Hawking radiation from AdS black holes

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We investigate Hawking radiation from black holes in $(d+1)$-dimensional anti–de Sitter space. We focus on s waves, make use of the geometrical optics approximation, and follow three approaches to analyze the radiation. First, we compute a Bogoliubov transformation between Kruskal and asymptotic coordinates and compare the different vacua. Second, following a method due to Kraus, Parikh, and Wilczek, we view Hawking radiation as a tunneling process across the horizon and compute the tunneling probability. This approach uses an anti–de Sitter version of a metric originally introduced by Painlevé for Schwarzschild black holes. From the tunneling probability one also finds a leading correction to the semiclassical emission rate arising from back reaction to the background geometry. Finally, we consider a spherically symmetric collapse geometry and the Bogoliubov transformation between the initial vacuum state and the vacuum of an asymptotic observer.

I. INTRODUCTION

Anti–de Sitter black holes play a major role in the AdS conformal field theory (CFT) conjecture [1] (see [2] for a review), and they have also received interest in the context of brane-world scenarios based on the setup of Randall and Sundrum [3,4]. The purpose of this paper is to briefly investigate the basic property of black holes, Hawking radiation, in anti–de Sitter spacetimes.

In the $(2+1)$-dimensional special case, the Bañados-Teitelboim-Zanelli (BTZ) black holes [5], Hawking radiation of a massles conformally coupled scalar field was investigated in detail by Hyun *et al.* [6]. Here we will instead take a more generic approach (while sacrificing detail), and extend some standard ways to analyze Hawking radiation from Schwarzschild black holes to the anti–de Sitter case in a generic dimension. The standard methods in question are to investigate eternal black holes and the role of different vacua based on different boundary conditions at the horizons, and to investigate radiation in a spherical collapse geometry while the quantum field is in a natural initial vacuum. Customarily, the focus is on finding the leading thermal characteristics of the radiation. For this purpose it is sufficient to invoke a geometric optics approximation which neglects the backscattering of outgoing waves from the spacetime curvature and essentially reduces the problem to two dimensions. The neglected effects would give rise to a graybody factor in the thermal emission spectrum. The graybody factors can be found by investigating the absorption by AdS black holes.

We will also investigate the back reaction to the geometry by taking into account the self-interaction effect analyzed by Kraus, Parikh, and Wilczek [9–11] (see also [12]). This approach views Hawking radiation as a tunneling process across the horizon. Here we extend the approach of [9–11] to the anti–de Sitter case and find similar results.

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II. HAWKING RADIATION FROM ETERNAL BLACK HOLES

One standard method to derive the Hawking radiation is to use a Bogoliubov transformation between two basis of annihilation and creation operators corresponding to mode expansions of the field operator in two preferred coordinate systems used to describe a black hole: the asymptotic coordinates and the Kruskal coordinates. In the asymptotic coordinates, the AdS_{d+1} black hole metric is (see, e.g., [7])

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 \, d\Omega_{d-1}^2, \tag{1}$$

where

$$F(r) = 1 - \frac{\mu}{r^{d-2}} + r^2, \tag{2}$$

and we work in units where the AdS radius $l=1$. The parameter μ is proportional to the Arnowitt-Deser-Misner (ADM) mass M of the black hole:

$$M = \frac{(d-1)A_{d-1}}{16\pi G_{d+1}} \mu, \tag{3}$$

where G_{d+1} is Newton's constant in $d+1$ dimensions and $A_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a unit $(d-1)$-sphere. The coordinates (1) are the anti–de Sitter analogue of the Schwarzschild coordinates, relevant for external observers at a fixed radial distance from the black hole. One can show that the flux of outgoing radiation observed by asymptotic fiducial observers at $r=$ const is in agreement with the rate at which the black hole emits its mass. We postpone this calculation to the end of Sec. IV, where we discuss the decay of AdS black holes in more detail.

There is a coordinate singularity at the radius of the horizon. The explicit formula for the horizon radius r_H can be found by solving the polynomial equation $F(r_H)=0$. For example, in $4+1$ dimensions,

$$r_H = \sqrt{\frac{1}{2}(\sqrt{1+4\mu}-1)}. \qquad (4)$$

Then, the Hawking temperature T_H can be found by looking at the periodicity of the Euclidean section of the metric near r_H. The generic result is

$$T_H = \frac{1}{4\pi} F'(r_H), \qquad (5)$$

where $' = d/dr$. E.g., in $4+1$ dimensions,

$$T_H = \frac{\sqrt{1+4\mu}}{2\pi r_H}. \qquad (6)$$

To investigate mode solutions of a field equation, it is useful to introduce the tortoise coordinate r_*:

$$r_* = \int_{r_H}^{r} \frac{d\hat{r}}{F(\hat{r})}. \qquad (7)$$

Again, in $4+1$ dimensions the explicit formula is

$$r_* = \frac{1}{\sqrt{1+4\mu}}\left[r_0 \arctan\left(\frac{r}{r_0}\right) + \frac{1}{2} r_H \ln\left(\frac{r-r_H}{r+r_H}\right)\right], \qquad (8)$$

where r_H is the horizon radius and r_0 is a shorthand notation denoting a radius:

$$r_0 = \sqrt{\frac{1}{2}(\sqrt{1+4\mu}+1)}. \qquad (9)$$

[In $4+1$ dimensions, the equation $F(r)=0$ has four complex zeros: two real zeros at $\pm r_H$ and two imaginary zeros at $\pm i r_0$.] It is also convenient to introduce the null coordinates

$$u = t - r_*,$$

$$v = t + r_*, \qquad (10)$$

where r_* is the tortoise coordinate. Then the metric takes the form

$$ds^2 = -F(r) du\, dv + r^2 d\Omega_{d-1}^2 \qquad (11)$$

and the solutions to the wave equation are infalling and outgoing partial waves.

In the region outside the past and future horizons, the Kruskal coordinates U, V are defined as follows:

$$U = -\exp(-2\pi T_H u),$$

$$V = \exp(2\pi T_H v). \qquad (12)$$

In U, V coodinates, the metric takes the form

$$ds^2 = -\frac{F(r)e^{-4\pi T_H r_*}}{(2\pi T_H)^2} dU\, dV + r^2 d\Omega_{d-1}^2. \qquad (13)$$

In $4+1$ dimensions, the explicit form is

$$ds^2 = -\frac{r_H^2}{1+4\mu}\left(1+\frac{r_0^2}{r^2}\right)(r_H+r)^2$$
$$\times \exp\left[-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right] dU dV + r^2 d\Omega_3^2. \qquad (14)$$

In Kruskal coordinates, the metric can be extended over the whole spacetime, except for the origin $r=0$, which corresponds to the true curvature singularity of the black hole.

Next, we will focus on s waves, adopt the geometric optics approximation, and truncate to two dimensions, following the classic paper by Unruh [8]. The discussion also overlaps with [13].[1]

For a quantum field in the black hole background, there are two canonical choices for a natural vacuum state. If one wants to mimic a situation where the black hole is created by collapsing matter, one requires the field to be in a vacuum corresponding absence of positive energy modes in the U and v coordinates near the past horizon $V=0$. This boundary condition refers only to the past of the asymptotic region of spacetime and is known as the Unruh vacuum. Another vacuum choice, the Hartle-Hawking vacuum, refers to a mixture of boundary conditions in the past and future horizons. Now one requires the absence of positive energy U modes near $V=0$ and the absence of positive energy V modes near $U=0$. Physically, this mimics a black hole in thermal equilibrium with an external heat bath. The task is to compare these two vacuua with the Boulware vacuum which is a natural vacuum for a fiducial observer in the asymptotic region. The Boulware vacuum corresponds to absence of positive energy u and v modes. To complete the discussion of vacua, we will also need to take into account the reflective boundary condition at the boundary of the anti–de Sitter space. We will do that at the end of this section.

To compute the Bogoliubov transformations between the different natural modes, we use the standard trick [8]. Let us focus on the outgoing modes first. To begin with, we define the modes

$$\phi_{+,\omega} = \exp(-i\omega u) = \theta(-U)(-U)^{i\omega/2\pi T_H} \quad (U<0), \qquad (15)$$

$$\phi_{-,\omega} = \exp(i\omega u) = \theta(U) U^{-i\omega/2\pi T_H} \quad (U>0). \qquad (16)$$

Then, to find a complete basis for positive energy U modes we consider the linear combinations which extend over the whole $V=0$ line:

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$$\phi_{1,\omega} = \phi_{+,\omega} + C_1 \phi^*_{-,\omega}$$

$$= \theta(-U)(-U)^{i\omega/2\pi T_H} + C_1 \theta(U) U^{i\omega/2\pi T_H}, \quad (17)$$

$$\phi_{2,\omega} = \phi_{-,\omega} + C_2 \phi^*_{+,\omega}$$

$$= \theta(U) U^{-i\omega/2\pi T_H} + C_2 \theta(-U)$$

$$\times (-U)^{-i\omega/2\pi T_H}. \quad (18)$$

We then demand that $\phi_{1,\omega}$ be a positive energy Kruskal mode: that for $\omega > 0$ it must be analytic in the lower half complex U plane. This condition is satisfied if the coefficient C_1 is

$$C_1 = \exp\left(-\frac{\omega}{2T_H}\right). \quad (19)$$

A similar condition for $\phi_{2,\omega}$ fixes the coefficient C_2:

$$C_2 = \exp\left(-\frac{\omega}{2T_H}\right). \quad (20)$$

Hence $C_1 = C_2 \equiv C$.

From this we can compute the (unnormalized) α and β Bogoliubov coefficients which denote the overlap of a positive energy U mode with positive and negative energy u modes:

$$\alpha = (\phi_{1,\omega}, \phi_{+,\omega}) = (\phi_{2,\omega}, \phi_{-,\omega}) \propto 1, \quad (21)$$

$$\beta = (\phi_{1,\omega}, \phi^*_{-,\omega}) = (\phi_{2,\omega}, \phi^*_{+,\omega}) \propto C \quad (22)$$

(for $i = 1,2$). The ratio of the two coefficients is thus

$$\left|\frac{\beta}{\alpha}\right|^2 = |C|^2 = \exp\left(-\frac{\omega}{T_H}\right). \quad (23)$$

Using the normalization condition of the Bogoliubov coefficients,

$$|\alpha|^2 - |\beta|^2 = 1, \quad (24)$$

we find that the average occupation number for positive energy u modes, seen by a fiducial observer when the quantum field is in a vacuum with respect to positive energy U modes, simplifies to the expected form of a Bose-Einstein distribution:

$$\bar{n}_\omega = |\beta|^2 = \frac{1}{\exp\left(\dfrac{\omega}{T_H}\right) - 1}, \quad (25)$$

where T_H is the Hawking temperature of the AdS_{d+1} black hole.

A similar relation holds between the V and v modes also. Now, we take into account the reflection from the boundary of AdS space. Reference [13] considered different possibilities for the boundary condition at infinity: Dirichlet, Neu-

mann, and Robin boundary conditions. What is the preferred boundary condition? Let us leave the geometric optics approximation for the moment and consider mode solutions to the exact wave equation for a free scalar field in the AdS black hole space time. In Minkowski signature, the mode solutions can fall into two categories, non-normalizable solutions $\phi^{(-)}_{\omega,\vec{k}}$ and normalizable solutions $\phi^{(+)}_{\omega,\vec{k}}$, with the asymptotic behavior

$$\phi^{(\pm)}_{\omega,\vec{k}}(t,r,\Omega) \to r^{-2h_\pm} \tilde{\phi}^\pm(t,\Omega) \quad (r \to \infty), \quad (26)$$

where $h_\pm$ are parameters related to the mass $\mu =$ and the dimension of the space d by

$$2h_\pm = \frac{1}{2}(d \pm \sqrt{d^2 + 4\mu^2}). \quad (27)$$

The quantized field ϕ is expanded as a linear combination of the normalizable modes. Their decay behavior at the boundary corresponds to reflection. The exact mode solutions $\phi^{(+)}$ are easy to find in $2+1$ dimensions in terms of hypergeometric functions [14,15]. Near the black hole horizon, the normalizable modes reduce to a form

$$\phi^{(+)} \sim (e^{-i\omega u} + e^{-i\omega v + i2\theta_0}) e^{-in\phi}, \quad (28)$$

where θ_0 is a phase shift factor; its exact form can be found in [14]. Thus, in the geometric optics approximation, the modes which take into account the reflection from the boundary and are appropriate to a fiducial observer are a linear combination of the positive energy u and v modes:

$$\phi_\omega = e^{-i\omega u} + e^{-i\omega v + i2\theta_0}. \quad (29)$$

For Kruskal modes, one must consider the corresponding linear combination of the positive energy U and V modes. Thus, in the eternal black hole geometry, the appropriate vacuum is the Hartle-Hawking vacuum. Then, a fiducial observer sees a thermal spectrum for both infalling and outgoing modes. In the eternal AdS geometry, the Unruh vacuum is not well defined with respect to the boundary condition at infinity. However, in the "realistic" geometry of a black hole created by a collapsing shell of matter, the Unruh vacuum would be the natural vacuum. We will return to this issue in Sec. IV, where we discuss Hawking radiation in the collapse geometry.

III. HAWKING RADIATION AS TUNNELING

Recently, a method to describe Hawking radiation as a tunneling process, where a particle moves in dynamical geometry, was developed by Kraus and Wilczek [9,10] and elaborated upon by Parikh and Wilczek [11]. This method also gives a leading correction to the emission rate arising from loss of mass of the black hole corresponding to the energy carried by the radiated quantum. This method was also investigated in the context of black holes in string theory [12], and it was demonstrated that in the string picture of microstates of the black hole, the correction to the emission rate corresponds to a difference between counting of

states in the microcanonical and canonical ensembles. However, in all these investigations the black holes have had asymptotically flat spacetime geometry. We now extend the investigation to black holes in AdS spacetime. We will base our treatment on the presentation of [11].

A convenient trick in the method of [9–11] is to write the black hole metric in a coordinate system where constant time slices are flat, without a singularity at the horizon. In these coordinates, the Schwarzschild metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dt\,dr + dr^2 + r^2\,d\Omega^2.$$
(30)

These coordinates were first introduced 80 years ago by Painlevé [16], but then disappeared from general knowledge, until they were independently rediscovered in [9] and used to investigate black hole quantum mechanics. We will now derive an analogue of the Painlevé coordinates for AdS black holes, which we shall refer to as the AdS-Painlevé coordinates.

By analogue to the asymptotically flat black holes, the AdS-Painlevé coordinates should have the property that constant time slices of the AdS black hole metric (1) will have the same geometry as constant time slices of a global AdS_{d+1} metric:

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{(1+r^2)} + r^2\,d\Omega^2_{d-1}.$$
(31)

Thus, we perform a coordinate transformation $t = \hat{t} + f(r)$, so the metric (1) takes the form

$$ds^2 = -F(r)d\hat{t}^2 + 2f'(r)F(r)d\hat{t}\,dr$$
$$+ \left(\frac{1}{F(r)} - F(r)[f'(r)]^2\right)dr^2 + r^2\,d\Omega^2,$$
(32)

and then demand that on constant $\hat{t}$ slices the metric reduce to

$$ds^2 = (1+r^2)^{-1}\,dr^2 + r^2\,d\Omega^2.$$
(33)

This implies that

$$f'(r) = \frac{1}{r^{(d-2)/2}F(r)}\sqrt{\frac{\mu}{1+r^2}},$$
(34)

so the AdS-Painlevé metric reads as follows:

$$ds^2 = -F(r)d\hat{t}^2 + \frac{2}{r^{(d-2)/2}}\sqrt{\frac{\mu}{1+r^2}}d\hat{t}\,dr + (1+r^2)^{-1}\,dr^2$$
$$+ r^2\,d\Omega^2.$$
(35)

Now we move on to discuss Hawking radiation. The (s-wave) quanta of a massless scalar field follow radial-light-like geodesics

$$\dot{r} = -\frac{\sqrt{\mu}}{r^{(d-2)/2}}\sqrt{1+r^2} \pm (1+r^2),$$
(36)

where the $(+)$ sign corresponds to an outgoing geodesic and the $(-)$ sign corresponds to an ingoing geodesic, respectively. Next we take into account the response of the background geometry to an emitted quantum of frequency ω. We keep the total mass M of the spacetime fixed, but in order to take into account the energy carried by the quantum, we replace μ in Eq. (36) by μ':

$$\mu' \equiv \frac{16\pi}{(d-1)A_{d-1}}(M-\omega).$$
(37)

Note that at the horizon,

$$\dot{r}|_{r_H} = 0.$$
(38)

As the particle travels across the horizon from r_{in} to r_{out}, its action[2] receives an imaginary contribution

$$\mathrm{Im}\,S = \mathrm{Im}\int_{r_{in}}^{r_{out}} p_r\,dr = \mathrm{Im}\int_{r_{in}}^{r_{out}}\int_H \frac{dH}{\dot{r}}dr,$$
(39)

where on the last line we switched the order of integration and used Hamilton's equation $\dot{r} = dH/dp_r$. Next, we substitute from Eq. (36) the radial velocity along the outgoing geodesic, and use $dH = d(M-\omega) = -d\omega$:

$$\mathrm{Im}\,S = -\mathrm{Im}\int_0^\omega d\omega' \int_{r_{in}}^{r_{out}} dr \frac{1+\sqrt{\frac{\mu'}{r^{d-2}}(1+r^2)^{-1}}}{F(r)}.$$
(40)

The only imaginary contribution to the radial integral comes from the pole at r_H. Then

$$\mathrm{Im}\,S = \pi \int_0^\omega d\omega' \frac{2r_H}{r_H^2 d + (d-2)}.$$
(41)

On the other hand, after solving for r_H as a function of μ, we can derive that

$$r_H^{d-2}\frac{dr_H}{d\mu} = \frac{r_H}{r_H^2 d + (d-2)}.$$
(42)

Substituting this into the integral yields

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[2]Note that in the local point particle description used in this section, the issue of boundary conditions at infinity does not arise.

$$\mathrm{Im}\,S = \pi \frac{(d-1)A_{d-1}}{16\pi} \int_{\mu-[16\pi/(d-1)A_{d-1}]\omega}^{\mu} d\mu' \, 2r_H^{d-2} \frac{dr_H}{d\mu'}$$

$$= \frac{1}{8} A_{d-1}[r^{d-1}(M) - r^{d-1}(M-\omega)] = \frac{1}{2}\Delta S_{BH}, \tag{43}$$

where $\Delta S_{BH} = S_{BH}(M) - S_{BH}(M-\omega)$ is the difference of the entropies of the black hole before and after the emission. Thus, the tunneling probability for the particle is

$$\Gamma = \exp(-\Delta S_{BH}). \tag{44}$$

If we Taylor expand ΔS_{BH} in ω, the leading term gives the thermal Boltzmann factors $\exp(-\omega/T_H)$ for the emanating radiation. The second term represents corrections from the response of the background geometry to the emission of a quantum. The same result holds for emission from asymptotically flat black holes [12].

IV. PARTICLE CREATION BY A COLLAPSING SPHERICAL SHELL IN AdS

We will now turn to a third way to analyze Hawking radiation from black holes and investigate particle creation by a collapsing spherical body which forms a black hole in AdS space. We will base our treatment on the discussion in [17], which in turn follows [18] and [19]. As in [17], the starting point is that we assume that in the remote past the spherical body is distended so much that it deforms the anti–de Sitter space. Thus, in the beginning we can assume that a quantum field is in a vacuum constructed with respect to global coordinates in AdS space. Now a convenient choice for the global coordinates is given by

$$ds^2 = (\sec\rho)^2(-dt^2 + d\rho^2) + (\tan\rho)^2 \, d\Omega_{d-1}^2. \tag{45}$$

The (normalizable) mode solutions can be found, e.g., in [20]:

$$\phi_{n,l}^{(+)} = e^{-i\omega t} Y_{l,\{m\}}(\Omega)(\cos\rho)^{2h_+}(\sin\rho)^l$$

$$\times P_n^{(l+d/2-1,2h_+-d/2)}(\cos 2\rho), \tag{46}$$

where $Y_{l,\{m\}}(\Omega)$ is a spherical harmonic on S^{d-1}, $P_n^{(l+d/2-1,2h_+-d/2)}$ is a Jacobi polynomial, and

$$2h_+ = \frac{d}{2} + \frac{1}{2}\sqrt{d^2 + 4m^2} \equiv \frac{d}{2} + \frac{1}{2}\nu. \tag{47}$$

As a result of the boundary conditions at the origin and at the boundary of AdS space, the spectrum is discrete, with

$$\omega = 2h_+ + 2n + 2l, \quad n = 0,1,2,\ldots. \tag{48}$$

Again, we focus on s waves $(l=0)$. As in flat space, we expect the quantum to experience a strong redshift as it propagates across the collapsing body. Thus, in the remote

past, we are most interested in high-frequency modes. In the high-frequency limit, the mode solution (46) reduces to a simplified form

$$\phi_n^{(+)} \sim (\cos\rho)^{(d-1)/2} e^{-i\omega_n t} \cos\left(\omega_n\rho - \frac{d\pi}{4}\right), \tag{49}$$

where we suppressed normalization factors. In other words, they take the form of a standing wave, a superposition of an ingoing and outgoing spherical wave, with a discrete spectrum. Note that using the radial coordinate r, the overall factor $(\cos\rho)^{(d-1)/2} \sim (1/r)^{(d-1)/2}$ as $r \gg 1$, so we recover the expected overall decay factor for the amplitude. Now, we will add into the picture the collapsing body and try to compute the redshift due to the passage of the wave across it. As in [17], we will use the geometric optics approximation and truncate the analysis to two dimensions to the t,r plane by suppressing the overall decay factor of the waveform. For simplicity, we assume that the collapsing body is a thin shell of radius R, with R monotonically decreasing in time. The truncated metric inside and outside the shell takes the form

$$ds^2 = -F_\pm \, dt_\pm^2 + F_\pm^{-1}(r)dr^2, \tag{50}$$

where $F_+(r) = 1 - \mu/r^{d-2} + r^2$ and $F_-(r) = 1 + r^2$. We then define the tortoise coordinates $r_*^\pm$,

$$r_*^\pm = \int \frac{dr}{F_\pm(r)}, \tag{51}$$

and the null coordinates

$$u = t_+ - r_*^+, \quad v = t_+ + r_*^+,$$

$$U = t_- - r_*^-, \quad V = t_- + r_*^-, \tag{52}$$

so that the interior and exterior metrics are conformal to a flat metric. The tortoise coordinate in the interior is $r_*^- = \arctan r$, so the origin $r=0$ corresponds to $r_*^- = 0$. In terms of the null coordinates U, V, the origin is then at $V - U = 0$. Note also that $r_*^- = \rho$, where ρ is the coordinate that appears in the global metric (45). The exterior and interior null coordinates are related by

$$v = \beta(V),$$

$$U = \alpha(u), \tag{53}$$

where $\alpha(u)$ and $\beta(V)$ are to be determined below. In the (t,r) coordinates, the passage of a wave across the shell turns to a reflection condition at the origin:

$$v = \beta(V) = \beta(U) = \beta(\alpha(u)). \tag{54}$$

Thus, in the asymptotic region (near the boundary), the waves have a phase structure

$$\tilde{\phi}^{(+)} \sim e^{-i\omega_n v} - e^{-i\omega_n\beta(\alpha(u))}. \tag{55}$$

To find the functions α, β, we match the interior and exterior metrics across the collapsing shell at $r = R(\tau)$. Here τ de-

notes the shell time, which is related to the time coordinates $t_\pm$ in the interior and exterior of the shell through

$$ds^2 = [-F_\pm \, dt_\pm^2 + F_\pm^{-1} \, dr^2]_{|r=R(\tau)} = -d\tau^2. \qquad (56)$$

It is easiest to consider the derivatives

$$\alpha'(u) = \frac{dU}{du} = \frac{\dot{U}}{\dot{u}},$$

$$\beta'(V) = \frac{dv}{dV} = \frac{\dot{v}}{\dot{V}} \qquad (57)$$

(at the shell), where $\cdot = d/d\tau$. Using the definition (56), we obtain

$$\frac{dU}{du} = \frac{F_+(R)[\sqrt{F_-(R)+\dot{R}^2}-\dot{R}]}{F_-(R)[\sqrt{F_+(R)+\dot{R}^2}-\dot{R}]},$$

$$\frac{dv}{dV} = \frac{F_-(R)[\sqrt{F_+(R)+\dot{R}^2}+\dot{R}]}{F_+(R)[\sqrt{F_-(R)+\dot{R}^2}+\dot{R}]}. \qquad (58)$$

As the radius of the shell approaches the horizon, we can approximate

$$F_+(R) \approx 4\pi T_H(R-r_H),$$

$$F_-(R) \approx F_-(r_H) \equiv A. \qquad (59)$$

Then, we can approximate

$$\frac{dU}{du} \approx -\frac{2\pi T_H(R-r_H)}{A\dot{R}}[\sqrt{A+\dot{R}^2}-\dot{R}], \qquad (60)$$

where we used $\sqrt{\dot{R}^2} = -\dot{R}$ since $\dot{R}<0$ as the shell is collapsing and $|\dot{R}| \neq 0$ as a function of shell time. In the above, $\dot{R} = \dot{R}_{|r_H}$. Next, we relate U to $R-r_H$ by expanding

$$U \approx U(r_H) + \frac{dU}{dR_{|R=r_H}}(R-r_H) \qquad (61)$$

and evaluating the derivative dU/dR at the horizon, using the chain rule and Eqs. (51), (53), (56), and (59). We obtain

$$(R-r_H) \approx [U-U(r_H)]\frac{A\dot{R}}{[\sqrt{A+\dot{R}^2}-\dot{R}]}. \qquad (62)$$

We substitute this into Eq. (60) and obtain

$$\frac{dU}{du} \approx -2\pi T_H[U-U(r_H)] \equiv -\kappa U + \text{const}, \qquad (63)$$

where κ is the surface gravity of the black hole. Integration then gives

$$\alpha(u) = e^{-\kappa u} + \text{const}. \qquad (64)$$

A similar calculation for dv/dV gives

$$\frac{dv}{dV} \approx -\frac{A}{\dot{R}[\sqrt{A+\dot{R}^2}+\dot{R}]} \equiv c \quad (=\text{const}). \qquad (65)$$

By integration,

$$\beta(V) = cV + \text{const}. \qquad (66)$$

Thus, we find that in the asymptotic region the waves have a phase structure

$$\tilde{\phi}^{(+)} \sim e^{-i\omega_n v} - e^{-i\omega_n c(e^{-\kappa u}+\text{const})}. \qquad (67)$$

To obtain modes where the outgoing wave is of standard form, we invert functionally and write

$$\tilde{\phi}^{(+)} \sim e^{i\omega_n\kappa^{-1}\ln[(v_0-v)/c]} - e^{-i\omega_n u}, \qquad (68)$$

which is valid only for $v<v_0$. Now, we move back to $d+1$ dimensions and compare with the high-frequency limit of the global modes (49). Note that far in the asymptotic region, $r \to \infty$, the exterior tortoise coordinate reduces to the same form as ρ, $r_*^+ \approx \rho$, so we can write the global mode as

$$\phi^{(+)} \sim (\cos\rho)^{(d-2)/2}(e^{-i\omega_n v - id\pi/2} - e^{-i\omega_n u}). \qquad (69)$$

We want to compare this with the modes that we found in the collapsing shell geometry:

$$\tilde{\phi}^{(+)} \sim (\cos\rho)^{(d-2)/2}(e^{i\omega_n\kappa^{-1}\ln[(v_0-v)/c]} - e^{-i\omega_n u})$$

$$(v<v_0), \qquad (70)$$

where we added the overall decay factor. The Bogoliubov transformation follows the discussion in [17], and as a result we find that the outgoing modes are thermally excited, if the field is in a global vacuum. Thus, the global vacuum resembles an Unruh vacuum.

We can now easily perform a check that the coordinates u,v are indeed the appropriate coordinates for external observers measuring the radiation. We evaluate the expectation value of the energy-momentum tensor in the u,v coordinate frame. Since we have focused only on the s waves, the expectation value of the (renormalized) stress tensor is given by the Schwarzian derivative (see [17], Sec. 7.1)

$$\langle T_{uu}\rangle_{\text{ren}} = \frac{1}{12\pi}[p'(u)]^{1/2}\partial_u^2[p'(u)]^{-1/2}, \qquad (71)$$

where

$$p(u) \equiv c(e^{-\kappa u}+\text{const}) \qquad (72)$$

from Eq. (67). The other components of the stress tensor vanish,

$$\langle T_{vv}\rangle_{\text{ren}} = \langle T_{uv}\rangle_{\text{ren}} = \langle T_{vu}\rangle_{\text{ren}} = 0, \qquad (73)$$

signaling only outgoing flux. Substituting Eq. (72) we find

$$\langle T_{uu} \rangle_{\text{ren}} = \frac{\kappa^2}{48\pi}, \tag{74}$$

which is the energy density of a thermal flux of radiation at temperature

$$T_H = \frac{\kappa}{2\pi}. \tag{75}$$

On the other hand, the decay rate for the black hole is given by

$$\frac{dM}{dt} = -L = -\frac{1}{2\pi} \sum_{l=0}^{\infty} \int_0^{\infty} d\omega \frac{\omega \Gamma_{\omega l}}{e^{\omega/T_H} - 1}. \tag{76}$$

When we evaluate the luminosity L in the s-wave approximation ($l=0$) and in the geometric optics approximation with no backscattering ($\Gamma_{\omega l}=1$), we obtain the result

$$\frac{dM}{dt} = -\frac{\kappa^2}{48\pi}. \tag{77}$$

Thus the decay rate (with respect to the time coordinate t of the asymptotic observer) agrees with the observed flux of radiation (74).

In principle, the above result only applies to the onset of Hawking radiation as the black hole has formed and does not address the issue of subsequent evolution. As a result of the reflecting boundary conditions at the boundary of the AdS space [21], one might consider the subsequent evolution of a black hole as analogous to that of a black hole in a finite box. Indeed, in the AdS$_3$ case it has been suggested [22] that an evaporation black hole will eventually settle in equilibrium with the thermal radiation. Let us quickly review the key arguments in [22]. In order to see that equilibrium with a remnant in a thermal bath is possible, one treats the system as a microcanonical ensemble with total energy,

$$E = E_{\text{rad}} + M, \tag{78}$$

fixed. E_{rad} denotes the energy of the radiation, and M denotes the mass of the remnant. Initially, there is no radiation and all energy is in the black hole, $E = M_0$. The total entropy of the system is

$$S_{\text{total}} = S_{\text{rad}} + S_{\text{remnant}} \sim \left(\frac{m_c}{m_p}\right) \left(\frac{M_0 - M}{m_c}\right)^{2/3} + \left(\frac{m_c}{m_p}\right)^{3/2} \left(\frac{M}{m_c}\right)^{1/2}, \tag{79}$$

up to irrelevant numerical coefficients. The equation involves two mass scales:

$$m_c = \frac{1}{G} \tag{80}$$

is a classical mass scale, and G is Newton's constant in 2 + 1 dimensions. The physical significance of m_c for black

holes is that the characteristic wavelength of Hawking radiation λ is smaller than the radius of the horizon if $M > m_c$ [22]. The other mass scale

$$m_p = \left(\frac{\hbar^2 \Lambda}{G}\right)^{1/3} \tag{81}$$

is the Planck scale in 2 + 1 dimensions. Its value depends on the cosmological constant Λ. Weak coupling approximation for gravity applies when $m_c \gg m_p$.

The total entropy is extremized ($\partial S/\partial M = 0$) when

$$\left(\frac{M}{m_c}\right)^{1/2} \left(1 - \frac{M}{M_0}\right)^{-1/3} = \left(\frac{M_0}{m_c}\right)^{1/3} \left(\frac{m_c}{m_p}\right)^{1/2}. \tag{82}$$

For an example, suppose that the equilibrium sets at $M = (m_c)^2/m_p$. Then the initial mass of the black hole was

$$M_0 = \left(1 + \frac{m_c}{m_p}\right) m_c. \tag{83}$$

So only a small fraction of the initial mass was lost before the equilibrium was reached. However, as noted in [22], consistency requires that the evaporation time to the equilibrium must be much larger than the time that it takes for the emitted particles to be reflected from the boundary. Only then is there sufficient time for the thermal background to build up for the reflected particles to be captured again by the black hole. This is a subtle point. If the emitted particles were massive, they would follow timelike geodesics and fall back to the black hole[3] in time of the order $t_{return} \sim \Lambda^{-1/2}$. Massless particles would propagate to the boundary and reflect back from there. However, in the AdS black hole spacetime, it takes an infinite time for massless particles to reach the boundary.[4] Thus, a thermal background can build up in finite time only when emission of massive particles is included. In that case, let us estimate the decay time. We assume that the emitted particles are very light, and use a (2 + 1)-dimensional version of Stefan's law,

$$\frac{dM}{dt} \sim -AT^3 \sim -M^2, \tag{84}$$

where the area of the black hole $A \sim r_H \sim \sqrt{M}$ and the Hawking temperature is $T_H \sim r_H$. The decay time from the initial mass M_0 to the equilibrium mass M is [22]

$$\Delta t \sim \left(\frac{m_c}{m_p}\right)^{2/3} \left(1 - \frac{M_0}{M}\right) \Lambda^{-1/2}. \tag{85}$$

For our example case (83),

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[3]Timelike geodesics for AdS$_3$ black holes have been investigated in [23].
[4]Contrary to what was claimed in [22].

$$\Delta t \sim \left(\frac{m_c}{m_p}\right)^{2/3} \Lambda^{-1/2} \gg \Lambda^{-1/2}, \tag{86}$$

so the decay time is much larger than the time it takes for the emitted particles to fall back into the black hole.

To sum up, for black holes created by spherically collapsing matter and for Hawking radiation of massless fields in the s-wave approximation, the Unruh vacuum is the appropriate vacuum. However, if the emission involves massive particles, the black hole evaporation proceeds in the asymptotically AdS spacetime as if the black hole was in a finite box. Then it has been noted [22] that at least in $2+1$ dimensions, the evaporation can stabilize, with the end point of the system being a stable remnant in equilibrium with radiation.

At equilibrium, the appropriate vacuum state is the Hartle-Hawking state. The equilibrium analysis in dimensions $d > 2$ is more involved. In the context of AdS/CFT correspondence, when the asymptotic spacetime is $\mathrm{AdS}_{d+1} \times S^{d+1}$, it has been investigated in [7].

Note added. As we were finalizing this paper, the paper [24] appeared, discussing Hawking radiation in the optical collapse geometry for spherically symmetric black holes.

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