state vector corresponding to the Wilson line operator $\hat{U}_\alpha$ as

$$U_\alpha = \hat{U}_\alpha |0\rangle, \tag{17}$$

where $|0\rangle$ is a vacuum state. To be as close as possible to the connection representation formalism, we require

$$\tilde{P}_X^i |0\rangle = 0. \tag{18}$$

Since $\tilde{P}_X^i$ are commutative, condition (18) is consistent. Condition (18) may lead to troubles if one acts by the inverse triad on the vacuum state. To avoid problems, one may consider a more general vacuum state with a nontrivial internal geometry

$$\tilde{P}_X^i |0\rangle = \langle \tilde{P}_X^i \rangle |0\rangle. \tag{19}$$

Consistency with the second class constraints requires that $\langle \tilde{P}_X^i \rangle$ is expressed through $\langle \tilde{E} \rangle$ and $\langle \chi \rangle$ as in Eq. (3). After the calculations, one can take $\langle \tilde{P}_X^i \rangle \to 0$. The vacuum state (19) may also be interesting in its own right (see discussion in Sec. V). We shall primarily use the simplest vacuum (18), but shall also comment, at some points, about which modifications would appear if the vacuum (19) was used instead.

We have constructed a natural generalization of the the Wilson line states for the case of a noncommutative Lorentz connection. Let us recall that the unitary representations of the Lorentz group are infinite dimensional. Therefore, it is much harder to address orthogonality, completeness, and other functional properties of the loop states than in the standard su(2) case. We will not discuss these properties here. Instead, we will concentrate on the algebraic aspect of the problem.

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[1]Being expressed through $\tilde{P}_{(\beta)}$ the operator $g(\Sigma)$ reads $\beta^2 g^{XY} \tilde{P}_{(\beta)X}(\Sigma) \tilde{P}_{(\beta)Y}(\Sigma)/(\beta^2 - 1)$. The printed version of Ref. [10] contains a mistake in this formula.

To find the area spectrum, we study the action of the smeared triad on a state created by the Wilson line. Consider the simplest situation when the path α has with the surface Σ, one intersecting point c, which breaks α in two parts, α_1 and α_2. Then the action is given by

$$\tilde{P}_X(\Sigma)\hat{U}_\alpha(a,b)|0\rangle = -\int_\Sigma d^2\sigma \int_\alpha ds\, \varepsilon_{ijl}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\frac{\partial x^k}{\partial s} \times \delta^3[\vec{x}(\sigma),\vec{x}(s)]\hat{U}_{\alpha_1}(a,c) \times [A_k^Y T_Y, \tilde{P}_X^l]\hat{U}_{\alpha_2}(c,b)|0\rangle. \quad (20)$$

Here the vacuum state (18) has been used. For the vacuum (19), an additional term $\langle \tilde{P}_{(\beta)X}(\Sigma)\rangle U_\alpha(a,b)$ appears on the right-hand side of Eq. (20).

In the standard loop approach [4,5] one has to consider the action of the smeared triad $\tilde{E}$ on the Wilson line with su(2) connection A_i^a. Therefore, Eq. (20) should be replaced by an analogous one with the commutator of the canonical variables $[A_i^a, \tilde{E}_b^j]$ on the right-hand side. This commutator is proportional to δ_i^j. Because of this fact, the explicit x dependence can be canceled, and the right-hand side of $\tilde{E}U_\alpha$ becomes, in the standard loop approach, a purely algebraic expression. As a result the area operator (that is essentially $\tilde{E}$ applied twice) can be easily diagonalized. In the present case $\{A_k^Y, \tilde{P}_X^l\}_D$ is *not* proportional to δ_k^l. Consequently, the area operator acting on the Wilson line U_α with the canonical connection A, is not just a matrix in the Lorentz indices and cannot be made diagonal that easily. A way to bypass this difficulty is suggested in Sec. III B.

B. Shifted connection

We have seen that to enable the diagonalization of the area operator, the commutator of the connection and P should be a unit matrix in the spatial indices. It is known that if one adds a vector to a connection, the resulting object will again transform as a connection. We are going to use this arbitrariness in the choice of the connection to diagonalize the area operator. We are interested in a new connection $\mathcal{A}_i^X$ such that (i) it is a true Lorentz connection, i.e., $\mathcal{A}_i^X - A_i^X$ is tensorial in both indices, (ii) the Dirac brackets $\{\mathcal{A}_k^Y, \tilde{P}_X^l\}_D$ is proportional to δ_k^l, and (iii) $\mathcal{A}_i^X - A_i^X$ is proportional to the first class constraints. These requirements appear to be very strong. There is just one connection that satisfies all of them. To show this, let us note that all the triad (or tetrad) components have dimension 0, while the connection has mass dimension 1. Consequently the Gauss constraint has dimension 1, and the diffeomorphism and Hamiltonian constraints have dimension 2. Therefore, it is clear that

$$\mathcal{A}_i^X = A_i^X + \alpha_i^{XY}(Q)\mathcal{G}_Y, \quad (21)$$

where $\alpha_i^{XY}(Q)$ does not contain derivatives or connections. The coefficient functions $\alpha_i^{XY}(Q)$ have to be tensorial in order to ensure correct diffeomorphism and Lorentz transformation properties of $\mathcal{A}$:

$$\{\mathcal{G}_X, \mathcal{A}_i^Y\}_D = \delta_X^Y\partial_i - f_{XZ}^Y\mathcal{A}_i^Z, \quad (22)$$

$$\{\mathcal{D}(\vec{N}), \mathcal{A}_i^X\}_D = \mathcal{A}_j^X\partial_i N^j + N^j\partial_j\mathcal{A}_i^X. \quad (23)$$

$\mathcal{D}(\vec{N})$ is defined in Eq. (C1). Thus $\mathcal{A}_i^X$ is the true so(3,1) connection. There is still a six-parameter family of the connections that satisfies Eqs. (22) and (23). This ambiguity is fixed uniquely by the second condition (ii). We arrive at the following Lorentz connection:

$$\mathcal{A}_i^X = A_i^X + \frac{1}{2[1+(1/\beta^2)]}R_S^X I_{(Q)}^{ST} R_T^Z f_{ZW}^Y \underset{\sim}{P}{}_i^W \mathcal{G}_Y. \quad (24)$$

The connection $\mathcal{A}_i^X$ has a very simple bracket with $\tilde{P}_Y^j$:

$$\{\mathcal{A}_i^X, \tilde{P}_Y^j\}_D = \delta_i^j I_{(P)Y}^X. \quad (25)$$

At this point we already observe independence of the right-hand side of Eq. (25) from β. It should be stressed that this β independence is *not* a prerequirement in our construction. This is rather a consequence of conditions (i)–(iii) above. We also observe

$$\{\mathcal{A}_i^X, \underset{\sim}{P}{}_j^Y\}_D = -\underset{\sim}{P}{}_j^X \underset{\sim}{P}{}_i^Y, \quad (26)$$

$$\{\mathcal{A}_i^X, I_{(P)}^{YZ}\}_D = 0. \quad (27)$$

Due to this relation, the projectors $I_{(P)}$ and $I_{(Q)}$ behave very similar to c numbers.

The Dirac bracket of two connections has a very complicated form and will not be presented here. However, an important observation can be made by considering the Jacobi identity

$$\{\{\mathcal{A}_i^X, \mathcal{A}_j^Y\}_D, \tilde{P}_Z^k\}_D = \{\{\mathcal{A}_i^X, \tilde{P}_Z^k\}_D, \mathcal{A}_j^Y\}_D - \{\{\mathcal{A}_j^Y, \tilde{P}_Z^k\}_D, \mathcal{A}_i^X\}_D = 0. \quad (28)$$

It follows from Eq. (28) that $\{\mathcal{A}_i^X, \mathcal{A}_j^Y\}_D$ does not depend on the connection. It is a function of $\tilde{Q}$ and its derivatives, i.e., this bracket contains only commuting objects on the right-hand side. Therefore, there will be no ordering ambiguity if we replace the Dirac brackets with $\mathcal{A}_i^X$ by the corresponding operator relation. We will use this as a new quantization rule. In particular,

$$[\mathcal{A}_i^X, \tilde{P}_Y^j] = i\hbar\,\delta_i^j I_{(P)Y}^X. \quad (29)$$

Note, that the commutators with the new connection (24) are insufficient to define all commutators involving the canonical connection. The reason is that the (classical) field $\mathcal{A}_i^X$ satisfies the condition

$$g_{YZ}(\delta_i^k I_{(Q)X}^Y - \tilde{Q}_i^Y\tilde{Q}_X^k)\mathcal{A}_k^Z = I_{(Q)X}^Y f_{YZ}^W \underset{\sim}{Q}{}_i^Z \partial_j \tilde{Q}_W^j \quad (30)$$

and has fewer independent components than A_i^X. From Eq. (24) it is clear that the missing components are contained in the Gauss constraint. For practical purposes it is, therefore, enough to know the commutators with $\mathcal{A}_i^X$ *and* the commutators with the Gauss constraint, which are defined either by the structure constants of the Lorentz group or by the matrix elements in corresponding representations. These quantization rules have one more important advantage. They ensure that quantum transformation laws are identical to the classical ones. So there will be no gauge anomaly for the Lorentz group.

IV. AREA SPECTRUM

The shifted connection $\mathcal{A}$ can be used as an argument for the Wilson line. Let us evaluate the action of the area operator (15) on the states created by such Wilson lines. It is given by

$$\mathcal{S}\hat{U}_\alpha[\mathcal{A}]|0\rangle=\hbar\,\hat{U}_{\alpha_1}[\mathcal{A}]\sqrt{-I_{(P)}^{XY}T_XT_Y}\hat{U}_{\alpha_2}[\mathcal{A}]|0\rangle, \quad (31)$$

where we used Eqs. (20) and (29) and the prescription [4,5] for taking the square root of the operator (assuming that the latter is still valid for the Lorentz gauge group). The vacuum state is supposed to be the trivial one [Eq. (18)].

Consider the matrix operator $I_{(P)}^{XY}T_XT_Y$. It can be rewritten as

$$I_{(P)}^{XY}T_XT_Y=g^{XY}T_XT_Y-I_{(Q)}^{XY}T_XT_Y, \quad (32)$$

where the first term is a quadratic Casimir of the Lorentz algebra:

$$g^{XY}T_XT_Y=C_2[\mathrm{so}\ (3,1)]. \quad (33)$$

In order to study the second term in Eq. (32), let us introduce the generators

$$q_a:=\frac{1}{\sqrt{1-\chi^2}}\left(\delta_{ab}-\frac{1-\sqrt{1-\chi^2}}{\chi^2}\chi_a\chi_b\right)\underset{\sim}{E}{}_i^b\tilde{Q}_X^iT^X. \quad (34)$$

One can check directly that

$$I_{(Q)}^{XY}T_XT_Y=-q_aq_a, \quad (35)$$

$$[q_a,q_b]=-\varepsilon_{ab}{}^cq_c. \quad (36)$$

Consequently, q_a generates the so (3) subalgebra of so (3,1), and $I_{(Q)}^{XY}T_XT_Y$ is the Casimir operator of this subalgebra:

$$q_aq_a=-C_2[\mathrm{so}\ (3)]. \quad (37)$$

In a suitable basis in the defining representation of so(3,1), the generators q_a annihilate the vector $v_\chi=(1-\chi^2)^{-1/2}(1,\chi_a)$. All vectors v_χ belong to the same orbit of the Lorentz group. Therefore, the subalgebras spanned by $\{q_a\}$ for different χ, are conjugate in so (3,1), and the spectrum of so (3) representations obtained after the restriction so $(3,1)\downarrow$so (3) from a given representation of so (3,1), does not depend on χ. Eigenvalues of the Casimir operator (37) are also χ independent.

The spectrum of the area operator acting on Wilson lines reads

$$\mathcal{S}\sim\hbar\sqrt{-C_2[\mathrm{so}\ (3,1)]+C_2[\mathrm{so}\ (3)]}. \quad (38)$$

This formula represents the main result of our paper.

One can think naively that the Lorentz invariance of the area spectrum (38) is broken due to the presence of the Casimir operator of a subgroup. However, this is not the case. Under local Lorentz transformations, the Wilson line changes as $U(x,y)\to\mathcal{U}(x)U(x,y)\mathcal{U}^{-1}(y)$, where $\mathcal{U}(x)$ is an element of the Lorentz group taken in an appropriate representation. The matrix operator $\sqrt{-I_{(P)}^{XY}T_XT_Y}$ changes in a similar way: $\sqrt{-I_{(P)}^{XY}T_XT_Y}\to\mathcal{U}(x)\sqrt{-I_{(P)}^{XY}T_XT_Y}\mathcal{U}^{-1}(x)$. Thus proper (covariant) transformation properties of Eq. (31) are recovered.

As expected, the area spectrum (38) does not depend on the Immirzi parameter β.

V. DISCUSSION

In this paper we analyzed the area operator spectrum in a manifestly Lorentz covariant formalism. We have constructed a generalization of the Wilson line states for the case of noncommutative connection. As usual, there is certain arbitrariness in the choice of the connection. Namely, any connection can be shifted by a vector and would still remain a connection. We use this arbitrariness to define a connection $\mathcal{A}$ such that $\{\mathcal{A}_i^X,\tilde{P}_Y^j\}_D\sim\delta_i^j$. Because of the rather simple commutation relations (29), we are able to explicitly find the area spectrum (38). Since the right-hand side of Eq. (29) does not depend on the Immirzi parameter β, there is no dependence on β in the spectrum (38) as well.

Note, that the connection $\mathcal{A}$ is unique only if we require that it coincides with A on the surface of the constraints. A different idea might be to fix the connection by considering its spacetime properties. Because of the rather complicated form of the Dirac brackets with the Hamiltonian constraint, this is, technically, a very involved calculation. We hope that the results obtained in this way will agree with our results.

We must admit that there is no proof in this paper that the area spectrum with *any* connection does not depend on β. We cannot perform direct calculations with a connection other than $\mathcal{A}$. We may, however, *interpret* the shift $A\to\mathcal{A}$ as a diagonalization of the area operator. Our results suggest, that in a Lorentz covariant quantization, the dependence of the physical quantities on the Immirzi parameter, ultimately disappears.

In addition to the explicit Lorentz covariance there is another advantage of our approach. The Hamiltonian constraint (4) is a polynomial in the canonical variables (as for the Ashtekar or Euclidean cases). Due to this, the corresponding regularized quantum operator may be similar to the first term of Thiemann's constraint operator [18]. That would eliminate difficulties created by the second term. Note that the spin foam formulation of loop quantum gravity takes into account the first term of Thiemann's Hamiltonian only [2,15].

Let us comment on the choice of the vacuum state. The connection representation implies that the trivial vacuum (18) is chosen. Such representation does not exist in our case due to the noncommutativity of the connection fields. Therefore, we must choose a vacuum state explicitly. The possibility of a more general vacuum state (19) can be taken into account. (A similar possibility has been already discussed in Ref. [19]). For the vacuum state (19) we have no problem with the action of the inverse triad on the vacuum, but we lose explicit background independence. The physical consequences of different vacua have yet to be clarified.

Even without a relation to the Immirzi parameter problem, the quantization of gravity in manifestly Lorentz covariant terms is an important task. We have considered here the algebraic part of the problem, while the functional analysis has been completely ignored. We do not know how to construct a complete orthogonal basis in the space of states out of the Wilson lines. Consequently, we may only guess which representations do actually contribute to the area spectrum (38).

The area spectrum (38) now contains the Casimir operator of a noncompact Lorentz group. Since unitary representations of the Lorentz group are labeled by a pair of indices (ρ,j), and the index ρ is continuous, we may expect that the area spectrum becomes continuous as well. This would be a new feature for the loop quantum gravity, though a continuous spectrum appears in the spin foam models [14]. However, in view of the remarks in the previous paragraph, this feature should be taken with a great amount of care.

Recently, a manifestly so(3,1)-covariant formalism has been developed in the framework of spin foam models [14]. It has been suggested to use the so-called simple representations of the Lorentz group only. The Immirzi parameter has also been included in this approach [20,21]. The area spectrum obtained in the spin foam models is different from our expression (38). The reason is that we use different quantization rules. We should stress that our commutation relations are *derived* from the gravitational action rather than postulated. Therefore, our quantization rules may provide a more solid ground for the Lorentz-invariant spin foam models. Despite complicated Dirac brackets, our final commutation relations (29) are rather simple. It should be possible to use them in the spin foam approach.

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APPENDIX A: MATRIX ALGEBRA

In the basis (3) the so (3,1) structure constants are

$$f^{A_3}_{A_1A_2}=0,\quad f^{A_3}_{A_1B_2}=-\varepsilon^{A_1B_2A_3},\quad f^{A_3}_{B_1B_2}=0,$$

$$f^{B_3}_{B_1B_2}=-\varepsilon^{B_1B_2B_3},\quad f^{B_3}_{A_1B_2}=0,\quad f^{B_3}_{A_1A_2}=\varepsilon^{A_1A_2B_3}. \tag{A1}$$

Here we split the six-dimensional index X into a pair of three-dimensional indices, $X=(A,B)$, so that $A,B=1,2,3$. ε is the Levi-Civita symbol, $\varepsilon^{123}=1$.

All triad multiplets are connected by numerical matrices:

$$\tilde{P}^i_X=\Pi^Y_X\tilde{Q}^i_Y,\quad \Pi^Y_X=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\delta^b_a, \tag{A2}$$

$$\tilde{P}^i_X=\frac{R^Y_X}{1+(1/\beta^2)}\tilde{P}^i_{(\beta)Y},$$

$$R^Y_X=\begin{pmatrix}1&-\dfrac{1}{\beta}\\ \dfrac{1}{\beta}&1\end{pmatrix}\delta^b_a. \tag{A3}$$

They, as well as their inverse, commute with each other and, furthermore, they commute with the structure constants in the following sense:

$$f^{XYZ'}\Pi^Z_{Z'}=f^{XY'Z}\Pi^Y_{Y'}. \tag{A4}$$

Other useful relations can be found in Ref. [10].

APPENDIX B: INVERSE MULTIPLETS AND PROJECTORS

The inverse triad multiplets are introduced as the following fields:

$$\underset{\sim}{P}{}^X_i=\left(\frac{\delta^a_b-\chi^a\chi_b}{1-\chi^2}\underset{\sim}{E}{}^b_i,-\frac{\varepsilon^a{}_{bc}\underset{\sim}{E}{}^b_i\chi^c}{1-\chi^2}\right),$$

$$\underset{\sim}{Q}{}^X_i=\left(\frac{\varepsilon^a{}_{bc}\underset{\sim}{E}{}^b_i\chi^c}{1-\chi^2},\frac{\delta^a_b-\chi^a\chi_b}{1-\chi^2}\underset{\sim}{E}{}^b_i\right). \tag{B1}$$

They satisfy

$$\{\mathcal{G}_X,\underset{\sim}{P}{}^Y_i\}=-f^Y_{XZ}\underset{\sim}{P}{}^Z_i,\quad \tilde{P}^i_X\underset{\sim}{P}{}^X_j=\delta^i_j,\quad \tilde{Q}^i_X\underset{\sim}{P}{}^X_j=0. \tag{B2}$$

Similar properties are valid for $\underset{\sim}{Q}{}^X_i=\Pi^X_Y\underset{\sim}{P}{}^Y_i$.

The projectors (12) read

$$I^Y_{(P)X}=\begin{pmatrix}\dfrac{\delta^b_a-\chi_a\chi^b}{1-\chi^2}&\dfrac{\varepsilon_a{}^{bc}\chi_c}{1-\chi^2}\\ \dfrac{\varepsilon_a{}^{bc}\chi_c}{1-\chi^2}&-\dfrac{\delta^b_a\chi^2-\chi_a\chi^b}{1-\chi^2}\end{pmatrix} \tag{B3}$$

and $I^Y_{(Q)X}=\delta^Y_X-I^Y_{(P)X}$. Besides, one can note the relations that are very helpful in calculations:

$$I^{XY}_{(P)}=-\Pi^X_ZI^{ZW}_{(Q)}\Pi^Y_W, \tag{B4}$$

$$f^{WYZ}I^X_{(P)W}\tilde{Q}^i_Y\tilde{Q}^j_Z=0, \tag{B5}$$

$$f^{WYZ}I^X_{(Q)W}\tilde{Q}^i_Y\tilde{Q}^j_Z=f^{XYZ}\tilde{Q}^i_Y\tilde{Q}^j_Z. \tag{B6}$$

The commutators of the second class constraints form the following triangular matrix:

$$\Delta=\begin{pmatrix} 0 & D_1 \\ -D_1 & D_2 \end{pmatrix}, \quad \Delta^{-1}=\begin{pmatrix} D_1^{-1}D_2D_1^{-1} & -D_1^{-1} \\ D_1^{-1} & 0 \end{pmatrix}, \tag{B7}$$

where

$$D_1^{(ij)(kl)}=\{\phi^{ij},\psi^{kl}\}=\frac{4\beta^2}{1+\beta^2}(\tilde{Q}\tilde{Q})^{\{i[j\}}(\tilde{Q}\tilde{Q})^{\{k]l\}}, \tag{B8}$$

$$(D_1^{-1})_{(kl)(mn)}=\frac{1}{8}\left(1+\frac{1}{\beta^2}\right)[(\tilde{Q}\tilde{Q})_{kl}(\tilde{Q}\tilde{Q})_{mn} - (\tilde{Q}\tilde{Q})_{km}(\tilde{Q}\tilde{Q})_{ln}-(\tilde{Q}\tilde{Q})_{kn}(\tilde{Q}\tilde{Q})_{lm}]. \tag{B9}$$

An explicit form of D_2 is not needed since all brackets are expressed in terms of D_1^{-1} only [see Eq. (11)].

APPENDIX C: CONSTRAINT ALGEBRA

Define the smeared constraints:

$$\mathcal{G}(n)=\int d^3x\, n^X\mathcal{G}_X, \quad H(\underset{\sim}{N})=\int d^3x\, \underset{\sim}{N}H,$$

$$\mathcal{D}(\vec{N})=\int d^3x\, N^i(H_i+A_i^X\mathcal{G}_X). \tag{C1}$$

They obey the following algebra:

$$\begin{aligned}
&\{\mathcal{G}(n),\mathcal{G}(m)\}_D=\mathcal{G}(n\times m),\\
&\{\mathcal{D}(\vec{N}),\mathcal{D}(\vec{M})\}_D=-\mathcal{D}([\vec{N},\vec{M}]),\\
&\{\mathcal{D}(\vec{N}),\mathcal{G}(n)\}_D=-\mathcal{G}(N^i\partial_i n),\\
&\{H(\underset{\sim}{N}),\mathcal{G}(n)\}_D=0,\\
&\{\mathcal{D}(\vec{N}),H(\underset{\sim}{N})\}_D=-H(\mathcal{L}_{\vec{N}}\underset{\sim}{N}),\\
&\{H(\underset{\sim}{N}),H(\underset{\sim}{M})\}_D=\mathcal{D}(\vec{K})-\mathcal{G}(K^jA_j),
\end{aligned} \tag{C2}$$

where

$$\begin{aligned}
&(n\times m)^X=f^X_{YZ}n^Ym^Z, \quad \mathcal{L}_{\vec{N}}\underset{\sim}{N}=N^i\partial_i\underset{\sim}{N}-\underset{\sim}{N}\partial_iN^i,\\
&[\vec{N},\vec{M}]^i=N^k\partial_kM^i-M^k\partial_kN_i,\\
&K^j=(\underset{\sim}{N}\partial_i\underset{\sim}{M}-\underset{\sim}{M}\partial_i\underset{\sim}{N})\tilde{Q}^i_X\tilde{Q}^j_Yg^{XY}.
\end{aligned} \tag{C3}$$

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