Trajectories for the wave function of the universe from a simple detector model

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Inspired by Mott's analysis of particle tracks in a cloud chamber, we consider a simple model for quantum cosmology which includes, in the total Hamiltonian, model detectors registering whether or not the system, at any stage in its entire history, passes through a series of regions in configuration space. We thus derive a variety of well-defined formulas for the probabilities for trajectories associated with the solutions to the Wheeler-DeWitt equation. The probability distribution is peaked about classical trajectories in configuration space. The "measured" wave functions still satisfy the Wheeler-DeWitt equation, except for small corrections due to the disturbance of the measuring device. With modified boundary conditions, the measurement amplitudes essentially agree with an earlier result of Hartle derived on rather different grounds. In the special case where the system is a collection of harmonic oscillators, the interpretation of the results is aided by the introduction of "timeless" coherent states—eigenstates of the Hamiltonian which are concentrated about entire classical trajectories.

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I. INTRODUCTION

The focus of attention in quantum cosmology is the Wheeler-DeWitt equation

$$H\Psi = 0. \tag{1.1}$$

Here, the wave function Ψ is a functional of the gravitational and matter fields on a three surface, and it describes the quantum state of a closed cosmological model [1]. The most striking and conceptually problematic aspect of this equation is that it does not involve time explicitly, severely complicating efforts to extract predictions from it [2,3]. Among the many attempts to understand this feature, one is to claim that "time" and indeed entire histories of the universe are already contained among the arguments of the wave function; hence no time label is required [4,5]. While these claims seem to be true at some level in simple models of quantum cosmology, it presents us with the interesting challenge of reformulating standard quantum theory without the explicit use of time, and then demonstrating the emergence of time and of classical trajectories. Although the Wheeler-DeWitt equation in the form Eq. (1.1) is unlikely to be the last word in quantum gravity, it does seem likely that whatever replaces it will still be of a timeless nature. The loop variables program of Ashtekar and others, for example, certainly preserves this feature [6]. It is therefore of interest to investigate this feature in simple models.

Many attempts to use and make sense of Eq. (1.1) have been made. These attempts focus on simple (minisuperspace) models, in which one has an N-dimensional configuration space $\mathcal{C}$ with coordinates $\mathbf{x}$, and the Hamiltonian operator has the form

$$H = -\tfrac{1}{2}\nabla^2 + V(\mathbf{x}). \tag{1.2}$$

The signature of the metric is typically hyperbolic so the Wheeler-DeWitt equation is like a Klein-Gordon equation in curved space with a spacetime dependent mass term. Associated with it is a Klein-Gordon current

$$J = i(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi). \tag{1.3}$$

Like the Klein-Gordon equation, however, this does not produce a positive probability density except in very special cases (namely when there is a Killing vector associated with H). It also vanishes for real wave functions. One might also consider the Schrödinger inner product

$$\langle\Psi_1|\Psi_2\rangle = \int_{\mathcal{C}} d^N\mathbf{x}\, \mu(\mathbf{x})\Psi_1^*(\mathbf{x})\Psi_2(\mathbf{x}), \tag{1.4}$$

where $\mu(\mathbf{x})$ is an appropriate measure, but the norm $\langle\Psi|\Psi\rangle$ typically diverges. In practice, most uses of the Wheeler-DeWitt equation rely on something like the "WKB interpretation," in which in the oscillatory regime the wave function is written in the form $\Psi = Ce^{iS}$, where S is a solution to the Hamilton-Jacobi equation. It is argued that this wave function corresponds to an ensemble of classical trajectories satisfying the first integral $p = \nabla S$, with $|C|^2$ giving a measure on the ensemble. Although probably correct it is somewhat heuristic and can only be used in the oscillatory regime. (See Ref. [7], for example, for a discussion of these issues.)

Recent more successful work with the Wheeler-DeWitt or similar equations involves the induced inner product (also known as Rieffel induction or refined algebraic quantization) [8,9]. Here, one considers eigenvalues of the Wheeler-DeWitt operator

$$H|\Psi_{Ek}\rangle = E|\Psi_{E'k'}\rangle, \tag{1.5}$$

where k is the degeneracy label for each E. The spectrum is typically continuous in E in which case the states are normalized via Eq. (1.4) according to

$$\langle\Psi_{Ek}|\Psi_{E'k'}\rangle = \delta(E-E')\,\delta_{kk'} \tag{1.6}$$

and one can now see why $\langle\Psi|\Psi\rangle$ diverges. The induced inner product is then, loosely speaking, to drop the factor $\delta(E-E')$ as E and E' are set to zero. This procedure can be defined rigorously and induces an inner product on the zero energy eigenstates. (This procedure is not of course necessary when the spectrum is discrete.)

Related to these issues is the prevalent idea that any operations performed on the wave function in the computation of physically interesting probabilities should commute with H [3,10,11]. Mathematically, this is to respect the symmetry of the theory, reparametrization invariance, expressed by the constraint equation (1.1). Physically, it is connected with the fact that the universe is a genuinely closed system, and all realistic measurements are carried out from the inside, so cannot displace the system from a zero energy eigenstate of H.

Given these preliminaries, turn now to the questions one would like to ask of the wave function of the system in order to extract useful cosmological predictions from it. We are interested in the notion that the wave function corresponds in some way to a set of trajectories. Let us therefore ask the question: ''What is the probability that the system is found in a series of regions in configuration space, $\Delta_1, \Delta_2, \dots \Delta_n$?'' Note that the question is stated in such a way that does not involve time. There is no requirement that the system enter one of the regions at a particular ''time,'' or that the regions are entered in a particular order. We cannot ask this because the Wheeler-DeWitt equation does not know about such an ordering parameter. In the classical case the corresponding situation consists of a statistical ensemble of classical trajectories with the same fixed energy. The trajectories are simply curves in configuration space and it is straightforward to determine the probability that a given curve passes through a given region at any stage in its entire history. The question is more involved in quantum theory, since quantum theory is somewhat resistant to the notion of a trajectory (in the nonrelativistic case, it involves specifying positions at different time, which do no commute). It is, nevertheless, important to develop this notion, since the timeless nature of the Wheeler-DeWitt equation cries out for an interpretation in terms of entire histories of the universe. The aim of this paper is to offer one possible way of giving meaning to the above question in the quantum theory of simple cosmological models.

Intuitively, one would expect that the question can be formulated and answered using a simple toolbox of parts: the quantum state $|\Psi\rangle$ satisfying the constraint, projection operators onto the regions Δ_k, or maybe projections onto some class of operators which commute with the constraint H. One might also expect to find the Green function associated with the Wheeler-DeWitt equation, which has the form

$$G(\mathbf{x},\mathbf{y}) = i\int_0^\infty d\tau \langle \mathbf{x}|e^{-i(H-i\epsilon)\tau}|\mathbf{y}\rangle = \left\langle \mathbf{x}\left|\frac{1}{(H-i\epsilon)}\right|\mathbf{y}\right\rangle \tag{1.7}$$

(this is the analogue of the Feynman propagator). It might possibly also involve one of the other types of propagators obtained by integrating τ over an infinite range in this expression [12–14]. The question is then exactly how one stitches all these components together to make a plausible probability distribution describing trajectories passing through a series of regions.

The decoherent histories approach offers an approach to answering this question and it does indeed use some of the above elements [15]. It is particularly adapted to this sort of situation since it directly addresses the issue of defining a quantum notion of ''trajectory'' and this approach is currently being investigated in this context [16] (see also Ref. [1]). Other approaches involving observables—operators commuting with the constraint—have also been considered [9,17,18,19]. Most importantly, Kiefer and Zeh [20] and Barbour [4,21,22] have devoted much effort to elucidating the emergence of trajectories and of time from the timeless Wheeler-DeWitt equation.

The approach we adopt here stems from Barbour's observation [22] that a substantial insight into the Wheeler-DeWitt equation may be found in Mott's 1929 analysis of α particle tracks in a Wilson cloud chamber [23]. Mott's paper concerned the question of how the α particle's outgoing spherical wave state, e^{ikR}/R, could lead to straight line tracks in a cloud chamber. His explanation was to model the cloud chamber as a collection of atoms that may be ionized by the passage of the α particle. They therefore act as measuring devices that measure the α particle's trajectory. The probability that certain atoms are ionized is indeed found to be strongly peaked when the atoms lie along a straight line through the point of origin of the α particle.

Although Mott seems to have had in mind a time-evolving process, he actually solved the time-independent equation

$$(H_0 + H_d + \lambda H_{\mathrm{int}})|\Psi\rangle = E|\Psi\rangle. \tag{1.8}$$

Here H_0 is the α particle Hamiltonian, H_d is the Hamiltonian for the ionizing atoms, and H_{int} describes the Coulomb interaction between the α particle and the ionizing atoms (where λ is a small coupling constant). Now the interesting point, as Barbour notes, is that Mott derived all the physics from this equation with little reference to time. Mott's calculation is therefore an excellent model for many aspects of the Wheeler-DeWitt equation. Barbour has elucidated this very eloquently, showing how it sheds light on a number of different aspects [4,21,22,24].

Barbour's discussion is largely qualitative. The point of the present paper, by contrast, is to extract quantitative information from the comparison between the Mott calculation and the Wheeler-DeWitt equation. Mott derived the straight line tracks by looking at the wave function associated with two atoms being in the ionized state, for the special case of an outgoing wave initial state. But since this is elementary quantum mechanics, it is a simple matter to generalize it to arbitrary initial states and other types of detector models, and to derive a detailed expression for the probability distribution. Therefore, *Mott's calculation points the way toward a general expression for the probability distribution for the system passing through a series of regions in configuration space without reference to time*. This is what we will work out in detail in this paper.

We consider a system in an N dimensional configuration space R^N with coordinates $\mathbf{x}$ described by a Hamiltonan H_0. Although we are ultimately interested in quantum cosmology, most of the results apply to both simple quantum-mechanical models and to quantum-cosmological models with Hamiltonian given by Eq. (1.2). The precise form of the Hamiltonian, including the signature of the metric, will therefore be left general, except where explicitly stated otherwise. The system is coupled to a set of detectors via an interaction $H_{\rm int}$ and the state of the whole system is given by the solution to Eq. (1.8). Mott used the electronic degrees of freedom of atoms as detectors. However, the essence of the calculation is maintained with a much simpler detector model. The detector we use consists of a two state system with $H_d=0$ and detector states $|0\rangle$ and $|1\rangle$, where

$$a|0\rangle=0, \quad a|1\rangle=|0\rangle, \quad a^\dagger|0\rangle=|1\rangle, \quad a^\dagger|1\rangle=0. \tag{1.9}$$

We take

$$H_{\rm int}=\sum_x f_k(\mathbf{x})(a_k+a_k^\dagger). \tag{1.10}$$

Here, $f_k(\mathbf{x})$ is spatially localized in the region Δ_k. One could, for example, take f_k to be a window function which is 1 in Δ_k and 0 outside it, but we will not restrict to this choice. If the detector in Δ_k is ''initially'' in the ground state $|0\rangle$, it will be displaced into the excited state $|1\rangle$ if the particle's trajectory $\mathbf{x}(t)$ enters Δ_k and remains in the ground state otherwise. Of course, in the timeless context of the Wheeler-DeWitt equation, ''initially'' has no meaning. We will impose the condition that the detector is in the ground state in the absence of coupling to the system. This does not, however, fix the solution completely, as we shall see, so further boundary conditions are required. We will not, in fact, commit to a particular set of boundary conditions in this paper, but will explore the general properties of the solutions.

Ultimately, it would be desirable to use this scheme in an interesting cosmological model and to apply some of the well-known boundary condition proposals, such as the no-boundary proposal of Hartle and Hawking [25]. This particular proposal involves taking the wave function given by a path integral over metrics and matter fields on a compact four geometry. With a detector coupled in as here, this path integral would also include a sum over detector states on a compact four geometry, and it would generate a solution to the Wheeler-DeWitt analogue of Eq. (1.8).

The detector described here is far from realistic, not least of all because it can return to its ground state if the particle spends too much time in the detector region. We will discuss its problems and possible improvements. We note, however, that similarly simple detector models have been profitably used elsewhere, e.g., in the Coleman-Hepp model [26]. (See also Ref. [27]).

Furthermore, although this model is very simple, it does possess at least some key features that are cosmologically relevant. In the real universe, we make measurements at the present time of, for example, the microwave background and the distribution of galaxies. These physical features of the present universe constitute ''records''—physical states which persist for a long time and which are correlated with the state of the universe at early times. Hence, through measurements of these quantities in the present we can make deductions about the behavior of the universe around the time of the big bang.

The detector model above is a crude model of this process. In the case of a classical system coupled to the quantum detector, a trajectory passing through the detector region causes the detector to register and, to within certain limits of detector accuracy (which may be estimated), the detector remains in the ''registered'' state along the rest of the trajectory. An observer in the late universe may then determine the detector state and from this deduce that the trajectory entered the region at some stage in the past. The fully quantum picture of this scenario is more subtle, in that the absence of a time parameter makes it difficult to speak about the detector registering ''after'' it passes through the region. Nevertheless, on the basis of the corresponding classical picture, it seems reasonable to interpret the part of the system wave function, which is correlated with the ''registered'' state of the detector, to be the amplitude for the system passing through the region. This is exactly what we will do here.[1]

It is perhaps worth noting that the question considered here bears a close resemblance to the arrival time and tunneling time questions in nonrelativistic quantum mechanics [28,29]. There also, time enters in a nontrivial way, and equivalent classical approaches to the problem are inequivalent at the quantum level. A variety of approaches have been brought to bear on these problems including, as here, explicit detector models.

In Sec. II, we solve the system Eq. (1.8) using the simple two state detector model.

Using the results, we then ask, in Sec. III, some simple questions of the detected wave function. Does the detected wave function still obey the Wheeler-DeWitt equation? We find that it does exactly outside the detector region, and that it does approximately (for small λ) inside the detector region. We compute the probabilities for detection and see, as Mott essentially saw, that they are strongly peaked when the detectors lie along a classical trajectory. We also observe that the resulting amplitude for detection bears a very close resemblance to a formula written down by Hartle [30] (without using an explicit detector model) and we discuss the connections with his result. We discuss the implementation of boundary conditions (this is the only point at which the indefinite signature of the minisuperspace metric is significant). We also discuss various other aspects of the solution in relation to detection and the timelessness of the solutions.

The results of Sec. III indicate that the wave function of the system may, in some sense, be regarded as a superposition of states each of which is peaked about an entire classi-

[1]It is of course of interest to calculate the extent to which the ''registered'' state of the detector is correlated with the particle passing through the region. To answer this question fully requires the calculation of a probability distribution for both the detectors and the histories of the system. This involves the more elaborate machinery of the decoherent histories approach [1], but will not be pursued here.

cal history. To demonstrate this explicitly, we specialize, in Sec. IV, to the case of a collection of harmonic oscillators (hence the Hamiltonian in this case is the N-dimensional harmonic oscillator Hamiltonian). We introduce a new type of coherent states, the "timeless coherent states." These states are eigenstates of the Hamiltonian, and are therefore time independent, but are peaked about classical trajectories. Any eigenstate may be expanded in terms of theses states, and we show that a series of detections along a classical path essentially projects the state down onto a timeless coherent state.

Since the detector is so simple, its dynamics may be solved exactly and this is carried out in Sec. V. This calculation confirms that the detector model is only physically realistic in the perturbative regime (when the particle spends only a short time in each detector region). Although the solution is exact and leaves the boundary conditions general, it turns out that it is not very useful for the Mott solution, since the boundary conditions lead to a rather inelegant solution. On the other hand, it is by no means clear that one is required to take the Mott boundary conditions for the analogous situation in quantum cosmology, and given the freedom to choose different conditions, an elegant alternative solution for the detector amplitude is obtained. It is in fact almost the same as the amplitude Hartle proposed [30].

In Sec. VI, we briefly describe a more elaborate detector model, in which the detector is a simple harmonic oscillator coupled to the particle with the same coupling Eq. (1.10). The solution has a nice path integral representation and suffers fewer shortcomings than the two-state detector. It also clearly illustrates the peak about classical paths.

We summarize and conclude in Sec. VII.

II. DETECTION AMPLITUDE FROM THE TWO-STATE DETECTOR

We now solve the eigenvalue equation

$$H|\Psi\rangle = E|\Psi\rangle \tag{2.1}$$

with H given by Eq. (1.8), and the detector is the simple two state detector described in Sec. I, with $H_d=0$. We will for convenience refer to this equation as the Wheeler-DeWitt equation (and it is convenient to retain a nonzero value of E). It now reads

$$(H-E)|\Psi\rangle = (H_0+\lambda H_{\rm int}-E)|\Psi\rangle = 0. \tag{2.2}$$

We initially use only two detectors, so

$$H_{\rm int} = \sum_{k=1}^{2} f_k(\mathbf{x})(a_k+a_k^\dagger) \tag{2.3}$$

but the generalization to a arbitrary number of detectors is straightforward. We solve perturbatively by writing

$$|\Psi\rangle = |\Psi^{(0)}\rangle + \lambda|\Psi^{(1)}\rangle + \lambda^2|\Psi^{(2)}\rangle + \cdots. \tag{2.4}$$

We require that in the absence of coupling to the detectors, the detectors are in the state of no detection, $|0\rangle$. This means that

$$|\Psi^{(0)}\rangle = |\psi\rangle|0\rangle|0\rangle \tag{2.5}$$

and $|\psi\rangle$ is the state we are trying to measure. Inserting in Eq. (2.2) and equating powers of λ, we get

$$(H_0-E)|\Psi^{(0)}\rangle = 0, \tag{2.6}$$

$$(H_0-E)|\Psi^{(1)}\rangle = -H_{\rm int}|\Psi^{(0)}\rangle, \tag{2.7}$$

$$(H_0-E)|\Psi^{(2)}\rangle = -H_{\rm int}|\Psi^{(1)}\rangle, \tag{2.8}$$

and similarly to higher orders. The first relation shows that $|\psi\rangle$ must obey the unperturbed eigenvalue equation, as expected. Inserting Eq. (2.5) into Eq. (2.7), we get

$$(H_0-E)|\Psi^{(1)}\rangle = -f_1(\mathbf{x})|\psi\rangle|1\rangle|0\rangle - f_2(\mathbf{x})|\psi\rangle|0\rangle|1\rangle. \tag{2.9}$$

This is readily solved by writing

$$|\Psi^{(1)}\rangle = |\Psi^{(1)}_{00}\rangle|0\rangle|0\rangle + |\Psi^{(1)}_{10}\rangle|1\rangle|0\rangle + |\Psi^{(1)}_{01}\rangle|0\rangle|1\rangle + |\Psi^{(1)}_{11}\rangle|1\rangle|1\rangle \tag{2.10}$$

and we discover that

$$(H_0-E)|\Psi^{(1)}_{00} = 0, \tag{2.11}$$

$$(H_0-E)|\Psi^{(1)}_{10}\rangle = -f_1(\mathbf{x})|\psi\rangle, \tag{2.12}$$

$$(H_0-E)|\Psi^{(1)}_{01}\rangle = -f_2(\mathbf{x})|\psi\rangle, \tag{2.13}$$

$$(H_0-E)|\psi^{(1)}_{11}\rangle = 0. \tag{2.14}$$

Equations (2.12) and (2.13) may be solved with the assistance of the Green's function G, defined by Eq. (1.7) (with H replaced by H_0-E). It obeys the equation

$$(H_0-E)G = 1. \tag{2.15}$$

[For convenience we use an operator notation in which G is the operator with coordinate representation $G(\mathbf{x,y}) = \langle \mathbf{x}|G|\mathbf{y}\rangle$ and the right-hand side of Eq. (2.15) would be the δ function $\delta(\mathbf{x,y})$ in the coordinate representation.] We then obtain

$$|\Psi^{(1)}_{10}\rangle = -Gf_1(\mathbf{x})|\psi\rangle + |\phi_1\rangle, \tag{2.16}$$

$$|\Psi^{(1)}_{01}\rangle = -Gf_2(\mathbf{x})|\psi\rangle + |\phi_2\rangle, \tag{2.17}$$

where $|\phi_{1,2}\rangle$ are solutions to the homogeneous equation

$$(H_0-E)|\phi_{1,2}\rangle = 0. \tag{2.18}$$

To fix some of these solutions more precisely, we need to appeal to boundary conditions. This is a subtle issue and depends very much on the precise context. Mott was concerned with the particular case of an outgoing spherical wave and imposed boundary conditions appropriate to this case.

This led him to set $|\phi_{1,2}\rangle$ and $|\Psi_{11}^{(1)}\rangle$ to zero (since otherwise it represents a stream of incoming particles fired at an already excited detector) [23]. We are not obviously compelled to make the same choice of boundary conditions in quantum cosmology, and we will return to a discussion of this important issue in Sec. III C. But for the moment, we work with the Mott solution.

At first order only one detector is stimulated into the excited state. The system wave function correlated with detector state $|1\rangle$ now is

$$|\psi_1\rangle = -\lambda G f_1 |\psi\rangle \tag{2.19}$$

and the probability that the detector registers is therefore

$$p_1 = \langle \psi_1 | \psi_1 \rangle = \lambda^2 \langle \psi | f_1 G^\dagger G f_1 | \psi \rangle. \tag{2.20}$$

[When the spectrum of H_0 is continuous, expressions of this type need to be regularized along the lines of Eq. (1.6), but we will carry out this explicitly only when we need to calculate it in more detail.]

To get two detectors to register, we need to go to second order. We now have, from Eq. (2.8), and the solution Eq. (2.10)

$$\begin{aligned}(H_0 - E)|\Psi^{(2)}\rangle &= -(f_1(a_1 + a_1^\dagger) + f_2(a_2 + a_2^\dagger))|\Psi^{(1)}\rangle \\ &= f_2 G f_1 |\psi\rangle |1\rangle |1\rangle + f_1 G f_2 |\psi\rangle |1\rangle |1\rangle \\ &\quad + \cdots,\end{aligned} \tag{2.21}$$

where the omitted terms on the right-hand side are proportional to $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, and $|1\rangle|0\rangle$, and will not be needed. Again we may solve them by expanding as in Eq. (2.10). Here we write down only the term required, which describes two detectors being excited

$$|\Psi^{(2)}\rangle = |\Psi_{11}^{(2)}\rangle |1\rangle |1\rangle + \cdots \tag{2.22}$$

and it is readily seen that the solution is

$$|\Psi_{11}^{(2)}\rangle = (G f_2 G f_1 + G f_1 G f_2)|\psi\rangle. \tag{2.23}$$

Again, following the spirit of the Mott solution, possible homogeneous solutions are set to zero. We now have the system wave function correlated with two detectors registering: it is

$$|\psi_2\rangle = \lambda^2 (G f_2 G f_1 + G f_1 G f_2)|\psi\rangle \tag{2.24}$$

and the probability is $\langle \psi_2 | \psi_2 \rangle$.

The analysis is readily extended to an arbitrary number of detectors, but it is easy to anticipate the result from Eq. (2.24): for n detectors, the amplitude is

$$|\psi_n\rangle = \lambda^n (G f_n G f_{n-1} \cdots G f_2 G f_1)|\psi\rangle + \text{symmetrizations}, \tag{2.25}$$

where "symmetrizations" means add all possible permutations of $1,2,3,\cdots n$. It is clear that these terms are there to ensure that there is no preference in the order in which each of the detectors registers, reflecting the genuinely timeless nature of the underlying dynamics. Equation (2.25) is the main result of this section.

III. PROPERTIES OF THE SOLUTION

We have shown that the wave function for the whole system when there are, for example, two detectors takes the form

$$|\Psi\rangle = |\psi_0\rangle|0\rangle|0\rangle + |\psi_1\rangle(|1\rangle|0\rangle + |0\rangle|1\rangle) + |\psi_2\rangle|1\rangle|1\rangle. \tag{3.1}$$

We can now ask various questions of the detected wave function $|\psi_2\rangle$, or more generally, Eq. (2.25).

A. Does the detected wave function obey the Wheeler-DeWitt equation?

As stated in Sec. I, a prevalent idea in quantum gravity is that all observables should commute with the total Hamiltonian [3,10,11]. Related to this is the notion that "measurements" of the wave function (whatever this may mean in general) should not displace the system from its eigenstate of the Hamiltonian. Given that we have presented here an explicit model of detection, it is perhaps of interest to ask to what extent this idea holds up.

We have, using Eq. (2.15), and taking the simple case $n=2$,

$$(H_0 - E)|\psi_2\rangle = \lambda^2 (f_2 G f_1 + f_1 G f_2)|\psi\rangle. \tag{3.2}$$

In configuration space, the right-hand side is zero, except in the detector regions Δ_1, Δ_2, because the functions f_1 and f_2 are localized there. In these regions, it is of order λ^2, which we regard as small in comparison to the terms on the left. Hence the measured wave function approximately obeys the Wheeler-DeWitt equation. This is no surprise. In standard quantum mechanics, a physically measured system does not obey the Schrödinger equation but has corrections due to the measuring device. That it obeys the Wheeler-DeWitt equation only approximately is not in conflict with exact reparametrization invariance, since the wave function for the entire system always obeys the Wheeler-DeWitt equation exactly. On the other hand, one wonders whether it might not be possible to devise a detection scheme in which the detected amplitude obeys the Wheeler-DeWitt equation exactly. For example, given the simple toolbox of parts outlined in Sec. I [such as the Green's function Eq. (1.7)], it would not be unreasonable to guess that the detection amplitude might be of the form Eq. (2.25) but with the G given by Eq. (1.7) replaced by the one obtained by integrating τ over an infinite range. This gives a solution to the Wheeler-DeWitt equation, and as a consequence, the modified detection amplitude would also obey the Wheeler-DeWitt equation exactly.

B. Amplitudes for classical paths

Of greater interest is the question of the configurations about which the amplitude Eq. (2.25) (or the associated probability) is peaked. The amplitude may be written in an explicit coordinate representation as

$$\langle \mathbf{x}_f | \psi_n \rangle = \lambda^n \int d^N \mathbf{x}_n \cdots d^N \mathbf{x}_2 \, d^N \mathbf{x}_1 \, G(\mathbf{x}_f, \mathbf{x}_n) f_n(\mathbf{x}_n) \times G(\mathbf{x}_n, \mathbf{x}_{n-1}) f_{n-1}(\mathbf{x}_{n-1}) \cdots G(\mathbf{x}_2, \mathbf{x}_1) f_1(\mathbf{x}_1) \psi(\mathbf{x}_1) \tag{3.3}$$

(plus symmetrizations). It has the form of approximate projections onto the regions Δ_k (exact projections if the f_k are window functions) with evolution between regions described by the fixed energy propagator $G(\mathbf{x}_{k+1}, \mathbf{x}_k)$. It is analogous to the amplitude for a history of positions at a sequence of times in nonrelativistic quantum mechanics, which is known to be peaked about classical trajectories [31]. But note that here the evolution is at fixed values of the energy, and there is no reference to time.

We can estimate the form of Eq. (3.3) using a WKB approximation for the fixed energy propagator [32]. It is given by an expression of the form

$$G(\mathbf{x}'', \mathbf{x}') = C(\mathbf{x}'', \mathbf{x}') e^{iS(\mathbf{x}'', \mathbf{x}')}, \tag{3.4}$$

where C is a slowly varying prefactor and $S(\mathbf{x}'', \mathbf{x}')$ is the fixed energy Hamilton-Jacobi function, i.e., the action of the classical solution from $\mathbf{x}'$ to $\mathbf{x}''$ with fixed energy E [33]. The initial and final momenta of this classical solution are

$$\mathbf{p}'' = \nabla_{\mathbf{x}''} S(\mathbf{x}'', \mathbf{x}'), \quad \mathbf{p}' = -\nabla_{\mathbf{x}'} S(\mathbf{x}'', \mathbf{x}'). \tag{3.5}$$

In terms of Eq. (3.4), the amplitude Eq. (3.3) may be written

$$\langle \mathbf{x}_f | \psi_n \rangle = \lambda^n \int d^N \mathbf{x}_n \cdots d^N \mathbf{x}_2 \, d^N \mathbf{x}_1 \prod_{k=1}^{n} C(\mathbf{x}_{k+1}, \mathbf{x}_k) f_k(\mathbf{x}_k) \times \exp\left(i \sum_{k=1}^{n} S(\mathbf{x}_{k+1}, \mathbf{x}_k) \right) \psi(\mathbf{x}_1), \tag{3.6}$$

where $\mathbf{x}_{n+1} = \mathbf{x}_f$.

Consider the integrals over $\mathbf{x}_2 \cdots \mathbf{x}_n$ with $\mathbf{x}_1$ (and $\mathbf{x}_{n+1}$) fixed. Suppose for the moment that the functions f_k are absent so the integrals are unrestricted. By the stationary phase approximation, the dominant contribution to the integral comes from the values of $\mathbf{x}_2 \cdots \mathbf{x}_n$ for which the phase is stationary, i.e., for which

$$\nabla_{\mathbf{x}_j} \sum_{k=1}^{n} S(\mathbf{x}_{k+1}, \mathbf{x}_k) = 0 \tag{3.7}$$

for $j = 2, \cdots n$. This equation means that

$$\nabla_{\mathbf{x}_j} S(\mathbf{x}_{j+1}, \mathbf{x}_j) + \nabla_{\mathbf{x}_j} S(\mathbf{x}_j, \mathbf{x}_{j-1}) = 0. \tag{3.8}$$

Using Eq. (3.5), this implies that the final momentum of the classical path from $\mathbf{x}_{j-1}$ to $\mathbf{x}_j$ is equal to the initial momentum of the classical path from $\mathbf{x}_j$ to $\mathbf{x}_{j+1}$. It is not difficult to see that this in turn implies that the point $\mathbf{x}_j$ must lie on the classical path from $\mathbf{x}_{j-1}$ to $\mathbf{x}_{j+1}$. Hence, the stationary phase points of the whole integral Eq. (3.6) lie on the classical path from $\mathbf{x}_1$ to $\mathbf{x}_f$. The approximate value of the integral is the integrand of Eq. (3.6) with the stationary phase point values inserted.

Now consider what happens when the restricting functions f_k are present. If the regions Δ_k (about which the f_k are concentrated) include the stationary phase points (and if the regions are larger than the fluctuations about these points) then, since the integral takes its dominant contribution from these points, the presence of the f_k makes little difference and the integral is given once again by its stationary phase value. On the other hand, if one or more of the f_k lie far away from the stationary phase points then, since the integral is prevented from taking a contribution from these points, its value must be much smaller than the stationary phase value. We thus see that the amplitude Eq. (3.3) will be largest when the regions Δ_k are chosen to include the stationary phase points of the integral. As we have shown, the stationary phase points lie along the classical path from $\mathbf{x}_1$ to $\mathbf{x}_f$. It follows that the amplitude Eq. (3.2) will be largest when the regions Δ_k are chosen to lie along a classical path.

Mott's argument for straight line paths in an expression analogous to Eq. (2.25) relied on the explicit from the Green function in the free particle case, and on a special initial state [23]. Here we see that the peaking about classical paths can be seen, at least heuristically, from elementary properties of path integrals, for a broad class of Hamiltonians and initial states. Bell has also discussed the Mott calculation at some length [34]. He notes that the first projector f_1 spatially localizes Mott's initial wave function, but in a realistic ionizing event, the resultant uncertainty in momentum can still be extremely small. As a consequence the angular spread of the wave packet in its subsequent evolution can be extremely small, hence the appearance of a straight line track.

C. Comparison with a result of Hartle

The result Eq. (2.24) is very closely related to a result of Hartle [30]. He considered a simple model quantum cosmology with a Hamiltonian quadratic in the momenta, as here, and asked for the amplitude that the system passes through two regions of configuration space: Δ_1, Δ_2. Using some simple arguments about propagators and elementary principles of quantum theory, he showed that the amplitude is (in the language of the present paper)

$$(Gf_2Gf_1 + Gf_1Gf_2)|\psi\rangle - (G^\dagger f_2 G^\dagger f_1 + G^\dagger f_1 G^\dagger f_2)|\psi\rangle, \tag{3.9}$$

where ψ is a solution to the Wheeler-DeWitt equation and f_1 and f_2 are taken to be exact projectors onto the regions Δ_1, Δ_2. Other than the factor of λ^2 in Eq. (2.24) which is not important, Hartle's result differs from Eq. (2.24) by the subtraction of an identical term but with G replaced by $G^\dagger$ (which is generally not the same as G). Hartle argues that this should be there on the grounds that, in an expression like Eq. (2.25) with G represented by Eq. (1.7), the time parametrization should not have a preferred direction with $\mathbf{y}$ the "initial" point of the parametrization and $\mathbf{x}$ the "final" point, hence we should sum the amplitude over both possible parametrization directions.

To understand why this term can be there in the present calculation, let us first elaborate on the Green's function. G, as defined by Eq. (1.7), may be written more explicitly as

$$G(\mathbf{x},\mathbf{y},E)=\sum_n \frac{u_n^*(\mathbf{x})u_n(\mathbf{y})}{E-E_n+i\epsilon}, \tag{3.10}$$

where $u_n(\mathbf{x})$ are eigenfunctions of H_0 with the eigenvalue E_n. $G(\mathbf{x},\mathbf{y},E)$ is real when E lies in the discrete part of the spectrum and complex when E lies in the continuous part (see, for example, Ref. [35]). Hence in the free particle case considered by Mott, $G\neq G^\dagger$, but for the harmonic oscillator (considered in Sec. IV) $G=G^\dagger$. Quantum cosmological models usually have a spectrum which is at least in part continuous, so we expect $G\neq G^\dagger$ in general.

Now recall the comments after Eq. (2.18), where it was noted that we are by no means obliged in quantum cosmology to take the same boundary conditions as Mott. Since $G^\dagger$ also satisfies Eq. (2.15), we may use it in place of G to generate solutions to the detector amplitude. Because $G-G^\dagger=i\delta(H_0-E)$, it is easily seen that the difference between using G and $G^\dagger$ is a homogeneous solution in Eqs. (2.16) and (2.17). It is also true that, in higher order perturbations, we may use G or $G^\dagger$ or some combination. Therefore, there is a more general class of solutions for the detection amplitude which are sums of terms of the form Eq. (2.25) with some of the G's replaced by $G^\dagger$'s (and with a suitable overall normalization). In particular, Hartle's amplitude falls into this enlarged class of solutions, so there is no conflict with his result.

Mott made a particular choice of solution appropriate to the physical situation he was investigating. In the case of relativistic field theory in Minkowski space, one would normally impose some sort of causality requirement to fix the solution more precisely, and therefore to choose between G and $G^\dagger$ (since G is essentially the Feynman propagator). One could require, for example, that the wave function for the whole system be affected by the detector only in the future light cone of the detector region. In quantum cosmology, however, although the metric has hyperbolic signature like Minkowski space, it is by no means clear that one is obliged to impose an analogous requirement, and in fact it is difficult to see exactly how to do this in general since the configuration space is usually not globally hyperbolic. (See, however, Ref. [12]). Instead, one might expect to fix the choice of G or $G^\dagger$ by appealing to cosmological boundary conditions. The no-boundary proposal of Hartle and Hawking, for example, picks out a wave function that is real [25]. It is sometimes claimed that this corresponds to a "time-symmetric" wave function [36]. It therefore stays most closely to the timeless nature of the Wheeler-DeWitt equation and in some sense represents a complete abandonment of any fundamental notion of causality. It is now interesting to note that Hartle's detection wave function Eq. (3.8) is in fact real if ψ is real. In many ways this therefore seems like the most natural solution to take in the case of the Wheeler-DeWitt equation. We do not, however, in this paper commit to any particular choice of boundary condition, as stated in Sec. I.

In summary, therefore, a more general solution to the detector model is a sum of terms of the form Eq. (2.25) involving both G and $G^\dagger$, and this more general solution includes Hartle's result Eq. (3.14). Note also that the replacement of G by $G^\dagger$ does not affect the discussion of the peaking of the amplitude about classical trajectories.

D. On timelessness and detection

It is perhaps worth elaborating on a feature of the detector model which appears at first sight to be incompatible with the timelessness of the Wheeler-DeWitt equation. We have coupled the system to a series of detectors via the interaction Hamiltonian Eq. (1.8). This Hamiltonian describes a situation in which, along a trajectory $\mathbf{x}(t)$, the detector is in the ground state "before" the trajectory enters the detector region and in an excited state "after" it has passed through the region. Along a *classical* trajectory $\mathbf{x}(t)$ in which there is a notion of time, and of before and after, this is undeniably correct. (The parameter t simply labels the points along the curve—it may be taken to be, for example, the distance along the curve from some reference point.) But how are we to understand how the detector works in the genuinely timeless world described by the Wheeler-DeWitt equation? There is no before and after and there is no preferred direction of time.

The above results effectively imply that each solution to the Wheeler-DeWitt equation may be regarded as a superposition of states, each of which is concentrated along an entire classical trajectory in configuration space (and we will see this in more detail in Sec. IV). What then seems to be happening in the present model including the detector, is that each trajectory carries a label indicating whether or not it passes through the detector region at any stage along its entire length. In the (perhaps restrictive) language of time, at each point along the trajectory, the label allows the trajectory to "know" whether it passed through the detector region in the past, or will pass through it in the future. But it is only on adopting this temporal language that the situation seems paradoxical. The paradox vanishes when ones speaks the vocabulary of entire trajectories in configuration space, and one can see this in the solution, Eq. (3.1). The wave function of the entire system is written as a correlated state in which the state correlated with the detectors in the $|1\rangle$ state is the state Eq. (2.25): the detected state is indeed concentrated on trajectories that pass through $\Delta_1,\cdots\Delta_n$.

The issues discussed in this section may be of relevance to the perennial debate on the question of time asymmetry in quantum cosmology [20,37].

IV. COHERENT STATES FOR TIMELESS DYNAMICS

Since the peaking about classical paths is the most important property of the amplitude Eq. (2.25), it is worth exploring it in more detail for the special case of a collection of harmonic oscillators, where it is possible to show very clearly how the solutions to the Wheeler-DeWitt equation correspond to superpositions of states peaked about classical paths. We will introduce a class of coherent states appropriate to the timeless theories considered here and which are natural analogs of the standard coherent states of the harmonic oscillator.

The Hamiltonian for a set of N identical harmonic oscillators is

$$H_0=\frac{1}{2}(\mathbf{p}^2+\mathbf{x}^2). \tag{4.1}$$

In this case the spectrum of H_0 is discrete and we have

$$\delta(H_0-E)=\int_0^{2\pi}\frac{dt}{2\pi}e^{-i(H_0-E)t}. \tag{4.2}$$

Since $\delta(H_0-E)$ is now in fact a true projection operator we may write $\delta(H_0-E)^2=\delta(H_0-E)$ without having to worry about regularization through the induced inner product, as in the continuous case. (In this expression E is allowed to take only the discrete values corresponding to the spectrum of H_0). The Green function G is given as before by Eq. (1.7) with H replaced by H_0-E. For a one-dimensional oscillator, Eq. (4.2) is equivalent to

$$\delta(H_0-E)=|E\rangle\langle E|, \tag{4.3}$$

where $|E\rangle$ is the energy eigenstate. In more than one dimension the energy eigenstates are degenerate, so Eq. (4.2) has the form

$$\delta(H_0-E)=\sum_d |E,d\rangle\langle E,d|, \tag{4.4}$$

where $|E, d\rangle$ are the energy eigenstates with degeneracy label d.

The standard coherent states (see Ref. [38], for example) are denoted $|\mathbf{p}, \mathbf{x}\rangle$ and they have the important property that they are preserved in form under unitary evolution,

$$e^{-iH_0t}|\mathbf{p},\mathbf{x}\rangle=|\mathbf{p}_t,\mathbf{x}_t\rangle, \tag{4.5}$$

where $\mathbf{p}_t$, $\mathbf{x}_t$ are the classical solutions matching $\mathbf{p}$, $\mathbf{x}$ at $t=0$, hence they are strongly peaked about the classical path. We are interested in finding a set of states which are analogs of these states for the timeless case. That is, they should be eigenstates of H_0, and should be peaked about the classical paths of given fixed energy in phase space. It is not difficult to see that a set of states doing the job are

$$\begin{aligned}|\phi_{\mathbf{px}}\rangle&=\delta(H_0-E)|\mathbf{p},\mathbf{x}\rangle\\&=\int_0^{2\pi}\frac{dt}{2\pi}e^{-i(H_0-E)t}|\mathbf{p},\mathbf{x}\rangle\\&=\int_0^{2\pi}\frac{dt}{2\pi}e^{iEt}|\mathbf{p}_t,\mathbf{x}_t\rangle.\end{aligned} \tag{4.6}$$

These states are not in fact normalized to unity but we shall see that it is useful to work with them as they are. Since the states $|\mathbf{p}_t,\mathbf{x}_t\rangle$ are concentrated at a phase space point for each t, clearly integrating t over a whole period produces a state that is concentrated along the entire classical trajectory. Each state is labeled by a fiducial phase space point $\mathbf{p}$, $\mathbf{x}$ which determines the classical trajectory the state is peaked about. Under evolution of the fiducial point $\mathbf{p}$, $\mathbf{x}$ to another point, $\mathbf{p}_s$, $\mathbf{x}_s$, say, along the same classical trajectory, the state changes by a phase

$$|\phi_{\mathbf{px}}\rangle\rightarrow|\phi_{\mathbf{p}_s\mathbf{x}_s}\rangle=e^{iEs}|\phi_{\mathbf{px}}\rangle \tag{4.7}$$

as may be seen from Eqs. (4.5) and (4.6). We will refer to these states as *timeless coherent states*. Their properties are in fact very similar to the usual coherent states.

Two timeless coherent states of different energy are exactly orthogonal. The more interesting case is that in which they have the same energy, and then they are approximately orthogonal if they correspond to sufficiently distinct classical solutions. This is because we have

$$\begin{aligned}\langle\phi_{\mathbf{p}'\mathbf{x}'}|\phi_{\mathbf{px}}\rangle&=\langle\mathbf{p}',\mathbf{x}'|\delta(H_0-E)|\mathbf{p},\mathbf{x}\rangle\\&=\int_0^{2\pi}\frac{dt}{2\pi}e^{iEt}\langle\mathbf{p}',\mathbf{x}'|\mathbf{p}_t,\mathbf{x}_t\rangle.\end{aligned} \tag{4.8}$$

From the properties of the standard coherent states we know that

$$|\langle\mathbf{p}',\mathbf{x}'|\mathbf{p}_t,\mathbf{x}_t\rangle|\leq 1 \tag{4.9}$$

with equality if and only if $\mathbf{p}'=\mathbf{p}_t$ and $\mathbf{x}'=\mathbf{x}_t$. Moreover, the overlap of two coherent states is exponentially small if they are centered around phase space points that are sufficiently far apart. It follows that if $\mathbf{p}'$, $\mathbf{x}'$ does not lie on, or close to, the trajectory $\mathbf{p}_t$, $\mathbf{x}_t$, the overlap $\langle\mathbf{p}',\mathbf{x}'|\mathbf{p}_t,\mathbf{x}_t\rangle$ will always be exponentially small for all t. The integral over t in Eq. (4.8) will then give a result that is much smaller than the case in which $\mathbf{p}'$, $\mathbf{x}'$ does lie on, or close to, the trajectory $\mathbf{p}_t$, $\mathbf{x}_t$ (because in the latter case the overlap $\langle\mathbf{p}',\mathbf{x}'|\mathbf{p}_t,\mathbf{x}_t\rangle$ becomes close to unity for some value of t). The timeless coherent states are therefore approximately orthogonal for sufficiently distinct classical trajectories.

Note that the coherent states $|\mathbf{p},\mathbf{x}\rangle$ are in fact already *approximate* eigenstates of H_0, with eigenvalue $\frac{1}{2}(\mathbf{p}^2+\mathbf{x}^2)$, as long as $|\mathbf{p}|$, $|\mathbf{x}|$ are much larger than the coherent state quantum fluctuations. Given E, it therefore seems reasonable to choose the values $\mathbf{p}$, $\mathbf{x}$ in the fiducial coherent state so that $E=\frac{1}{2}(\mathbf{p}^2+\mathbf{x}^2)$, when constructing the timeless states Eq. (4.6). With this choice, the timeless coherent states are approximately normalized to unity, $\langle\phi_{\mathbf{px}}|\phi_{\mathbf{px}}\rangle\approx 1$.

The standard completeness relation for the coherent states is

$$\int\frac{d^N\mathbf{p}d^N\mathbf{x}}{(2\pi)^N}|\mathbf{px}\rangle\langle\mathbf{px}|=1. \tag{4.10}$$

Multiplying both sides by $\delta(H_0-E)$ from the left and right, and using Eq. (4.6), we get

$$\int\frac{d^N\mathbf{p}d^N\mathbf{x}}{(2\pi)^N}|\phi_{\mathbf{px}}\rangle\langle\phi_{\mathbf{px}}|=\delta(H_0-E). \tag{4.11}$$

Since $\delta(H_0-E)|\psi\rangle=|\psi\rangle$ on any solution to the eigenvalue equation $(H_0-E)|\psi\rangle=0$, this is as good as a completeness relation on the set of solutions to the eigenvalue equation (which is all we are interested in). We may therefore write any solution $|\psi\rangle$ as a superposition of timeless coherent states

$$|\psi\rangle = \int \frac{d^N\mathbf{p}d^N\mathbf{x}}{(2\pi)^N}|\phi_{\mathbf{px}}\rangle\langle\phi_{\mathbf{px}}|\psi\rangle. \tag{4.12}$$

It is then tempting to interpret $\langle\phi_{\mathbf{px}}|\psi\rangle$ as the amplitude that a system in the state $|\psi\rangle$ will be found on the classical trajectory corresponding to the timeless coherent state $|\phi_{\mathbf{px}}\rangle$. Again using the fact that $\delta(H_0-E)|\psi\rangle=|\psi\rangle$ it is easy to see that this is in fact the same as $\langle\mathbf{p},\mathbf{x}|\psi\rangle$, which is the amplitude for finding the system at the phase space point $\mathbf{p}$, $\mathbf{x}$ labeling the trajectory. The probability is then simply $|\langle\mathbf{p},\mathbf{x}|\psi\rangle|^2$. This is a simple and intuitively appealing result: the classical trajectory is completely fixed by its initial values $\mathbf{x}$, $\mathbf{p}$, hence we expect that the probability for being found on a certain classical trajectory is the same as the probability for being found at the initial phase space point that labels it.

This interpretation is put forward with a small note of caution, however, since the sum over $\mathbf{p}$, $\mathbf{x}$ in Eq. (4.12) is not only over states that are only approximately orthogonal (as with the usual coherent states) but, because of the property Eq. (4.7), includes some redundancy in the summation. In particular, $|\phi_{\mathbf{px}}\rangle\langle\phi_{\mathbf{px}}|$ is invariant along the classical phase space trajectory of its fiducial point. Since this only produces some sort of constant factor, it may not make any difference, and indeed, the above interpretation appears to produce sensible results. Still, it would be desirable to include, if possible, some sort of ''gauge fixing'' which factors out this redundancy. This will be explored elsewhere.

Given all of this background, we may now reconsider the detector amplitude Eq. (2.25). We expand the initial state $|\psi\rangle$ in the timeless coherent states, as in Eq. (4.12), so we need only consider the amplitude Eq. (2.25) operating on a timeless coherent state $|\phi_{\mathbf{px}}\rangle$. Consider the integral representation of it, Eq. (4.6). When f_1 operates on the state, it gives zero if the trajectory $\mathbf{p}_t$, $\mathbf{x}_t$ never passes through the region Δ_1. If it does pass through, it has the effect of restricting the time integral to the amount of time Δt_1 the trajectory spends in the region Δ_1

$$f_1|\phi_{\mathbf{px}}\rangle \approx \int_{\Delta t_1}\frac{dt}{2\pi}e^{iEt}|\mathbf{p}_t,\mathbf{x}_t\rangle. \tag{4.13}$$

Next, to consider the operation of G, we write it as

$$G=i\int_0^\infty d\tau e^{-i\tau(H-i\epsilon)}, \tag{4.14}$$

where $H=H_0-E$. Operating with this on Eq. (4.13), we obtain

$$Gf_1|\phi_{\mathbf{p},\mathbf{x}}\rangle \approx i\int_0^\infty d\tau\int_{\Delta t_1}\frac{dt}{2\pi}e^{i\tau(E+i\epsilon)+iEt}|\mathbf{p}_{t+\tau},\mathbf{x}_{t+\tau}\rangle, \tag{4.15}$$

where we have used Eq. (4.5). The integration over τ, since $|\mathbf{p}_t,\mathbf{x}_t\rangle$ is periodic with period 2π, means that Eq. (4.15) is concentrated along the same entire classical trajectory as the original timeless coherent state. Furthermore, from the form of the time integrals in Eq. (4.15), one can see that the amplitude is proportional to the original timeless coherent state Eq. (4.6) [although the time integrations may need regulating along the lines of Eq. (1.6)].

The operation of f_2 produces zero if the trajectory fails to pass through Δ_2, and a result similar to Eq. (4.13) if it passes through. The subsequent operation of G again produces a state concentrated along the entire trajectory. Proceeding in a similar manner to the end of the chain, we therefore find that the detector amplitude Eq. (2.25) with a timeless coherent state as the initial state is zero unless all the detection regions $\Delta_1,\Delta_2,\cdots\Delta_n$ lie along the trajectory of the timeless coherent state, and in that case the amplitude is proportional to the original state Eq. (4.6).

It follows that the detector amplitude Eq. (2.25) operating on a general state [expanded in timeless coherent states as in Eq. (4.12)] consists of a superposition of only those timeless coherent states that pass through all the detector regions $\Delta_1,\Delta_2\cdots\Delta_n$. This could be quite a large sum of states if the regions Δ_k are large. However, it will consist of essentially just one timeless coherent state if the detector regions lie along a classical trajectory and if their size is just bigger than the spatial width of the wave packet. The detection amplitude along this classical trajectory is then proportional to $\langle\phi_{\mathbf{px}}|\psi\rangle$, in agreement with the analysis based on Eq. (4.12).

Rovelli has written down a coherent state of the type considered here [17] in the context of a very similar model, although its properties were not explored and exploited as they are here. Klauder, in his approach to the quantization of constrained systems using coherent states, considered the projection of the standard coherent states onto the constraint subspace, hence in essence wrote down states of the form Eq. (4.6) [39]. He did not, however, consider their use as an interpretational tool. Wave packet solutions to the Wheeler-DeWitt equation, which approximately track the classical trajectories for more interesting cosmological models, have been considered in Refs. [40], [41].

V. EXACT SOLUTION TO THE DETECTOR DYNAMICS

In this section we discuss an alternative method of a solution of the eigenvalue equation (2.2). As we shall see, it does not in fact give a very elegant representation of the Mott solution, which is why it was not used above. If, however, one is permitted to use different boundary conditions, as may be reasonable in quantum cosmology, then it provides an alternative possible solution to the detector dynamics.

We consider first the case of a single detector. The key observation is that a solution to the eigenvalue equation (2.2) may be generated using either of the expressions:

$$|\Psi\rangle = \delta(H-E)|\phi\rangle \tag{5.1}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau e^{-i\tau(H_0+\lambda H_{\rm int}-E)}|\phi\rangle. \tag{5.2}$$

Here $|\phi\rangle$ is an arbitrary fiducial state in the joint system-detector Hilbert space. It is ambiguous up to the addition of a term of the form $(H-E)|\phi'\rangle$. For the moment we keep $|\phi\rangle$ general and take

$$|\phi\rangle=|\chi_0\rangle|0\rangle+|\chi_1\rangle|1\rangle. \tag{5.3}$$

To evaluate Eqs. (5.1) or (5.2) we introduce the eigenstates of $a+a^\dagger$, which are

$$|\pm\rangle=\frac{1}{\sqrt{2}}(|0\rangle\pm|1\rangle). \tag{5.4}$$

The expression Eq. (5.1) is then readily evaluated with the result

$$|\Psi\rangle=|\psi_{\rm nd}\rangle|0\rangle+|\psi_d\rangle|1\rangle, \tag{5.5}$$

where

$$|\psi_{\rm nd}\rangle=\tfrac{1}{2}(\delta(h+\lambda f_1)+\delta(h-\lambda f_1))|\chi_0\rangle+\tfrac{1}{2}(\delta(h+\lambda f_1)-\delta(h-\lambda f_1))|\chi_1\rangle, \tag{5.6}$$

$$|\psi_d\rangle=\tfrac{1}{2}(\delta(h+\lambda f_1)-\delta(h-\lambda f_1))|\chi_0\rangle+\tfrac{1}{2}(\delta(h+\lambda f_1)+\delta(h-\lambda f_1))|\chi_1\rangle \tag{5.7}$$

(recalling that $h=H-E$). These expressions are perhaps more easily appreciated using a path integral representation

$$\psi_{\rm nd}(\mathbf{x}_f)=\int_{-\infty}^{\infty}d\tau e^{i\tau E}\int\mathcal{D}\mathbf{x}(t)\exp(iS[\mathbf{x}(t)])\times\left[\cos\left(\lambda\int_0^\tau dt f_1(\mathbf{x}(t))\right)\chi_0(\mathbf{x}_0)+i\sin\left(\lambda\int_0^\tau dt f_1(\mathbf{x}(t))\right)\chi_1(\mathbf{x}_0)\right] \tag{5.8}$$

$$\psi_d(\mathbf{x}_f)=\int_{-\infty}^{\infty}d\tau e^{i\tau E}\int\mathcal{D}\mathbf{x}(t)\exp(iS[\mathbf{x}(t)])\times\left[i\sin\left(\lambda\int_0^\tau dt f_1(\mathbf{x}(t))\right)\chi_0(\mathbf{x}_0)+\cos\left(\lambda\int_0^\tau dt f_1(\mathbf{x}(t))\right)\chi_1(\mathbf{x}_0)\right]. \tag{5.9}$$

Equations (5.8) and (5.9) represent the exact solution to Eq. (2.2) with one detector in place.

Turn now to the question of the fiducial state. The condition Eq. (2.5), suggests that we should take

$$|\phi\rangle=|\chi_0\rangle|0\rangle, \tag{5.10}$$

where $|\psi\rangle=\delta(H_0-E)|\chi_0\rangle$, and hence that $|\chi_1\rangle=0$. In fact, in the induced inner product scheme, we may take $|\chi_0\rangle=|\psi\rangle$, since effectively $\delta(H_0-E)^2=\delta(H_0-E)$. This therefore yields a path integral expression for the amplitude for detection and has the property that the factoring condition Eq. (2.5) is satisfied when $\lambda=0$. So let us first explore the properties of Eqs. (5.8) and (5.9) with $\chi_1=0$.

The nature of the sums over paths $\mathbf{x}(t)$ in Eqs. (5.8) and (5.9) is governed by the quantity

$$\tau_d=\int_0^\tau dt f_1(\mathbf{x}(t)) \tag{5.11}$$

appearing in the sine and cosine factors. With f_1 normalized to be dimensionless, this quantity has the dimensions of time, and is essentially the time spent by the path $\mathbf{x}(t)$ in the region Δ_1 around the detector. [This is not of course a physically measurable time. The paths $\mathbf{x}(t)$ in the path integral have a well-defined notion of time, but after summing over the total time duration of each path τ the final result is time independent.] Now, in the amplitude for no detection, Eq. (5.8), the factor $\cos(\lambda\tau_d)$ is 1 for $\tau_d=0$, and decreasing for τ_d increasing from zero. In the path integral, it therefore has the effect of suppressing paths that pass close to the detector, and favors paths that stay away from it. Similarly, in the detection amplitude, Eq. (5.9), the factor $\sin(\lambda\tau_d)$ is zero for paths that spend no time near the detector, and nonzero for paths that enter the detector region. It therefore enhances the amplitude for paths entering the detector region. The sine and cosine factors therefore, in a very crude way, enforce restrictions on the paths corresponding to entering or not entering the detector region, as one would expect. We can also see, however, that these factors only do their job well if $\lambda\tau_d$ is somewhat smaller than 1, indicating that the detector model is only physically sensible in the perturbative regime about $\lambda=0$, as expected.

It is now important to check the agreement between the exact result above and the perturbative result of Sec. II. It is not difficult to show that the exact solution Eq. (5.9) with $\chi_1=0$ does not in fact agree with the perturbative solution Eq. (2.19). It differs by the presence of homogeneous solutions in the small λ limit of Eq. (5.9). We will not go into detail but it may be shown using Eq. (5.7) and the identity

$$\delta(h+\lambda f)=\delta(h)+i\lambda(GfG-G^\dagger fG^\dagger)+O(\lambda^2) \tag{5.12}$$

[which is proved using the exponential representation Eq. (5.2)]. Recall that in the perturbative solution the homogeneous solutions were removed at each order in perturbation theory essentially by inspection. Since no corresponding condition has been imposed here, it is not surprising that these spurious solutions crop up. We can, however, get agreement if we choose $|\chi_1\rangle=i\lambda fG^\dagger|\chi_0\rangle$ in Eq. (5.9) [again proved using Eq. (5.12)], although there is no obvious independent reason for making this choice. Furthermore, the presence of the Green's function $G^\dagger$ in the fiducial state rather destroys the elegance of the path integral representation compared to the case $\chi_1=0$.

On the other hand, although the Mott solution is not readily obtained, the removal of the homogeneous solutions in the perturbative solution is a subtle matter of boundary conditions, above and beyond the basic factoring condition Eq. (5.10). As discussed in Sec. III, in the case of quantum cosmology one is not obviously obliged to take the same conditions, and indeed one can argue that, beyond Eq. (5.10), the boundary conditions are up for grabs. Indeed, one could effectively choose boundary conditions by *proposing* that the solution is given by the formula Eq. (5.1) with the factoring condition Eq. (5.10). As one can see from Eq. (5.12), this proposal again produces solutions with a time symmetric flavor to them (i.e., with an equal number of G's and $G^\dagger$'s in the solution), a feature which persists to the case of more than one detector. Not surprisingly, this choice does in fact produce, in essence, the Hartle amplitude Eq. (3.8).

Therefore, in this section we have produced another candidate expression for the detection amplitude, which is arguably the more appropriate one for quantum cosmology. Furthermore, the means of generating it, Eqs. (5.1), (5.2), are readily generalizable to more complicated situations.

VI. AN IMPROVED DETECTOR MODEL

We now briefly consider a more elaborate detector model that consists of a harmonic oscillator instead of the simple two state system. So we take, in the case of a single detector,

$$H_d = \omega a^\dagger a, \quad H_{\rm int} = f(\mathbf{x})(a + a^\dagger). \tag{6.1}$$

The energy eigenstate of the total Hamiltonian is again calculated using Eq. (5.2), with the factored fiducial state Eq. (5.10), where $|0\rangle$ is the harmonic oscillator ground state. In terms of a path integral, denoting the harmonic oscillator coordinates by q,

$$\Psi(\mathbf{x}_f, q_f) = \int_{-\infty}^{\infty} d\tau e^{iE\tau} \int \mathcal{D}\mathbf{x}(t)\mathcal{D}q(t)\exp(iS_0[\mathbf{x}(t)] + iS_d[q(t)] + i\lambda S_{\rm int}[q(t),\mathbf{x}(t)])\chi_0(\mathbf{x}_0)u_0(q_0), \tag{6.2}$$

where $u_0(q) = \langle q|0\rangle$. The integral over q is conveniently rewritten as

$$\int \mathcal{D}q(t)\exp(iS_d[q(t)] + i\lambda S_{\rm int}[q(t),\mathbf{x}(t)])u_0(q_0) = \langle q_f|T\exp(-i\tau(H_d + \lambda H_{\rm int}))|0\rangle, \tag{6.3}$$

where T denotes time ordering. The right-hand side of Eq. (6.3) is just the unitary evolution of the vacuum state for a driven harmonic oscillator. Using the properties of coherent states [38], Eq. (6.3) is equal to $\langle q_f|z(\tau)\rangle$, where $|z\rangle$ is a standard coherent state and

$$z(\tau) = -i\lambda \int_0^\tau dt e^{i\omega(\tau - t)} f(\mathbf{x}(t)). \tag{6.4}$$

To find the amplitudes for detection and no detection we expand the total state Eq. (6.2) in terms of the eigenstates of the harmonic oscillator. Since all states other than the ground state correspond to detection, there is no single amplitude corresponding to detection (although there is a probability). It is therefore easier to look at the amplitude for no detection, which is obtained by overlapping Eq. (6.2) with the ground state $u_0(q_f)$, yielding,

$$\psi_{\rm nd}(\mathbf{x}_f) = \int_{-\infty}^{\infty} d\tau e^{iE\tau} \int \mathcal{D}\mathbf{x}(t)\langle 0|z(\tau)\rangle \times \exp(iS_0[\mathbf{x}(t)])\chi_0(\mathbf{x}_0). \tag{6.5}$$

From the properties of coherent states [38], we have

$$\langle 0|z(\tau)\rangle = \exp(-\tfrac{1}{2}|z(\tau)|^2). \tag{6.6}$$

The probability for no detection is $\langle \psi_{\rm nd}|\psi_{\rm nd}\rangle$ and the probability for detection is simply $1 - \langle \psi_{\rm nd}|\psi_{\rm nd}\rangle$.

The result Eq. (6.5) clearly has the desired properties. For paths $\mathbf{x}(t)$ which never enter the detection region, $z(\tau) = 0$ and the path integral is unaffected. Paths that enter the region, on the other hand, generally have $z(\tau) \neq 0$, and they are exponentially suppressed. This is therefore a much improved detector model in comparison to Eq. (5.8). Its validity is not restricted to the perturbative regime. The generalization to many detectors is trivial and essentially the same result concerning peaking about classical trajectories is then obtained.

Of course, this detector is still not fully satisfactory because it can happen that $z(\tau) = 0$ even for paths that enter the region, because of the oscillatory nature of $z(\tau)$, hence we again encounter the issue of detector recurrences. Again this will be avoided if the time the trajectory spends in the detector region is short (less than ω^{-1}).

A more challenging detector improvement avoiding the recurrence problem would be to construct one that, were it used in standard unitary quantum mechanics, would be irreversible, i.e., involves an essentially infinite number of degrees of freedom. Such a detector was introduced in the related context of measuring arrival times in Ref. [42] and it would be interesting to incorporate it into the situation considered here.

VII. SUMMARY

The aim of this paper was to give substance to the appealing intuitive notion that solutions to the Wheeler-DeWitt equation (1.1) correspond to entire histories of the universe with time emerging as a parameter along each trajectory. The concrete technical results—the detection amplitude and the introduction of a set of timeless coherent states—are compatible with this notion. There are, however, many subtle aspects to this notion [4], and we do not claim to have an exhaustive demonstration of the emergence of trajectories from the Wheeler-DeWitt equation.

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