Scalar hair on the black hole in asymptotically anti–de Sitter spacetime

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We examine the no-hair conjecture in asymptotically anti–de Sitter (AdS) spacetime. First, we consider a real scalar field as the matter field and assume static spherically symmetric spacetime. Analysis of the asymptotics shows that the scalar field must approach the extremum of its potential. Using this fact, it is proved that there is no regular black hole solution when the scalar field is massless or has a "convex" potential. Surprisingly, while the scalar field has a growing mode around the local minimum of the potential, there is no growing mode around the local maximum. This implies that the local maximum is a kind of "attractor" of the asymptotic scalar field. We give two examples of the new black hole solutions with a nontrivial scalar field configuration numerically in the symmetric or asymmetric double well potential models. We study the stability of these solutions by using the linear perturbation method in order to examine whether or not the scalar hair is physical. In the symmetric double well potential model, we find that the potential function of the perturbation equation is positive semidefinite in some wide parameter range and that the new solution is stable. This implies that the black hole no-hair conjecture is violated in asymptotically AdS spacetime.

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I. INTRODUCTION

The exterior gravitational field of a stationary source may have a large number of independent multipole moments. But when the source lies within a black hole event horizon (BEH), a radical simplification occurs as proposed by Ruffini and Wheeler [1]: after the gravitational collapse of the matter field, the resultant black hole approaches stationary spacetime, with all the multipole moments being uniquely determined by two parameters, M and a, which are physically interpreted as the mass and angular momentum of the black hole. When the source has a net charge Q, then of course its (electric and gravitational) multipole moments depend on Q as well. This statement is called the black hole no-hair conjecture.

In order to examine whether or not the black hole no-hair conjecture is true, some people tried to prove this conjecture and some tried to construct a counterexample after the proposal. In the former approach, several no-hair theorems were established. For example, the black hole uniqueness theorems in electrovacuum theories [2] strongly support the conjecture. The works of Chase [3], Bekenstein [4], Hartle [5] and Teitelboim [6] show that stationary black hole solutions are hairless in a variety of theories coupling classical fields to Einstein gravity.

In proving these theorems, one usually assumes particular matter fields, symmetries of spacetime and asymptotic conditions. In the latter approach, some of these assumptions were relaxed. Considering different matter fields yields several kinds of black hole solutions. One of the most impressive solutions is the colored black hole solution of the Einstein-Yang-Mills (EYM) system [7]. Although this solution was found to be unstable both in the gravitational sector [8] and in the sphaleron sector [9], non-Abelian hair is generic, and many other non-Abelian black holes were discovered after the colored black hole [10]. Ridgway and Weinberg derived the static but non-spherically symmetric black hole solution [11]. This solution is regarded as a magnetic monopole which has a black hole inside its core. When the monopole has more than one winding number, spherical symmetry is violated. Making use of this property [11], they calculate the deviation from the spherical symmetry perturbatively.

Most of the proofs of the no-hair theorems impose flatness as the asymptotic condition. Hence the following natural question arises: *Can we extend no-hair theorems to spacetimes with different asymptotic structures?* The authors first studied the scalar hair in the asymptotically de Sitter spacetime [12]. It is worth first commenting on the scalar hair in the asymptotically flat case here. Bekenstein [4,13] and Sudarsky [14] provided simple proofs of the no-scalar hair theorem in spherically symmetric spacetime in the case where the matter consists of a single scalar field with a convex potential, and in the extended case where the matter consists of multiple scalar fields with an arbitrary positive semidefinite potential. Heusler also proved the no-scalar hair theorem by using a scaling technique [15].

In the asymptotically de Sitter case, we assume static

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spherically symmetric spacetime. If the scalar field is massless or has a ''convex'' potential such as a mass term, it was proved that there is no regular black hole solution. By ''convex'' we mean $d^2V/d\phi^2>0$ for any ϕ where $dV/d\phi=0$, which implies that the potential has only one extremum for a finite value of ϕ and it is not maximum but minimum. For a general positive semidefinite potential, we searched for black hole solutions which support the scalar field with a double well potential, and found them by numerical calculations. When we take the zero horizon radius limit, the solution becomes a boson-star-like solution [16]. These black hole solutions are, however, unstable against the linear perturbations. As a result, we can conclude that the no-scalar hair conjecture holds in the case of scalar fields with a ''convex'' or a double well potential. We expect that this no-scalar hair theorem extends to general positive semidefinite potential.

What happens, then, if we consider the system with a negative cosmological constant, especially the asymptotically anti–de Sitter (AdS) spacetime? Recently, a tremendous amount of interest has focused on several issues related to the AdS spacetime. One of them is the AdS/CFT (conformal field theory) correspondence [17]. It states that conformal field theories in d dimensions R_d are described in terms of supergravity or string theory on the product space of asymptotically AdS$_{d+1}$ and a compact manifold. There are intimate relations between data on the boundary R_d of AdS$_{d+1}$ and data in the bulk AdS$_{d+1}$. The negative cosmological constant plays an important role also in the brane world scenarios [18,19], which were first proposed to solve the gauge hierarchy problem, i.e., the vast disparity between the weak scale and the Planck scale. In these scenarios we live in a four dimensional hypersurface embedded in five dimensional bulk AdS spacetime.

It should also be noted that the colored black hole solution in the EYM system is stabilized both in the gravitational and sphaleron sectors by putting the negative cosmological constant into the system [20]. Moreover, it was shown that there is a black hole solution with dyonic hair in AdS which nevertheless cannot exist in asymptotically flat spacetime [21]. Hence we expect that the negative cosmological constant will affect the existence and stability of black hole solutions.

In this paper, we examine scalar hair on the black hole in AdS spacetime. In Sec. II we introduce the model and the basic equations. In Sec. III we give a definition of the asymptotically AdS spacetime and examine the asymptotic behavior of the scalar field. By analyzing the asymptotic behavior of the scalar field, we find that the scalar field must approach the extremum of its potential. Using this fact, it is proved that there is no regular black hole solution when the scalar field is massless or has a ''convex'' potential. While the scalar field has a growing mode around the local minimum of the potential, there is no growing mode around the local maximum. It implies that the local maximum is a kind of ''attractor'' of the asymptotic scalar field. In Sec. IV we give two numerical examples of the new black hole solutions with nontrivial configuration of scalar field in the symmetric or asymmteric double well potential models. In Sec. V we study the stability of these solutions by using the linear perturbation method in order to examine whether the scalar hair is physical or not. In the symmetric double well potential model, we find that the potential function of the perturbation equation is positive semidefinite in some wide parameter range and that the new solution is stable. This implies that the black hole no-hair conjecture is violated in asymptotically AdS spacetime. We give conclusions and remarks in the final section.

II. MODEL AND BASIC EQUATIONS

We will consider the model given by the action

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}(R-2\Lambda)-\frac{1}{2}(\nabla\phi)^2-V(\phi)\right], \tag{1}$$

where ϕ is the real scalar field and $V(\phi)$ is its potential. We shall assume the cosmological constant Λ to be negative. The metric of a spherically symmetric spacetime can be expressed in the Schwarzschild type form

$$ds^2=-fe^{-2\delta}dt^2+f^{-1}dr^2+r^2d\Omega^2, \tag{2}$$

where

$$f=1-\frac{2Gm}{r}-\frac{\Lambda}{3}r^2, \tag{3}$$

and $d\Omega^2$ is the metric of the unit 2-sphere.

The mass function m and the lapse function δ depend on both the time coordinate t and the radial coordinate r. The mass function is the quasilocal mass defined in Ref. [22], which is the gravitational energy minus the energy due to the cosmological constant, i.e., the energy of the matter field. In a spherically symmetric spacetime, the mass function is nondecreasing in the outgoing null or spacelike direction in the timelike region ($f>0$), if the matter fields satisfy the dominant energy condition.

Varying the action (1) and substituting ansatz (2), we derive the field equations

$$m'=4\pi r^2\left[\frac{1}{2}f^{-1}e^{2\delta}\dot{\phi}^2+\frac{1}{2}f\phi'^2+V(\phi)\right], \tag{4}$$

$$\delta'=-4\pi r[f^{-2}e^{2\delta}\dot{\phi}^2+\phi'^2], \tag{5}$$

$$\dot{m}=4\pi r^2f\dot{\phi}\phi', \tag{6}$$

$$-[e^{\delta}f^{-1}\dot{\phi}]^{\cdot}+\frac{1}{r^2}[r^2e^{-\delta}f\phi']'=e^{-\delta}\frac{dV(\phi)}{d\phi}. \tag{7}$$

Here, we have used the dimensionless variables, $\sqrt{|\Lambda|}t\to t$, $\sqrt{|\Lambda|}r\to r$, $\sqrt{|\Lambda|}Gm\to m$, $\sqrt{G}\phi\to\phi$ and $GV/|\Lambda|\to V$. By using these variables, f is expressed as

$$f=1-\frac{2m}{r}+\frac{r^2}{3}. \tag{8}$$

We will use these dimensionless variables below. A dot and a prime in the field equations denote derivatives with respect to the dimensionless variables t and r, respectively.

For the boundary conditions of the metric functions, we first impose the existence of a regular BEH at $r=r_B$; i.e.,

$$2m(r_B)=r_B\left(1+\frac{1}{3}r_B^2\right), \tag{9}$$

$$\delta(r_B)<\infty. \tag{10}$$

Second, we impose the non-existence of singularity outside of the BEH, i.e., for $r>r_B$,

$$2m<r\left(1+\frac{1}{3}r^2\right). \tag{11}$$

As for the asymptotic behavior, we can expect roughly four possibilities depending on the form of the potential $V(\phi)$. Define $\lambda_{eff}:=\lambda_{add}-1:=8\pi V(\phi_\infty)-1$ where $\phi_\infty:=\phi(\infty)$. (i) If $\lambda_{eff}<0$, the spacetime approaches AdS. (ii) If $\lambda_{eff}>0$, the spacetime approaches de Sitter spacetime. In this case, there should appear a cosmological event horizon (CEH) at $r=r_C$. By imposing its regularity, the metric functions must satisfy

$$2m(r_C)=r_C\left(1+\frac{r_C^2}{3}\right), \tag{12}$$

$$\delta(r_C)<\infty. \tag{13}$$

(iii) If $\lambda_{eff}=0$, we expect that the asymptotically flat spacetime is realized. However, some fine tuning mechanism (dynamical or just by hand) should be needed for this case. (iv) Finally there is a possibility of the other behavior. We will briefly comment on this solution in Sec. VI. Later we will investigate the case (i) i.e., asymptotically AdS solution. As for the scalar field, we impose smoothness except at $r=0$ where a singularity exists.

III. ASYMPOTIC BEHAVIOR AND THE NO SCALAR-HAIR THEOREM IN THE "CONVEX" POTENTIAL CASE

In this paper, we focus on the static solution whose asymptotic structure is AdS spacetime with no CEH. First of all, we have to give the definition of the asymptotically AdS spacetime. In our metric ansatz, it is reasonable to define it as (i) $f\sim Ar^2$ where $A>0$ and (ii) $\delta\to\delta_\infty$. From the condition (i), the mass function behaves $m\lesssim Br^3$ [$B<1/6$ (or $B<|\Lambda|/6$ in the dimensional variables) in the $m\sim Br^3$ case]. By using Eq. (4),

$$\begin{aligned}m&=4\pi\int_{r_B}^{r}r^2\left[\frac{1}{2}f\phi'2+V(\phi)\right]dr+m_B\\&\sim4\pi\int_{r_B}^{r}\left[\frac{1}{2}Ar^4\phi'^2+V(\phi)r^2\right]dr+m_B,\end{aligned} \tag{14}$$

in the static case. Hence, $\phi'\lesssim O(r^{-1})$ and $V(\phi)\lesssim$const. Since Eq. (5) is integrated as

$$\delta=-4\pi\int_{r_B}^{r}r\phi'^2dr+\delta_B, \tag{15}$$

the scalar field must satisfy $\phi'<O(r^{-1})$ by condition (ii). As a result, the scalar field behaves as $\phi'<O(r^{-1})$ in the asymptotically AdS spacetime. The contribution from the gradient term to the mass function becomes subdominant when $V(\phi_\infty)>0$.

We study the asymptotic behavior of the scalar field by using its field equation (7) in detail under the static ansatz. Substituting the asymptotic form of the metric functions, we obtain

$$Ar^2\phi''+4Ar\phi'-\frac{dV}{d\phi}=0. \tag{16}$$

This is easily integrated as

$$Ar^4\phi'=\int r^2\frac{dV}{d\phi}dr. \tag{17}$$

Since the left hand side of this equation behaves as $<O(r^3)$, $dV/d\phi\lesssim O(r^{-\epsilon})$ and $dV/d\phi\to0$ as $r\to\infty$. This implies that ϕ_∞ takes the extremum value of the potential. If $V(\phi_\infty)>0$, the mass function behaves $m\sim4\pi V(\phi_\infty)r^3/3$. Hence $V(\phi_\infty)<1/8\pi$. Otherwise the spacetime will approach de Sitter or another exotic one.

By this asymptotic behavior, we can prove the no-hair theorem for the "convex" potential case. The case when the extremum is realized at $\phi=\infty$ is also included. There are two types of the "convex" potential. One is the case in which the potential has a minimum and the other is one in which it does not, i.e., the scalar field approaches its asymptotic value $\inf_\phi V(\phi)$ in the $\phi\to\infty$ (or $-\infty$) limit as $V(\phi)=e^{-\phi}$. The latter case is important since it appears in effective theory of superstring theories.

First we examine the former case where the potential has a minimum. We can set the minimum at $\phi=0$. If $\phi(r_B)=0$, the scalar field becomes trivial. Hence, we assume $\phi(r_B)>0$ without loss of generality. By Eq. (7),

$$f'\phi'=\frac{dV}{d\phi} \tag{18}$$

on the BEH. Since $f'>0$ and $dV/d\phi>0$ around the BEH, $\phi'(r_B)>0$. At the extrema of the scalar field, i.e., $\phi'=0$,

$$f\phi''=\frac{dV}{d\phi}. \tag{19}$$

f must be positive outside of the BEH unless the CEH appears. Hence $\phi''>0$ for $\phi>0$. These imply that the scalar field must increase monotonically. Since we assume that the scalar field does not diverge, there are three cases for the asymptotic behavior of the scalar field: (a) At finite radius $r=r_0$, the scalar field becomes $\phi>\phi_0$ where $V(\phi_0)=1/8\pi$. It should be noted that since the effective cosmological constant λ_{eff} becomes positive ($V>1/8\pi$) for $r>r_0$, there may appear the CEH. (b) The scalar field approaches its asymptotic value $\phi_\infty<\phi_0$. (c) $\phi_\infty=\phi_0$. We will examine these cases individually. *Case (a):* As we mentioned above, f' becomes negative by the positive λ_{eff}. Then the CEH appears at a finite radius. Although we are interested in the solution without the CEH, we will check the possibility of this solution. Equation (18) also holds at the CEH. Since $f'<0$ around the CEH, ϕ must decrease at the CEH. This contradicts the monotonicity of the scalar field. Hence there is no such solution. *Case (b):* In this case, the spacetime eventually approaches AdS spacetime and the left hand side of Eq. (7) becomes zero at infinity. However, $dV/d\phi$ is positive for $\phi_\infty>0$. This is a contradiction. *Case (c):* In this case, the spacetime becomes asymptotically flat since $\lambda_{eff}=0$. Hence, while the left hand side of Eq. (7) becomes zero, the right hand side is non-zero at $\phi=\phi_0$. This is a contradiction. One may consider the potential which approaches some constant value $V_{\rm inf}<1/8\pi$ for large ϕ. Then $dV/d\phi$ becomes zero at $\phi=\infty$. However this potential is not included in the definition of the "convex" potential. Furthermore, the scalar field must diverge in this case.

Next we examine the latter case, i.e., the potential having no minimum. It is easy to show that the potential is a monotonic function of ϕ, and we can prove the nonexistence of the relevant solution in a similar way as above. As a result, there is no solution for the "convex" potential except for the trivial one. Note that we did not assume the asymptotically AdS condition in the proof. So the scalar hair cannot be put on the black hole in any reasonable asymptotic condition if the cosmological constant is negative and the potential has a "convex" shape in general.

It is interesting to compare the asymptotic behavior with that of the EYM system with negative cosmological constant [20],

$$S_{EYM}=\int d^4x\sqrt{-g}\left[\frac{1}{4\pi G}(R-2\Lambda)-\frac{1}{16\pi g}\mathrm{tr}F^2\right]. \tag{20}$$

In the EYM system, the magnetic part of the Yang-Mills potential can take any value w_∞ asymptotically while the scalar field must approach the extremum of the potential. The equation of the Yang-Mills potential w is expressed as

$$\frac{1}{r^2}[r^2e^{-\delta}fw']'=\frac{e^{-\delta}}{r^2}[2rfw'^2-2w(1-w^2)]. \tag{21}$$

The right hand side of this equation corresponds to the contribution from the potential in our system [compare with Eq. (7)]. Note that this term vanishes as $r\to\infty$ independently of the value of w due to the factor r^{-2}. This is the key feature of the EYM system. On the other hand, our system does not have such a factor. Hence the scalar field must approach its extremum at infinity.

Let us examine the asymptotic behavior of the scalar field further. Since we know that the asymptotic value of the scalar field is the extremum of the potential, we next study how the scalar field approaches it. We assume $\phi\to\phi_\infty$ as $r\to\infty$. Defining $\varphi:=\phi-\phi_\infty$ and expanding Eq. (7), we obtain the linear equation

$$\varphi''+2\mu\varphi'+\omega^2\varphi=0, \tag{22}$$

where

$$\mu:=\frac{1}{r}+\frac{f'}{2f}, \tag{23}$$

$$\omega^2:=-\frac{1}{f}\left.\frac{d^2V}{d\phi^2}\right|_{\phi=\phi_\infty}. \tag{24}$$

The metric function f behaves as

$$f\to-\frac{\lambda_{eff}}{3}r^2. \tag{25}$$

Hence the friction coefficient μ and the frequency ω^2 are expressed as

$$\mu\to\frac{2}{r}, \tag{26}$$

$$\omega^2\to\frac{\alpha}{r^2}:=\frac{3}{\lambda_{eff}}\left.\frac{d^2V}{d\phi^2}\right|_{\phi=\phi_\infty}\frac{1}{r^2}. \tag{27}$$

Substituting these into Eq. (22), we obtain

$$\varphi''+\frac{4}{r}\varphi'+\frac{\alpha}{r^2}\varphi=0. \tag{28}$$

The solution of this equation is

$$\varphi=C_+r^{\lambda_+}+C_-r^{\lambda_-}, \tag{29}$$

where $C_\pm$ are constants and

$$\lambda_\pm:=\frac{-3\pm\sqrt{9-4\alpha}}{2}. \tag{30}$$

The behavior of the scalar field depends on the value of α. If $\alpha<0$, λ_+ becomes positive and the growing mode is dominant. Since negative α implies that $d^2V/d\phi^2|_{\phi=\phi_\infty}>0$ by definition, the scalar field leaves the local minimum of the potential unless one ensures the fine tuning with which the growing mode disappears. If $0<\alpha<9/4$, both of $\lambda_\pm$ are negative and the scalar field approaches the local maximum with the power λ_+ in general. If $\alpha=9/4$, the scalar field behaves as

$$\varphi=C_1r^{-3/2}+C_2r^{-3/2}\ln r, \tag{31}$$

and approaches the local maximum. This corresponds to the critical damping. If $\alpha>9/4$, there is no growing mode and the scalar field behaves as

$$\varphi \sim r^{-3/2} \cos\left(\frac{\sqrt{4\alpha - 9}}{2} \ln r \right). \tag{32}$$

Namely the scalar field oscillate around the local maximum with the frequency $\nu = \exp(-4\pi/\sqrt{4\alpha-9})$ and with its envelope decreasing with the power $-3/2$. As a result, the local maximum of the potential is a kind of attractor of the scalar field.

IV. BLACK HOLE SOLUTION WITH SCALAR HAIR

In this section we verify the expectations of the previous section by using concrete models and derive nontrivial black hole solutions with scalar hair. Here we will adopt the system with a symmetric double well potential

$$V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2 \tag{33}$$

as a first example of a positive semidefinite potential. If we normalize the constants as $\lambda / G|\Lambda| \to \lambda$ and $\sqrt{G} v \to v$, they become dimensionless variables. Here we assume $V(0) < 1/8\pi$. Since the top of the potential barrier $\phi = 0$ becomes the attractor of the asymptotic scalar field, the parameter α is calculated to be

$$\alpha = \frac{3\lambda v^2}{1 - 2\pi\lambda v^4} > 0. \tag{34}$$

There are two trivial solutions, $\phi \equiv \pm v$ and $\phi \equiv 0$. Both of them are Schwarzschild-AdS solutions. For the latter solution, the potential of the scalar field plays the role of the additional cosmological constant $\lambda_{add} = 2\pi\lambda v^4$. As is easily imagined, this solution is unstable against perturbations.

Now we search for non-trivial static solutions by using numerical analysis. We drop the time derivative terms of the field equations (4), (5), and (7), and integrate them from the BEH with the boundary conditions (9) and (10). We can restrict $\phi(r_B) > 0$ without loss of generality because the potential has reflection symmetry. Furthermore, we restrict $\phi(r_B) < v$; otherwise the situation is exactly the same as the ''convex'' potential case and there is no nontrivial solution. Since the equation of the scalar field (7) becomes singular on the BEH, we expand all terms in power series of $(r - r_B)$ to guarantee the regularity on the BEH, and use their analytic solutions for the first step of integration.

We found the black hole solutions with nontrivial scalar field configuration for any boundary value of the scalar field in the range $0 < \phi(r_B) < v$. This is one of the different properties from the positive cosmological constant case [12], where we have to choose the suitable boundary value at the BEH by the shooting method in order to satisfy the boundary condition at the CEH. In the present case we do not need a shooting parameter.

We show the configuration of the scalar field in Fig. 1. We choose the value of the parameters as $\lambda = 50$, $v = 0.1$, $r_B = 0.1$ and $\phi(r_B) = 0.5v$. In this case, $\alpha = 1.55 < 9/4$. As discussed in the previous section, we find that the scalar field decays with the power $\lambda_+ = -0.663$. We show the configuration of another solution with the different parameter $\lambda = 500$, $v = 0.1$, $r_B = 0.1$ and $\phi(r_B) = 0.5v$ in Fig. 2. In this case, $\alpha = 21.9 > 9/4$. The scalar field oscillates with damping. We show the ln plot of the same solution in Fig. 2(b). It shows that the envelope of the oscillation decays with power $r^{-3/2}$ and that the wavelength is proportional to $\ln r$. By Eq. (34), we find that the condition for oscillation with damping is

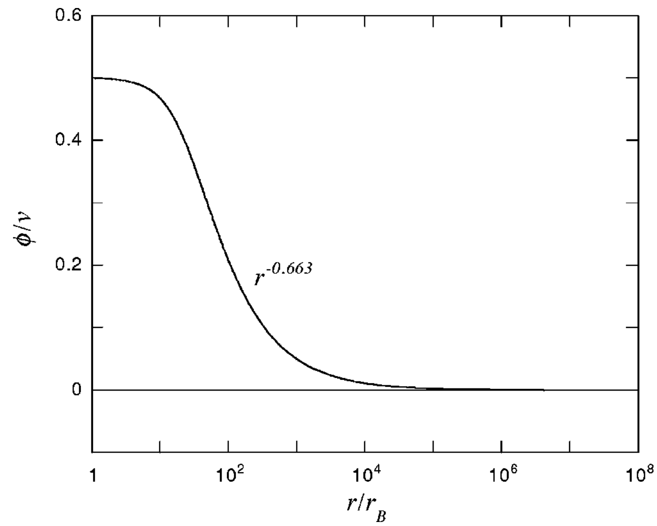


FIG. 1. The configuration of the scalar field ϕ in the symmetric double well potential model. We set the parameters $\lambda = 50$, $v = 0.1$, $r_B = 0.1$ and $\phi(r_B) = 0.5v$. The scalar field decays with the power $\lambda_+ = -0.663$.

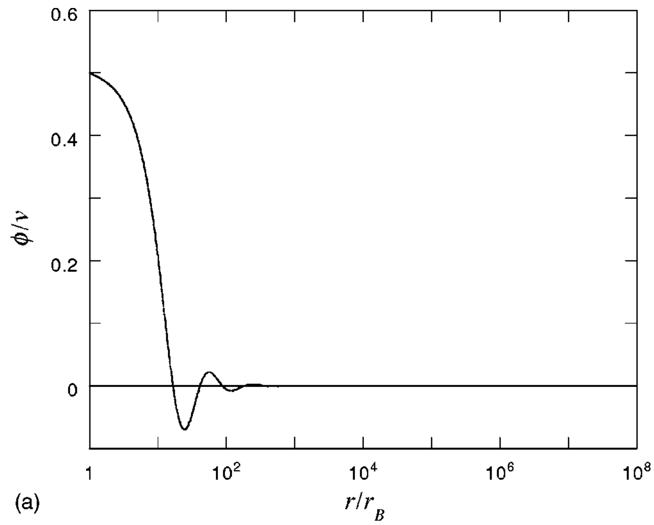
$$\lambda > \lambda_0 := \frac{3}{2v^2(3\pi v^2 + 2)}. \tag{35}$$

It should be noted that there are actually black hole solutions with the non-trivial scalar field in the asymptotically AdS spacetime in spite of there being no counterparts in asymptotically flat spacetime.

In the above example, the scalar field approaches the top of the potential barrier asymptotically. This is because the local maximum is an ''attractor'' and there is a growing mode around the bottom of the potential well. We may, however, make this growing mode vanish by tuning the value of the scalar field at the BEH. We will give another example. Since there is no such solution in the symmetric double well potential case, we adopt the following artificial asymmetric potential:

$$V = A\left[\frac{1}{4}\phi^4 - \frac{v_1 + v_2}{3}\phi^3 + \frac{v_1 v_2}{2}\phi^2 + \frac{1}{12} v_1^3 (v_1 - 2v_2) \right], \tag{36}$$

where A, v_1 and v_2 are constant. We show the form of this potential with the parameters $A = 1000$, $v_1 = 0.1$ and $v_2 = -0.01$ in Fig. 3. $\phi = 0$, $\phi = v_1$ and $\phi = v_2$ correspond to the top of the potential barrier, the global minimum and the local minimum, respectively. If we choose $\phi(r_B) > 0$ too large, the



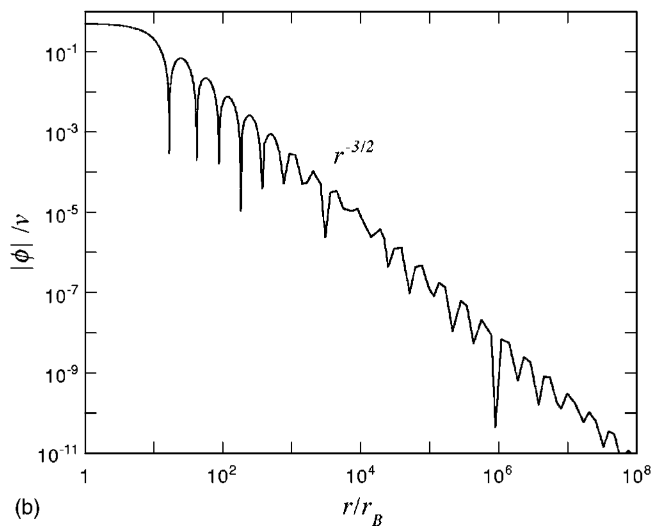


FIG. 2. The configuration of the scalar field ϕ in the symmetric double well potential model. We set the parameters $\lambda=500$, $v=0.1$, $r_B=0.1$ and $\phi(r_B)=0.5v$. (a) is the linear plot. The scalar field oscillates with damping. (b) is the ln plot. We find that the envelope of the oscillation decays with power $-3/2$ and that the wavelength is proportional to $\ln r$.

scalar field passes the local minimum v_2 as we integrate the field equations outward, and it diverges to minus infinity. If we choose $\phi(r_B)$ too small, the scalar field cannot reach v_2 but oscillates around the top of the potential barrier and decays to $\phi=0$. Hence if we choose a suitable value of $\phi(r_B)$ between these values, we will obtain the desirable solution. In this sense, $\phi(r_B)$ is a shooting parameter. By numerical calculation we find such non-trivial solution. Figure 4 shows the configuration of the scalar field with the potential depicted in Fig. 3. We set $r_B=0.1$. The scalar field approaches the local minimum v_2 of the potential. This solution is different from those in the symmetric double well potential model. Although the existence of this type of solution is interesting, it seems unstable against the time dependent perturbations as discussed later and cannot be used as a conter-example of the no-hair conjecture.

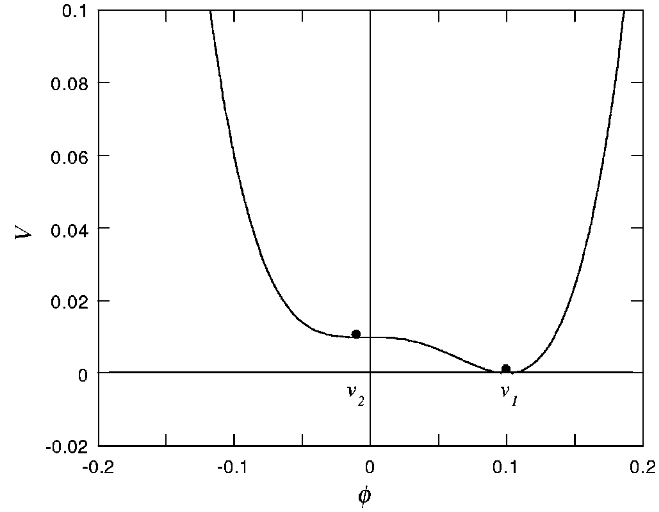


FIG. 3. The form of the asymmetric double well potential with the parameters $A=1000$, $v_1=0.1$, and $v_2=-0.01$. $\phi=0$, $\phi=v_1$ and $\phi=v_2$ correspond to the top of the potential barrier, the local minimum and the global minimum, respectively.

V. STABILITY ANALYSIS BY LINEAR PERTURBATION METHOD

In the previous section we found new black hole solutions. This means that the no-hair conjecture may not hold in asymptotically AdS spacetime. In this section we investigate the stability of the new solutions by using a linear perturbation method in order to check whether or not the scalar hair is really physical.

First we expand the field functions around the static solution ϕ_0, m_0 and δ_0 as follows:

$$\phi(t,r)=\phi_0(r)+\frac{\phi_1(t,r)}{r}\epsilon, \tag{37}$$

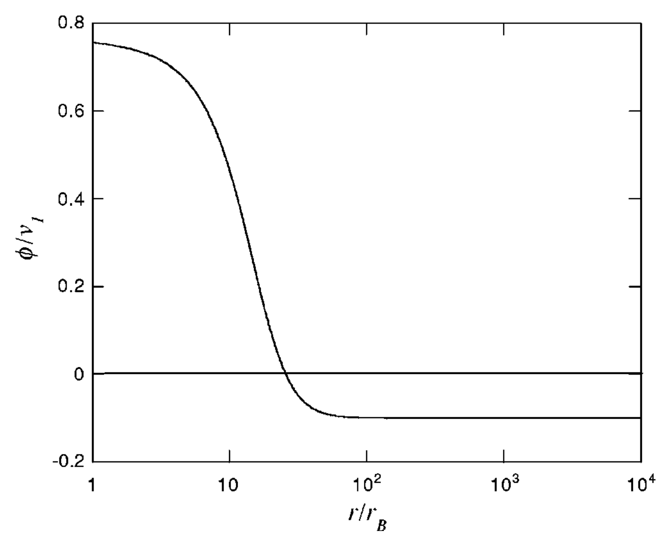


FIG. 4. The configuration of the scalar field ϕ in the asymmetric double well potential model. We set the parameters $A=1000$, $v_1=0.1$, $v_2=-0.01$ and $r_B=0.1$. The scalar field approaches the local minimum v_2 asymptotically.

$$m(t,r)=m_0(r)+m_1(t,r)\epsilon, \tag{38}$$

$$\delta(t,r)=\delta_0(r)+\delta_1(t,r)\epsilon. \tag{39}$$

Here ϵ is an infinitesimal parameter. Substituting them into the field equations (4)–(7) and dropping the second and higher order terms in ϵ, we find the following perturbation equation of the scalar field:

$$-\frac{d^2\xi}{dr^{*2}}+U(r)\xi=\sigma^2\xi, \tag{40}$$

where we set $\phi_1=\xi(r)e^{i\sigma t}$ and $m_1=\eta(r)e^{i\sigma t}$. If σ^2 is positive, ϕ oscillates around the static solution and the solution is stable. On the other hand, if it is negative, the perturbation ϕ_1 and m_1 diverge exponentially with time and then the solution is unstable. r^* is the tortoise coordinate defined by

$$\frac{dr^*}{dr}=\frac{e^{\delta_0}}{f_0}, \tag{41}$$

where $f_0 := 1-2m_0/r+1/3$. The potential function is

$$U(r)=e^{-2\delta_0}f_0\left[(1-8\pi r^2\phi_0'^2)\frac{f_0'}{r} -4\pi(1+8\pi r^2\phi_0'^2)f_0\phi_0'^2 +16\pi r\phi_0'\frac{dV}{d\phi}\bigg|_0+\frac{d^2V}{d\phi^2}\bigg|_0\right]. \tag{42}$$

First we discuss the general properties by using the asymptotic behavior of the potential function U without concrete potential form. To leading order, it behaves as

$$U(r)=\frac{2-\alpha}{9}\lambda_{eff}^2 e^{-2\delta_0}r^2. \tag{43}$$

Hence, when $\alpha>2$, the potential diverges to minus infinity. This means that there exists an infinite number of unstable modes and the black hole solution is unstable. On the other hand, when $\alpha<2$, the potential diverges to plus infinity. Although there seems to be no unstable modes in this case, $U(r)$ may have a well in the central region which is deep enough to produce negative eigenmodes. The previous example of the asymmetric double well potential case ($\alpha<0$) is such a case. It is expected that the solution has exactly one unstable mode. The number of unstable modes equals the number of times the scalar field goes over the potential barrier [12]. Hence the stability in the $\alpha<2$ case depends on the potential of the scalar field $V(\phi)$ we employ.

Next we examine the concrete model. We plot the potential function $U(r)$ of the symmetric double well potential model in Fig. 5. The values of the parameters are $\lambda=50$, $v=0.1$, $r_B=0.1$ and $\phi(r_B)=0.5v$ ($\alpha=1.55$). We find that the potential diverges to plus infinity as $\sim r^2$ as we analyzed. Note that it is positive semidefinite. This means that there is no unstable mode and that the black hole solution is stable. This is a very important result because it violates the no-scalar hair conjecture in the asymptotically AdS spacetime. Numerical calculation shows that the criterion of the stability is exactly $\alpha<2$ in the symmetric double well model as obtained by the asymptotic analysis. This condition is described as

$$\lambda<\lambda_{stable}:=\frac{2}{v^2(4\pi v^2+3)}. \tag{44}$$

Comparing this with Eq. (35), we find $\lambda_0>\lambda_{stable}$. Figure 6

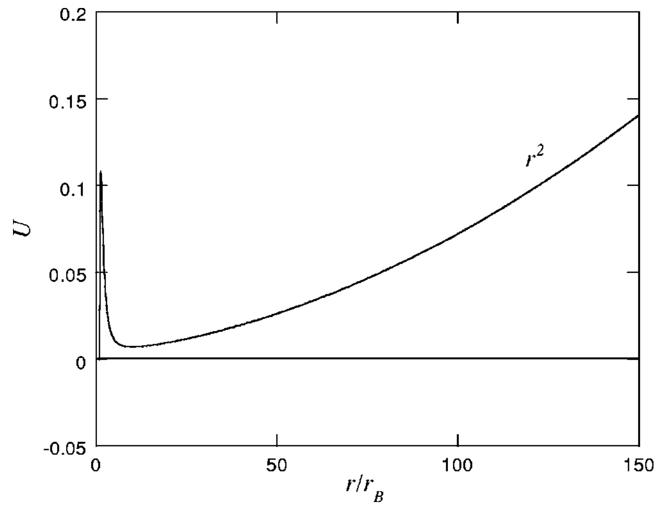


FIG. 5. The profile of the potential function U of the linear perturbation equation for the double well potential model. We set the parameters $\lambda=500$, $v=0.1$, $r_B=0.1$ and $\phi(r_B)=0.5v$. We find that the potential is positive semidefinite, which means that there is no unstable mode.

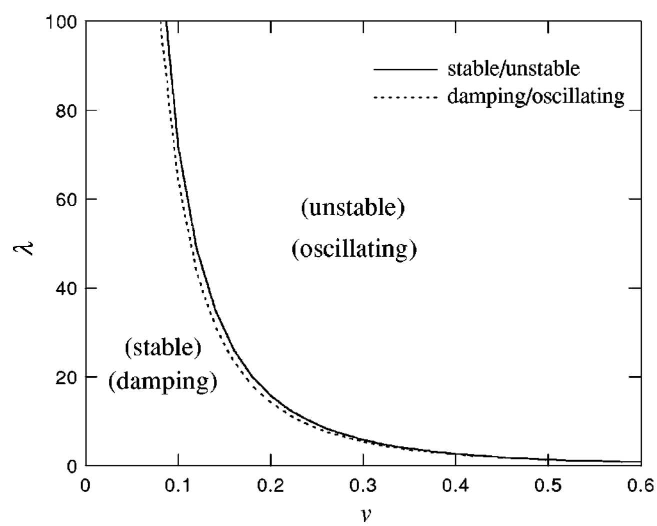


FIG. 6. The boundaries of the damping/oscillating behavior of the scalar field (solid line) and stability/instability of the new solution (dashed line) in the symmetric double well potential model. We find that $\lambda_{stable}<\lambda_0$ for all v.

shows the boundaries of the stablility/instablility and of the damping/oscillating behavior in the parameter space.

VI. CONCLUSION

We have examined the no-hair conjecture under the asymptotically AdS condition. As the first step, we consider the real scalar field as the matter field and assume the static spherically symmetric spacetime. By analyzing the asymptotic behavior of the scalar field, we find that the scalar field must approach the extremum of its potential. With this fact, it can be proved that there is no regular black hole solution when the scalar field is massless or has a ‘‘convex’’ potential-like mass term. While the scalar field has a growing mode around the local minimum of the potential, there is no growing mode around the local maximum. This implies that the local maximum is a kind of ‘‘attractor’’ of the scalar field asymptotically. When the variable α defined in Sec. III satisfies $\alpha > 9/4$, the scalar field oscillates with damping around its asymptotic value ϕ_∞. When $0 < \alpha < 9/4$, it decays without oscillation. We give two examples of the new black hole solutions with nontrivial scalar field configuration. The first one is the solution in the symmetric double well potential model. We can find the new solution without tuning the value of the scalar field at the BEH. The other example is the solution in the asymmetric double well potential model. We show the solution whose scalar field approaches the local minimum of its potential by tuning the boundary value of the scalar field. We study stability of the new solutions by using the linear perturbation method in order to examine whether or not the scalar hair is physical. While for the potential with $\alpha > 2$ the new solution is unstable, the asymptotic analysis shows that the solution may be stable for the $\alpha < 2$ case. In the symmetric double well potential model, we find that the potential function of the perturbation equation is positive semidefinite for the damping solution with $\alpha < 2$ and that the new solution is stable. This implies that the black hole no-hair conjecture is violated in asymptotically AdS spacetime.

In our present analysis, we show explicitly that the black hole can have a scalar hair only in a particular model, i.e., in the symmetric double well potential. It is expected, however, that there are similar solutions with scalar hair in different models, and we confirmed it. Furthermore, we can include the gauge field in our system as in the positive cosmological constant case [23]. Then the scalar hair is independent of the gauge hair unlike the dilaton hair which cannot exist without the gauge field [24]. In this sense our scalar hair is classified into the primary hair.

The new solution has a singularity at $r = 0$ which is hidden by the BEH. There will be also a regular solution without the BEH, i.e., the boson star solution. In the positive cosmological constant case, there exists such solutions by assuming the regular boundary condition at the origin instead of the black hole boundary condition, and they have interesting properties [16]. Unfortunately, they were found to be unstable against perturbations. In the negative cosmological constant case, however, it is expected that the boson star solution is stable since the black hole counterpart can be stable for the model with $\alpha < 2$. If so and if the Universe experienced a period with negative cosmological constant at least effectively, this boson star solution would affect the history of the Universe.

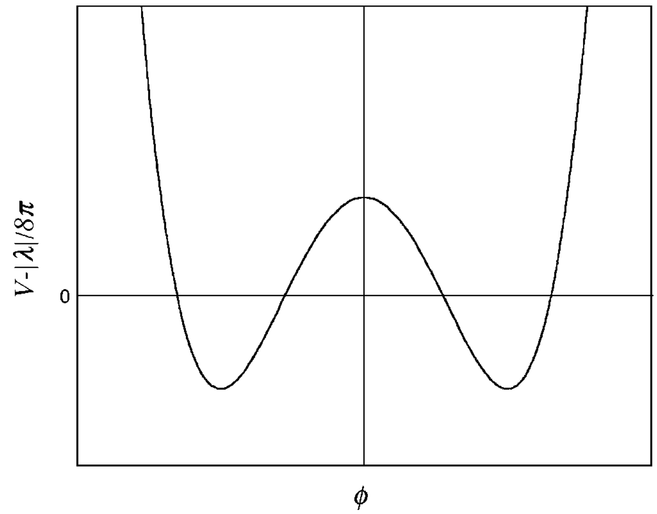


FIG. 7. The schematic figure of the double well potential. At the top of the potential barrier, λ_{eff} becomes positive.

In Sec. IV, we showed the black hole solution which satisfies the asymptotically AdS condition. This condition is guaranteed by imposing $\lambda_{eff} < 0$ on the potential of the scalar field, i.e., the additional cosmological constant produced by the potential at the local maximum is smaller than the absolute value of the original cosmological constant. Then, what happens if this condition is violated? In order to examine this question, we study the model which includes the potential depicted in Fig. 7. By examining the asymptotic behavior of the scalar field as in Sec. III, we find that the local maximum (minimum) of the potential is the attractor when $f > 0$ ($f < 0$). We choose a boundary value of the scalar field at the BEH and integrate outward. The scalar field damps to the local maximum first and continues to take that value for some range of the radial coordinate. Since the effective cosmological constant λ_{eff} is positive, there will appear the CEH at some radius r_C. However, the boundary condition of the scalar field at the CEH is not satisfied in general and the scalar field diverges. In order to evade that, we have to choose a suitable boundary value of $\phi(r_B)$. Then we can integrate over the CEH. After that the attractor moves to the local minimum of the potential because $f < 0$ and the scalar field oscillates around one of the local minima. Then, since the effective cosmological constant becomes negative there, another event horizon will appear [25]. However, the boundary condition at the event horizon again cannot be satisfied in general. Since there remains no free parameter to satisfy the boundary condition, the scalar field inevitably diverges. As a result, we cannot expect physically reasonable asymptotic structure.

Although our analysis is restricted in the 4-dimensional spacetime, it can be extended to the higher dimensional case.

In particular, we are interested in the 5-dimensional brane world scenario, which consists of a 4-dimensional thin/thick wall where we live and the bulk spacetime with negative cosmological constant. The static black hole solution has not been obtained yet in this context because the brane world has nontrivial S^1/Z_2 symmetry. However, our analysis shows the possibility that the scalar field can be a new primary hair of the black hole on the brane world. The black hole in asymptotically AdS spacetime has more variety than in the asymptotically flat case.

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[25] Here, the event horizon means the hypersurface with $f=0$. There is no exact definition of the event horizon for this solution since there is not the physically reasonable asymptotic region in this solution.