Can Schwarzschildean gravitational fields suppress gravitational waves?

Edward Malec
Institute of Physics, Jagiellonian University, 30-59 Kraków, Reymonta 4, Poland

Gerhard Schäfer
Theoretical Physics Institute, Friedrich Schiller University, 07743 Jena, Max-Wien Pl. 1, Germany

(Received 9 March 2001; published 25 July 2001)

Gravitational waves in the linear approximation propagate in Schwarzschild space-time similarly to electromagnetic waves. A fraction of the radiation scatters off the curvature of the geometry. The energy of the backscattered part of an initially outgoing pulse of the quadrupole gravitational radiation is estimated by compact formulas depending on the initial energy, the Schwarzschild radius, and the location and width of the pulse. The backscatter becomes negligible in the short wavelength regime.

DOI: 10.1103/PhysRevD.64.044012 PACS number(s): 04.30.Nk, 04.40.−b, 95.30.Sf

I. INTRODUCTION

Backscattering has been investigated for a long time for various wave equations (see, for instance, Ref. [1]). In general relativity, this topic has been studied since the early 1960s [2,3]. This paper continues the program that started with the study of the backscatter of scalar [4] and electromagnetic fields [5,6]. Here we investigate the propagation of even-parity gravitational waves in a (fixed) background Schwarzschild space-time, assuming a nonstationary source. The discussion, however, proceeds without any reference to the source. We only deal with field quantities. It is assumed that the initial data are those of an isolated pulse (burst) of a gravitational wave. The main question that is answered is what fraction of the initially outgoing radiation may undergo backscattering before reaching null infinity? The strength of the backscattering is assessed by bounding the fraction of the initial burst energy that will not reach a distant observer in the main pulse. The even-parity waves are the only waves which are radiated during the axisymmetric collision of nonspinning black holes [9], and since in this case the Schwarzschild space-time is a valid starting point for an approximation scheme, it gives us an opportunity to bound the strength of the phenomenon in a fairly realistic astrophysical context.

The following five sections of the paper give a theoretical description of the backscattering effect. Section II describes the notation and the Zerilli equation. In Sec. III, the initial data are bounded by the initial energy, and a solution is sought in the form of a superposition of an outgoing radiation (defined by initial data) and a backscattered term. The evolution of the backscattered term can be bounded by solutions of two differential inequalities. The bounds that are derived in Sec. IV and in the Appendix deal with a general situation; no assumption is made about the initial radiation. In Sec. V, we discuss initial data that are of compact support, the relative width of the support being small. Such data correspond to radiation that is dominated by short wavelengths. In this case stronger estimates are derived. They imply that in the limit of short wavelengths (the relative width of the support tending to zero) the backscattering effect becomes negligible. In Sec. VI, the "small relative width condition" of Sec. V is supplemented by the assumption that the initial burst is far away from the horizon.

II. FORMALISM

The space-time geometry is defined by a Schwarzschildean line element,

$$ds^2=-\left(1-\frac{2m}{R}\right)dt^2+\frac{1}{1-\frac{2m}{R}}dR^2+R^2d\Omega^2, \quad (2.1)$$

where t is a time coordinate, R is a radial coordinate that coincides with the areal radius, and $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$ is the line element on the unit sphere: $0\leqslant\phi<2\pi$ and $0\leqslant\theta\leqslant\pi$. Throughout this paper G, the Newtonian gravitational constant, and c, the velocity of light, are set equal to 1.

As explained in Sec. I, we restrict ourselves to even-parity axial perturbations. Their propagation is ruled, in the linear approximation, by the Zerilli equation [7]. Formulated in terms of the gauge-invariant amplitude Ψ defined by Moncrief [8], this equation reads, in the case of $l=2$ multipole [9],

$$(-\partial_t^2+\partial_{r*}^2)\Psi=V\Psi, \quad (2.2)$$

where the potential V is given by

$$V(R)=6\left(1-\frac{2m}{R}\right)^2\frac{1}{R^2}+\left(1-\frac{2m}{R}\right)\frac{63m^2\left(1+\frac{m}{R}\right)}{2R^4\left(1+\frac{3m}{2R}\right)^2}, \quad (2.3)$$

and where

$$r^*=R+2m\ln\left(\frac{R}{2m}-1\right) \quad (2.4)$$

is the tortoise radial coordinate.

Consider a set of functions $\Psi_i(r^*-t)$, $i=0$, 1, and 2, that satisfy the following linear relations:

$$\partial_t \Psi_1 = 3\Psi_0,$$
$$\partial_t \Psi_2 = \Psi_1 - m\partial_t \Psi_1. \quad (2.5)$$

The combination

$$\tilde{\Psi} \equiv \Psi_0(r^* - t) + \frac{\Psi_1(r^* - t)}{R} + \frac{\Psi_2(r^* - t)}{R^2} \quad (2.6)$$

solves Eq. (2.2) in Minkowski space-time ($m=0$); it represents purely outgoing radiation.

Let the initial data of a solution Ψ of Eq. (2.2) coincide with $\tilde{\Psi}$ at $t=0$. Then, initially, Ψ represents a purely outgoing wave. It should be noted that the assumption that initial data are (initially) purely outgoing is made in this paper only for the sake of clear presentation. In the linear approximation the propagation of the initially outgoing radiation is independent of whether or not ingoing radiation is present.

We decompose the sought solution $\Psi(r^*,t)$ into the known part $\tilde{\Psi}$ and an unknown function δ:

$$\Psi = \tilde{\Psi} + \delta. \quad (2.7)$$

Due to the choice of the initial data made above, one has $\delta = \partial_t \delta = 0$ at $t=0$.

III. ENERGY ESTIMATES

Let us assume that the quadrupole initial data are defined by a smooth triad of the functions Ψ_k ($k=0$, 1, and 2) with the initial support $[a,b]$ ($b<\infty$). That guarantees that the initial energy density multiplied by R^2, $\rho = ((\partial_t\Psi)^2 + (\partial_{r^*}\Psi)^2 + V\Psi^2)/\eta_R$, is smooth and vanishes on the boundary a. Here $\eta_R = 1 - 2m/R$ holds.

The energy content inside a part of a Cauchy hypersurface Σ_t that is exterior to a ball of a radius R can be defined as $E(R,t) \equiv \int_R^\infty dr\,\rho(r,t)$. Let us point out that in order to ensure a proper normalization of the energy flux at infinity, there should be a normalization constant in the definition of the energy $E(R,t)$. We decided to omit it, since later on we shall be interested only in the relative efficiency of the backscatter; thus the normalization factor cancels out. The total initial energy corresponding to the hitherto defined initial data is equal to $E(a,0)$.

Lemma 1. Defining

$$C_1 \equiv \frac{3}{2}\sqrt{(2+\sqrt{2/3})E(a,0)},$$

$$C_2 \equiv \sqrt{2E(a,0)},$$

$$C_3 \equiv \sqrt{\frac{2E(a,0)}{\eta_a(2\sqrt{6}+1)}}, \quad (3.1)$$

and introducing the two non-negative functions

$$\begin{aligned} g_1(R) = {} & \ln\left(\frac{-2m+R}{a-2m}\right) \\ & + 32m^5\left(\frac{-1}{5(-2m+R)^5} + \frac{1}{5(-2m+a)^5}\right) \\ & + 20m^4\left(\frac{-1}{(-2m+R)^4} + \frac{1}{(-2m+a)^4}\right) \\ & + 80m^3\left(\frac{-1}{3(-2m+R)^3} + \frac{1}{3(-2m+a)^3}\right) \\ & + 20m^2\left(\frac{-1}{(-2m+R)^2} + \frac{1}{(-2m+a)^2}\right) \\ & + 10m\left(\frac{-1}{-2m+R} + \frac{1}{-2m+a}\right) \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} g_2(R) = {} & R - a + 16m^4\left(\frac{-1}{3(-2m+R)^3} + \frac{1}{3(-2m+a)^3}\right) \\ & + (16m^3)\left(\frac{-1}{(-2m+R)^2} + \frac{1}{(-2m+a)^2}\right) + (24m^2) \\ & \times\left(\frac{-1}{-2m+R} + \frac{1}{-2m+a}\right) + 8m\ln\left(\frac{-2m+R}{-2m+a}\right), \end{aligned} \quad (3.3)$$

the following inequalities hold at $t=0$ and for $R \geqslant a$:

$$\frac{|\Psi_1(R)|}{R^{3/2}} \leqslant C_1 \eta_R^{3/2}\sqrt{g_1(R)},$$

$$\begin{aligned} \frac{|\Psi_2(R)|}{\sqrt{\eta_R}R^2} \leqslant {} & C_2\sqrt{g_2(R)} + C_1\frac{6m}{\sqrt{a}}\sqrt{g_1(R)}\left(1 - \sqrt{\frac{a}{R}}\right) \\ & + 6C_3\frac{m}{\sqrt{a}}\sqrt{1 - \left(\frac{a}{R}\right)^{2\sqrt{6}+1}} \\ & \times\left(\frac{1}{\sqrt{\eta_a}} - \frac{1}{\sqrt{\frac{R}{a} - \frac{2m}{a}}}\right), \end{aligned}$$

$$\frac{|\tilde{\Psi}(R)|}{\sqrt{R}} \leqslant C_3\sqrt{1 - \left(\frac{a}{R}\right)^{2\sqrt{6}+1}}, \quad (3.4)$$

$$\frac{|\Psi_0(R)|}{\sqrt{R}} \leqslant C_3 \sqrt{1-\left(\frac{a}{R}\right)^{2\sqrt{6}+1}} + C_1 \eta_R^{3/2} \sqrt{g_1(R)} + C_2 \sqrt{\frac{g_2(R)}{R}} + C_1 \frac{6m}{\sqrt{aR}} \sqrt{g_1(R)} \left(1-\sqrt{\frac{a}{R}}\right) + 6C_3 \frac{m}{\sqrt{aR}} \sqrt{1-\left(\frac{a}{R}\right)^{2\sqrt{6}+1}} \times \left(\frac{1}{\sqrt{\eta_a}} - \frac{1}{\sqrt{\frac{R}{a}-\frac{2m}{a}}}\right).$$

Proof. One can explicitly verify that

$$-\eta_R\left(\frac{\Psi_1}{R^2}+2\frac{\Psi_2}{R^3}\right) = \partial_t \tilde{\Psi} + \partial_{r*}\tilde{\Psi},$$
$$\eta_R\left(2\frac{\Psi_0}{R}+\frac{\Psi_1}{R^2}\right) = \partial_t \tilde{\Psi} + \partial_{r*}\tilde{\Psi} + \frac{2\eta_R\tilde{\Psi}}{R}. \tag{3.5}$$

Equations (3.5), using relations (2.6), result in

$$\partial_R\left(\eta_R^{-3/2}\frac{\Psi_1}{R^{3/2}}\right) = -\frac{3}{2R^{1/2}\eta_R^3}\left(\frac{\partial_t\tilde{\Psi}}{\eta_R^{1/2}} + \sqrt{\eta_R}\partial_R\tilde{\Psi} + \frac{2\tilde{\Psi}\sqrt{\eta_R}}{R}\right),$$
$$\partial_R\left(\eta_R^{-1/2}\frac{\Psi_2}{R^2}\right) = \frac{1}{\eta_R^2}\left(\frac{\partial_t\tilde{\Psi}}{\eta_R^{1/2}} + \sqrt{\eta_R}\partial_R\tilde{\Psi}\right) + \frac{3m}{R\eta_R^{3/2}}\left(\frac{\tilde{\Psi}}{R} - \frac{\Psi_1}{R^2}\right). \tag{3.6}$$

The integration from a to R and the use of the Schwarz inequality yields

$$\frac{|\Psi_1(R)|}{R^{3/2}} \leqslant \frac{3\eta_R^{3/2}}{2}\sqrt{(2+\sqrt{2/3})E(a,0)}\left(\int_a^R dr \frac{1}{r\eta_r^6}\right)^{1/2}. \tag{3.7}$$

Integrating $\int_a^R dr(1/r\eta_r^6)$, one immediately arrives at the first of the postulated inequalities.

In order to show the third inequality, notice that $|\tilde{\Psi}(R)|R^{\sqrt{6}} = |\int_a^R dr \partial_r(\tilde{\Psi}(r)r^{\sqrt{6}})|$. The latter expression is bounded from above, using the Schwarz inequality, by

$$\sqrt{2\int_a^R dr(\eta_r(\partial_r\tilde{\Psi})^2 + 6\eta_r\tilde{\Psi}^2/r^2)}\sqrt{\int_a^R dr\,\eta_r^{-1}r^{2\sqrt{6}}} \leqslant \sqrt{\frac{2E(a,0)}{\eta_a(2\sqrt{6}+1)}}R^{\sqrt{6}+0.5}\left[1-\left(\frac{a}{R}\right)^{\sqrt{6}+1}\right]^{1/2}, \tag{3.8}$$

where the inequality in Eq. (3.8) follows from the monotonicity of the energy as function of R. The first factor on the left hand side of this inequality is not greater than $\sqrt{2E(a,0)}$ since $6\eta_r/r^2 \leqslant V(r)/\eta_r$. The replacement of η_r^{-1} by η_a^{-1} and the integration of the other factor leads to the desired result.

The second of Eqs. (3.6) can be integrated. The Schwarz inequality and direct integration, as well as the $\tilde{\Psi}$ and Ψ_1 estimates, should be used in order to obtain the second inequality of Lemma 1. The Ψ_0 estimate, in turn, follows from the identity $\Psi_0 = \tilde{\Psi} - \Psi_1/R - \Psi_2/R^2$ and the preceding estimates.

IV. ESTIMATE OF THE DIFFUSED ENERGY

Let us define the strength of the backscattered radiation that is directed inward by

$$h_-(R,t) = \frac{1}{\eta_R}(\partial_t + \partial_{r*})\delta(R,t). \tag{4.1}$$

Let the outgoing null geodesic $\tilde{\Gamma}_{(R,t)}$ originate at (R,t). If a point lies on the initial hypersurface, then we will write $\tilde{\Gamma}_{(R,0)} \equiv \tilde{\Gamma}_R$. By $\tilde{\Gamma}_{(R_0,t_0),(R,t)}$ will be understood a segment of $\tilde{\Gamma}_{(R_0,t_0)}$ ending at (R,t).

A straightforward calculation shows that the rate of the energy change along $\tilde{\Gamma}_a$ is given by

$$(\partial_t + \partial_{r*})E(R,t) = -[\eta_R^2 h_-^2(R,t) + V\delta^2(R,t)]. \tag{4.2}$$

It is necessary to point out that in the case of the initial point $R_0 > a$ the result would be more complicated; the differentiation of the energy along $\tilde{\Gamma}_{R_0}$ would depend also on Ψ_0, Ψ_1 and Ψ_2. If, however, the outgoing null geodesics is $\tilde{\Gamma}_a$, then it starts from a where Ψ_0, Ψ_1 and Ψ_2 do vanish. Since these functions depend on the difference $r^* - t$, their values along outgoing geodesics are constant, and that allows one to conclude that they vanish at $\tilde{\Gamma}_a$.

The energy loss, that is, the amount of energy that diffused inward $\tilde{\Gamma}_a$ is equal to a line integral along $\tilde{\Gamma}_a$:

$$\delta E_a \equiv E(a,0) - E_\infty = \int_a^\infty dr\left[\eta_r h_-^2 + \frac{V\delta^2}{\eta_r}\right]. \tag{4.3}$$

Our goal is to find an estimate of δE_a of a single pulse of radiation based only on the information about the position and the energy of the initial pulse. Obviously, $0 \leqslant \delta E_a \leqslant E(a,0)$ holds. We are interested in deriving in this section a frequency independent bound, but later we obtain estimates that are frequency sensitive.

δ is initially zero, and its evolution is governed by the following equation:

$$(-\partial_t^2 + \partial_{r*}^2)\delta = V\delta + \left(V - 6\frac{\eta_R^2}{R^2}\right)\left(\Psi_0 + \frac{\Psi_1}{R} + \frac{\Psi_2}{R^2}\right) + \frac{2m\eta_R}{R^4}\left[-3\Psi_1 + 2\frac{\Psi_2}{R}\right]. \tag{4.4}$$

One can define an "energy" $H(R,t)$ of the field δ which is contained in the exterior of a sphere of radius R as follows:

$$H(R,t)=\int_R^\infty dr\left(\frac{(\partial_t\delta)^2}{\eta_r}+\eta_r(\partial_r\delta)^2+\delta^2\frac{V}{\eta_r}\right). \quad (4.5)$$

The rate of change of H along $\tilde{\Gamma}_{(R,t)}$ is given by

$$\begin{aligned}(\partial_t+\partial_r^*)H(R,t)&=-\eta_R\left[\eta_R\left(\frac{\partial_t\delta}{\eta_R}+\partial_R\delta\right)^2+\frac{V}{\eta_R}\delta^2\right]-4m\int_R^\infty dr\,\eta_r\frac{\partial_t\delta}{r^4}\\&\quad\times\left[-3\Psi_1+\frac{2\Psi_2}{r}+\frac{63m\left(1+\frac{m}{r}\right)}{4\left(1+\frac{3m}{2r}\right)^2}\left(\Psi_0+\frac{\Psi_1}{r}+\frac{\Psi_2}{r^2}\right)\right]\\&\leqslant 4m\int_R^\infty dr\frac{\partial_t\delta}{r^4}\left[-3\Psi_1+\frac{2\Psi_2}{r}+\frac{63m\left(1+\frac{m}{r}\right)}{4\left(1+\frac{3m}{2r}\right)^2}\right.\\&\quad\left.\times\left(\Psi_0+\frac{\Psi_1}{r}+\frac{\Psi_2}{r^2}\right)\right]. \quad (4.6)\end{aligned}$$

Herein the inequality follows from the omission of the non-positive boundary term. This allows one to estimate the maximal value H_M of the δ energy H: namely,

$$\sqrt{H_M}\leqslant 10.43\frac{m\sqrt{E(a,0)}}{a}+O(m^2). \quad (4.7)$$

The calculation is essentially simple, but the algebra is quite lengthy and some numerical integrations are required. Details are relegated to the Appendix. We would like to point out that the $O(m^2)$ terms become dominant only when the location of the initial radiation pulse is smaller than $6.6m$. At $a=15m$ the neglected terms contribute much less than the leading term proportional to m.

Now, the integration of the first part of Eq. (4.6) along $\tilde{\Gamma}_{(a,0)}$ yields

$$\begin{aligned}H(\infty)-H(0)&=-\int_a^\infty dR\left[\eta_R\left(\frac{\partial_t\delta}{\eta_R}+\partial_R\delta\right)^2+\frac{V}{\eta_R}\delta^2\right]\\&\quad-\int_a^\infty dR\,4m\int_R^\infty dr\frac{\partial_t\delta}{r^4}\left[-3\Psi_1+\frac{2\Psi_2}{r}\right.\\&\quad\left.+\frac{63m\left(1+\frac{m}{r}\right)}{4\left(1+\frac{3m}{2r}\right)^2}\left(\Psi_0+\frac{\Psi_1}{r}+\frac{\Psi_2}{r^2}\right)\right]. \quad (4.8)\end{aligned}$$

Initially, H vanishes (both δ and $\partial_t\delta$ vanish) and H is manifestly non-negative. The first integral on the right hand side of Eq. (4.8) is recognized to be just δE_a. The second integral in turn can be shown to be bounded—using the Schwarz inequality and then the results of the Appendix—by

$$2\sqrt{H_M}\left(10.43\frac{m\sqrt{E(a,0)}}{a}+O(m^2)\right).$$

Thus, Eq. (4.8) implies

$$\begin{aligned}\delta E_a&\leqslant 2\left(10.43\frac{m\sqrt{E(a,0)}}{a}+O(m^2)\right)\sqrt{H_M}\\&\leqslant\left[54.5\left(\frac{2m}{a}\right)^2+O(m^3)\right]E(a,0); \quad (4.9)\end{aligned}$$

the right hand side of the first inequality achieves a maximal value when H is maximal and that implies the second inequality. Thus, in summary, for the fraction of the energy that could diffuse through the null cone C_a, the following theorem holds:

Theorem. $\delta E_a/E(a,0)$ satisfies the inequality

$$\frac{\delta E_a}{E(a,0)}\leqslant 54.5\times\left(\frac{2m}{a}\right)^2+O(m^3/a^3). \quad (4.10)$$

We would like to point out that the above derivation is more efficient and simpler than the one used in Ref. [5] or [6] when δE_a was estimated directly on the basis of the estimates of δ and h_-. This alternative approach would require a laborious integration of the field equation, and the final estimate would be much worse.

V. WAVELENGTH OF THE INITIAL RADIATION AND THE BACKSCATTER

In this section we shall consider the backscatter of the radiation that is initially of compact support and, in addition, the condition $(a-b)/a\ll 1$ is satisfied. The leading contribution—only terms that are quadratic in m^2-—will be found.

Under the above conditions one infers from Eq. (3.4) that, on the initial hypersurface,

$$|\Psi_1(R)|b^{3/2}\leqslant C_1 b^{3/2}\sqrt{g_1(R)}\leqslant C_1\sqrt{\frac{b-a}{a}}b^{3/2} \quad (5.1)$$

and

$$|\Psi_2(R)|\leqslant C_2 b^2\sqrt{b-a} \quad (5.2)$$

are valid. With the same accuracy, inequality (A3) of the Appendix reads

$$(\partial_t+\partial_{r*})\sqrt{H(R,t)}\leqslant 2m\left(\int_R^{R(b)}dr\,\eta_r\frac{9\Psi_1^2}{r^8}\right)^{1/2}+2m\left(\int_R^{\infty}dr\frac{4\Psi_2^2}{r^{10}}\right)^{1/2}$$
$$\leqslant 6mC_1\sqrt{\frac{b-a}{a}}b^{3/2}\left(\int_R^{R(b)}dr\frac{1}{r^8}\right)^{1/2}+4mC_2b^2\sqrt{b-a}\left(\int_R^{R(b)}dr\frac{1}{r^{10}}\right)^{1/2}. \tag{5.3}$$

Herein, the integration extends from R, where $R\in\tilde{\Gamma}_a$, to $R(b)$, which is defined by $[R(b),t]\in\tilde{\Gamma}_b$. One has $R(b)-R=b-a$ up to the term m^0. The integral $\int_R^{R(b)}dr(1/r^8)$ is bounded from above by $(b-a)/R^8$, and the integral $\int_R^{R(b)}dr(1/r^{10})$ is bounded from above by $(b-a)/R^{10}$, again to lowest order in powers of m.

Thus one arrives at

$$(\partial_t+\partial_{r*})\sqrt{H(R,t)}\leqslant 4m(b-a)b^{3/2}\left(\frac{1.5C_1}{\sqrt{a}R^4}+\frac{C_2\sqrt{b}}{R^5}\right). \tag{5.4}$$

The integration of this inequality along the null geodesic $\tilde{\Gamma}_a$ yields

$$\sqrt{H_M}\leqslant m(b-a)\frac{b^{3/2}}{a^{7/2}}(2C_1+C_2)+O(m^2). \tag{5.5}$$

Taking into account the condition that $b-a\ll a$, one arrives at

$$H_M\leqslant\frac{4m^2}{a^2}\left(\frac{b-a}{a}\right)^2\left(\frac{b}{a}\right)^3\left(C_1+\frac{C_2}{2}\right)^2+O(m^3/b^3). \tag{5.6}$$

Since the amount of backscattered energy δE_a is bounded from above by $2H_M$, as shown in Sec. IV, one finally arrives at the following estimate

$$\frac{\delta E_a}{E(a,0)}\leqslant\frac{8m^2}{a^2}\left(\frac{b-a}{a}\right)^2\left(\frac{b}{a}\right)^3\left(\frac{3}{2}\sqrt{2+\sqrt{\frac{2}{3}}}+\frac{1}{\sqrt{2}}\right)^2+O(m^3/a^3)\leqslant 84\frac{m^2}{a^2}\left(\frac{b}{a}\right)^3\left(\frac{b-a}{a}\right)^2+O(m^3/a^3). \tag{5.7}$$

If $(b-a)/a<0.1$, then the above formula predicts

$$\frac{\delta E_a}{E(a,0)}\leqslant 0.84\frac{m^2}{a^2}. \tag{5.8}$$

It is clear that if the relative width of the initial pulse tends to zero, then the effect becomes negligible. This can be translated into the dependence on the wavelength of the radiation [6]: The compression of the support of a function leads to the decrease of its wavelength scale in its Fourier transform.

A careful analysis of the higher order terms would show that they give a contribution to (5.7) that also scales like $[(b-a)/a]^2$. In the case when $a\approx 2m$, Eq. (5.7) would be of the form

$$\frac{\delta E_a}{E(a,0)}\leqslant C(x)\left(\frac{b-a}{2m}\right)^2, \tag{5.9}$$

where $C(x)$ is a large number and $x\equiv 2m/a$. One can show that $\lim_{x\to 1}C(x)=\infty$, but on the other hand $C(x)$ is fixed, when $2m/a$ is fixed. Thus Eq. (5.9) implies that when $b\to a$, then the backscatter becomes negligible. Radiation that is dominated by infinitely short wavelengths does not backscatter.

VI. MORE ESTIMATES ON HIGH FREQUENCY RADIATION

We assume initial data of compact support $[a,b]$. The initial energy $E(a,0)$ (see the beginning of Sec. III) reads, expressed in terms of functions Ψ_0, Ψ_1 and Ψ_2, as follows:

$$E(a,0)=\int_a^b dr\rho=\int_a^b dr\left[\frac{6\{r[r\Psi_0(r)+\Psi_1(r)]+\Psi_2(r)\}^2}{r^6}+\left(\Psi_0'(r)+\frac{r\Psi_1'(r)+\Psi_2'(r)}{r^2}\right)^2+\frac{(-2\Psi_2(r)-r\{\Psi_1(r)+r[r\Psi_0'(r)+\Psi_1'(r)]+\Psi_2'(r)\})^2}{r^6}\right]. \tag{6.1}$$

The radiation energy in the wave zone is known to be $E(a,0)=C\int_a^b dr(\Psi_0')^2$. This can be compatible with Eq. (6.1) (modulo a normalization constant, which is not relevant here), if the terms with $(\Psi_0')^2$ give a leading contribution.

One notices, that, if $\Psi_\mu(R)$ $(\mu=0,1,2)$ are of compact support, then $|\Psi_\mu(R)|=|\int_a^R dr\partial_r\Psi_\mu(r)| \leqslant\sqrt{(R-a)\int_a^R dr\partial_r\Psi_\mu^2(r)}$. Combining this with Eq. (2.5), one arrives at

$$3|\Psi_0(r)|=|\Psi_1'(r)|\leqslant 3\sqrt{b-a}\sqrt{\int_a^b dr(\Psi_0')^2},$$

$$|\Psi_1(r)|=|\Psi_2'(r)|\leqslant 2(b-a)^{3/2}\sqrt{\int_a^b dr(\Psi_0')^2}, \tag{6.2}$$

$$|\Psi_2(r)|\leqslant 0.8(b-a)^{5/2}\sqrt{\int_a^b dr(\Psi_0')^2}.$$

Taking Eq. (6.2) into account, one concludes that if

$$C_1(b-a)/a\ll 1 \tag{6.3}$$

(C_1 is a constant of the order of 100), then the energy is well approximated by $E(a,0)=2\int_a^b dr(\Psi_0')^2$.

In such circumstances it is clear that our analysis can be greatly simplified. First of all the contribution coming from Ψ_2 to the backscatter is much smaller than that due to Ψ_1; note an additional power of $(b-a)/a$ in the relevant estimate of Eq. (6.2). Second, estimate (5.1) of Ψ_1 is now replaced by a stronger result:

$$|\Psi_0(r)|\leqslant\frac{1}{\sqrt{2}}(b-a)^{3/2}\sqrt{E(a,0)}. \tag{6.4}$$

The repetition of the calculation of Sec. V finally gives (taking into account the above conditions)

$$\frac{\delta E_a}{E(a,0)}\leqslant\left(\frac{2m}{a}\right)^2\left(\frac{b-a}{a}\right)^4. \tag{6.5}$$

VII. CONCLUSIONS

In our paper we derived upper bounds for the backscattering of gravitational quadrupole waves propagating outward from a central compact object. The calculations were restricted to situations where the initial configuration was either an arbitrarily shaped wave with support outside some radius *a* or the wave was a sharp pulse, i.e., its extension was small compared to its initial location *a*. The obtained upper bounds show that, for a given central object, the backscattering is weaker the farther outside from the central object the waves begin to propagate, and that is also weaker the more compact the pulses are, i.e., the higher the involved frequencies are. Both results confirm previous completely different calculations by Price, Pullin, and Kundu [10]. Backscattering should thus be strongest for pulses which begin to propagate outward close to the horizon of a black hole. This claim, however, needs further investigation for the following reasons. First, we gave bounds from above and not from below for the amount of backscattered energy. Second, the linear approximation may not be accurate enough very close to the horizon.

Results of Flanagan and Hughes [11] and Buonanno and Damour [12] showed that the merger part of the gravitational wave signal could be a significant part of the total energy emitted. The wave pulse during the merger phase can be inside 3 m. For a very compact pulse located in this region, inequality (5.9) of Sec. V can still yield a nontrivial bound, but in the general case our estimates fail. The main reason why we lose much in accuracy is that we are forced to use several times—for the sake of generality—the Schwarz inequality. The present bounds can be significantly improved if initial data are explicitly known, since in this case they can be numerically bounded by an exact expression involving the initial energy, and the Schwarz inequality would be used only once. On the other hand, it has been discovered that the backscattering can be quite strong when a signal propagates from within the photon sphere [13]. In a forthcoming paper we shall discuss, and compare with the results of our present paper, several aspects of the backscattering of gravitational waves where the sources of the gravitational waves are taken into account.

ACKNOWLEDGMENTS

This work was supported in part by KBN Grant No. 2 PO3B 010 16. One of the authors (E.M.) gratefully acknowledges financial support from the DAAD during his visit in Jena.

APPENDIX

In order to show estimate (4.7) one begins with the second inequality of Eq. (4.6). Note that $H(t=0)=0$, since $\delta(R,t=0)=\partial_t\delta(R,t)|_{t=0}$. Thence the integration of Eq. (4.6) along $\tilde{\Gamma}_{a,(R,t)}$ yields

$$H_M\leqslant\int_a^\infty\frac{dr}{\eta_r}|\mathcal{R}(r)|, \tag{A1}$$

where $\mathcal{R}(r)$ stands for the right hand side of Eq. (4.6). Our task consists in estimating the line integral of $|\mathcal{R}(r)|$.

In order to do this, one uses the estimates of Eq. (3.4). The calculation is quite long, and we will describe only the main points. In the first step one uses the Schwarz inequality on the right hand side of Eq. (4.6), in order to obtain an expression of the type

$$4m\left(\int_R^\infty dr\frac{(\partial_t\delta)^2}{\eta_r}\right)^{1/2}\times\left(\int_R^\infty dr\,\eta_r\frac{f^2(r)}{r^8}\right)^{1/2}, \tag{A2}$$

where $f(r)$ denotes $(-3\Psi_1)^2$ or $(2\Psi_2/r)^2$, or the squares of the terms that are proportional to $63m$. The first integral can be bounded by $\sqrt{H(R)}$; therefore, Eq. (4.6) and (A2) yield

$$
\begin{aligned}
&(\partial_t+\partial_{r*})\sqrt{H(R,t)} \\
&\leqslant 2m\left(\int_R^\infty dr\,\eta_r \frac{9\Psi_1^2}{r^8}\right)^{1/2} + 2m\left(\int_R^\infty dr \frac{4\Psi_2^2}{r^{10}}\right)^{1/2} \\
&+2m\left(\int_R^\infty dr\left(\frac{63m\left(1+\frac{m}{r}\right)}{4\left(1+\frac{3m}{2r}\right)^2}\right)^2\frac{\Psi_0^2}{r^8}\right)^{1/2} \\
&+2m\left(\int_R^\infty dr\left(\frac{63m\left(1+\frac{m}{r}\right)}{4\left(1+\frac{3m}{2r}\right)^2}\right)^2\frac{\Psi_1^2}{r^{10}}\right)^{1/2} \\
&+2m\left(\int_R^\infty dr\left(\frac{63m\left(1+\frac{m}{r}\right)}{4\left(1+\frac{3m}{2r}\right)^2}\right)^2\frac{\Psi_2^2}{r^{12}}\right)^{1/2}.
\end{aligned} \tag{A3}
$$

The integrands of Eq. (A3) are taken at a time t and $(R,t) \in \tilde{\Gamma}_a$; the integration extends over the part $r \geqslant R$ of the Cauchy hypersurface Σ_t. At this place one inserts the bounds on Ψ_0, Ψ_1 and Ψ_2. That requires some care; the estimates hold true on the initial hypersurface Σ_0, while here one needs estimates on Σ_t. This point is clarified later. It is useful to introduce dimensionless variables $x=R/a$ and $\tilde{m}=m/a$.

(i) First we shall consider the contribution that is due to $3\Psi_1$. Let r_0 be defined by $(r,t)\in\tilde{\Gamma}_{r_0}$. The insertion of the bound given in Eq. (3.4) bounds

$$
2m\left(\int_R^\infty dr\,\eta_r\frac{9\Psi_1^2}{r^8}\right)^{1/2}
$$

by

$$
6mC_1\left(\int_R^\infty dr\,\eta_r^4\frac{g_1(r_0)}{r^5}\right)^{1/2}.
$$

Note that $g_1(r)$ is an increasing function; therefore, if one replaces $g_1(r_0)$ with $g_1(r)$, then the integral that appears here cannot be smaller. In this way one utilizes the initial information [energy inequality (3.4)] in order to control the evolution. The integral in question can be performed explicitly, with the result

$$
6mC_1\left(\int_R^\infty dr\,\eta_r^4\frac{g_1(r)}{r^5}\right)^{1/2} = \frac{6mC_1}{a^2}\sqrt{G_1(x)}. \tag{A4}
$$

Here it holds that

$$
\begin{aligned}
-G_1(x) =& \frac{\tilde{m}^4(137-770\tilde{m}+1880\tilde{m}^2-2160\tilde{m}^3+960\tilde{m}^4)}{30(-1+2\tilde{m})^5x^8} - \frac{\tilde{m}^3(991-5110m+10840\tilde{m}^2-8880\tilde{m}^3-720\tilde{m}^4+3360\tilde{m}^5)}{105(-1+2m)^5x^7} \\
&+\frac{\tilde{m}^2(2981-13010\tilde{m}+18440\tilde{m}^2+7920\tilde{m}^3-41520\tilde{m}^4+27360\tilde{m}^5)}{420(-1+2\tilde{m})^5x^6} \\
&-\frac{\tilde{m}(4497-11370\tilde{m}-21720\tilde{m}^2+133040\tilde{m}^3-200240\tilde{m}^4+101600\tilde{m}^5)}{2100(-1+2\tilde{m})^5x^5} \\
&+\frac{375+4650\tilde{m}-35400\tilde{m}^2+93200\tilde{m}^3-110000\tilde{m}^4+49376\tilde{m}^5}{3360(-1+2\tilde{m})^5x^4} - \frac{11}{336\tilde{m}x^3} - \frac{11}{448\tilde{m}^2x^2} - \frac{11}{448\tilde{m}^3x} \\
&+\frac{11\ln\left(\frac{x}{-2\tilde{m}+x}\right)}{896\tilde{m}^4} - (280\tilde{m}^4-640\tilde{m}^3x+560\tilde{m}^2x^2-224\tilde{m}x^3+35x^4)\frac{\ln\left(\frac{-2\tilde{m}+x}{1-2\tilde{m}}\right)}{140x^8}.
\end{aligned} \tag{A5}
$$

This rather long expression is quite well approximated by $G_1=(1+4\ln x)/(16x^4)$ if $m/a \ll 1$. The integration of Eq. (A4) along a null cone C_a is done as follows. The integral $\int_1^\infty \sqrt{G_1(x)}$ is bounded from above:

$$
\int_1^\infty dx\,\eta_x^{-1}\sqrt{G_1(x)} \leqslant \left(\int_1^\infty dx x^2 G_1(x)\right)^{1/2}\left(\int_1^\infty dx x^{-2}\eta_x^{-2}\right)^{1/2}. \tag{A6}
$$

Numerical integration yields

$$
6C_1\frac{m}{a}\sqrt{\int_1^\infty \frac{(1+4\ln x)}{16x^2}} + O((m/a)^2) \approx 8.24\frac{m\sqrt{E(a,0)}}{a} + O[(m/a)^2]. \tag{A7}
$$

One can check that the neglected terms can give a contribution comparable to the leading term only at distances smaller than $6.6m$.

(ii) The calculation concerning the contribution of the Ψ_2 function is similar. The leading (proportional to m^0) term is $\int_R^\infty dr(\Psi_2^2/r^{10})$. $|\Psi_2|$ is bounded in terms of $g_2(x)$. $g_2(x)$ is an increasing function, and a reasoning similar to that made when discussing $g_1(x)$ leads to the conclusion that one that can again use the initial energy inequality given by Eq. (3.4). One finds that $\int_R^\infty dr(\Psi_2^2/r^{10})$ is bounded from above by

$$
\begin{aligned}
-\frac{1}{a^4}G_2(x) &:= -\frac{1}{a^4}\int_x^\infty dy\,\frac{g_2(y)}{y^6}(1-2\tilde{m}/y)^2 \\
&= \frac{4\tilde{m}^2(-3+44\tilde{m}-120\tilde{m}^2+96\tilde{m}^3)}{21(-1+2\tilde{m})^3x^7} \\
&\quad -\frac{2\tilde{m}(-21+236\tilde{m}-408\tilde{m}^2-192m^3+576\tilde{m}^4)}{63(-1+2\tilde{m})^3x^6} \\
&\quad +\frac{-21+88\tilde{m}+480\tilde{m}^2-1968\tilde{m}^3+1760\tilde{m}^4}{105(-1+2\tilde{m})^3\tilde{x}^5}-\frac{34}{105x^4} \\
&\quad -\frac{31}{630\tilde{m}x^3}-\frac{31}{840\tilde{m}^2x^2}-\frac{31}{840\tilde{m}^3x}+\frac{31\ln\left(\frac{x}{x-2\tilde{m}}\right)}{1680\tilde{m}^4} \\
&\quad -\frac{8\tilde{m}(60\tilde{m}^2-70\tilde{m}x+21x^2)\ln\left(\frac{-2\tilde{m}+x}{1-2m}\right)}{105x^7}.
\end{aligned}
\tag{A8}
$$

In the limit of $m\to 0$ the function $G_2(x)$ coincides with $(-4+5x)/(20x^5)$. Similarly as before, in order to obtain a term bounding $\sqrt{H_M}$, one should integrate $\sqrt{G_2(x)}/(1-2\tilde{m}/x)$ along a null cone C_a. This gives 0.15, up to terms $O(m)$, after manipulations similar to those done earlier. The $O(m)$ correction becomes dominant when $2m/a>0.3$. After a reasoning similar to that applied above in the case of Ψ_1 one finds that the total contribution due to the bound on the Ψ_2 function is equal to

$$4C_2\sqrt{\int_1^\infty dx\,\frac{(-4+5x)}{20x^3}}\frac{m\sqrt{E(a,0)}}{a}=2.19\frac{m\sqrt{E(a,0)}}{a}. \tag{A9}$$

In summary, one obtains

$$\sqrt{H_M}\leqslant 10.43\frac{m\sqrt{E(a,0)}}{a}+O(m^2). \tag{A10}$$

In the above analysis, in Eq. (A3), we neglected the terms proportional to $63m$. These give corrections of the order $O(m^2)$ to the right hand side of Eq. (A10). We checked that their contribution is small in the region $a>6.6m$. Our final result [Eq. (4.10)] tells us that $a>\sqrt{218}m\approx 15m$ is valid for a nontrivial estimate. Therefore, all higher order terms in Eq. (A10) can be safely neglected.

[1] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, 1923).
[2] C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[3] B. DeWitt and R. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960); W. Kundt and E. T. Newman, J. Math. Phys. **9**, 2193 (1968); R. G. McLenaghan, Proc. Cambridge Philos. Soc. **65**, 139 (1969); W. B. Bonnor and M. A. Rotenberg, Proc. R. Soc. London Ser. A **289**, 247 (1965); J. Bicak, Gen. Relativ. Gravit. **3**, 331 (1972); R. Price, Phys. Rev. D **5**, 2419 (1972); J. M. Bardeen and W. H. Press, J. Math. Phys. **14**, 7 (1973); B. Mashhoon, Phys. Rev. D **7**, 2807 (1973); **10**, 1059 (1974); G. Schäfer, Astron. Nachr. **311**, 213 (1990); L. Blanchet and G. Schäfer, Class. Quantum Grav. **10**, 2699 (1993); C. Gundlach, R. H. Price, and J. Pullin, Phys. Rev. D **49**, 883 (1994); **49**, 890 (1994); L. Blanchet, in *Relativistic Gravitation and Gravitational Radiation*, edited by J. A. Marck and J. P. Lasota (Cambridge University Press, Cambridge, England, 1997); L. Blanchet, Class. Quantum Grav. **15**, 113 (1998); W. B. Bonnor and M. Piper, *ibid.* **15**, 955 (1998); R. Mankin, T. Lass, and R. Tammelo, Phys. Rev. D **62**, 041501(R) (2000).
[4] E. Malec, N. O'Murchadha, and T. Chmaj, Class. Quantum Grav. **15**, 1653 (1998).
[5] E. Malec, Phys. Rev. D **62**, 084034 (2000).
[6] E. Malec, Acta Phys. Pol. B **32**, 47 (2001).
[7] F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).
[8] V. Moncrief, Ann. Phys. (N.Y.) **88**, 323 (1974).
[9] A. Garat and R. Price, Phys. Rev. D **61**, 044006 (2000).
[10] R. Price, J. Pullin, and X. Kundu, Phys. Rev. Lett. **70**, 1572 (1993).
[11] E. Flanagan and S. Hughes, Phys. Rev. D **57**, 4535 (1998).
[12] A. Buonanno and T. Damour, *Binary Black Holes Coalescence: Transition From Adiabatic Inspiral to Plunge*; contributed paper to the IX Marcel Grossmann Meeting in Rome, July 2000, gr-qc/0011052.
[13] J. Karkowski, E. Malec, and Z. Świerczyński, *Backscattering of Electromagnetic and Gravitational Waves Off Schwarzschild Geometry: Numerical Results*, gr-qc/0105042.