Tensor to scalar ratio of phantom dark energy models

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We investigate the anisotropies in the cosmic microwave background in a class of models which possesses a positive cosmic energy density but negative pressure, with a constant equation of state $w = (p/\rho) < -1$. We calculate the temperature and polarization anisotropy spectra for both scalar and tensor perturbations by modifying the publicly available code CMBFAST. For a constant initial curvature perturbation or tensor normalization, we have calculated the final anisotropy spectra as a function of the dark energy density and equation of state w and of the scalar and tensor spectral indices. This allows us to calculate the dependence of the tensor-to-scalar ratio on w in a model with phantom dark energy, which may be important for interpreting any future detection of long-wavelength gravitational waves.

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I. INTRODUCTION

For many years [1], arguments have been made in favor of the cold dark matter model with a cosmological constant (Λ CDM) because of its desirable property of having spatial flatness while accommodating a low matter density, conditions which seem to be favored by current astrophysical observations. A few years ago, however, an old alternative [2] resurfaced and has been receiving considerable attention [3]. This alternative, which we will refer to as the CDM with a scalar field (Φ CDM), borrows from the literature on inflation and postulates a slowly rolling scalar field whose energy density has only recently become cosmologically relevant. This slowly rolling scalar field provides a constituent of the universe which is smooth below the horizon scale¹ and whose evolution can be made arbitrarily slow. This is usually parametrized by the equation of state, $w \equiv p/\rho$, with the model becoming indistinguishable from the cosmological constant in the limit $w \rightarrow -1$. Generically 1 > w > -1 and w is typically a function of time.

Of particular interest to us was a paper by Caldwell [4] which introduced a Φ CDM model with w < -1. This form of dark energy, dubbed *phantom energy*, also has the remarkable property of satisfying $\rho_{\phi} + p_{\phi} < 0$, just as in the standard Φ CDM case. It was shown that this model is consistent with both recent observations and classical tests of cosmology, in some cases providing a better fit than the more familiar models with w > -1.

There are several known ways to achieve w < -1, e.g. Ref. [5], but Caldwell's model makes use of a non-canonical kinetic term, a method introduced in [6]. There is no obvious motivation from particle physics for considering such a radical extension to the theory, and indeed the model has been criticized on these grounds [5]. However we regard this as a useful toy model because it allows us to simply extend the equation of state parameter space continuously below w

=-1 without the additional complications inherent in modifications of the gravitational sector of the theory. (We shall discuss how our results depend on this detail later.) Observational constraints are often maximally likely at the value w = -1 and for the purposes of maximum likelihood analysis it is better not to have the region of maximum likelihood bumping against the edge of the parameter space. Thus it is advantageous to extend the domain of the parameter w, even if one is primarily interested, on theoretical grounds, in values $w \ge -1$.

One aspect which was not studied in [4] was the effect of gravitational wave perturbations in the cosmic microwave background (CMB), and in particular the dependence of the tensor-to-scalar ratio on the equation of state w. In this paper we study the tensor anisotropy spectrum in the phantom model. We show that the normalization of the tensor perturbations in the CMB is only weakly dependent on the equation of state. The behavior of the tensor-to-scalar ratio is therefore dominated by the w dependence of the scalar perturbations.

We are interested in the tensor contribution to the anisotropy spectrum because it is, in principle, observable and directly connected to the energy scale of inflation [7], a fundamental parameter of considerable importance. Our calibration of the relation between the amplitude of the anisotropy in the CMB and this fundamental energy scale is clearly necessary to interpret any future observations of this signal.

II. THE MODEL

Phenomenologically, the property of negative pressure and positive energy density can be achieved by considering the (unorthodox) Lagrangian density

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) \tag{1}$$

for the phantom component ϕ . We are adopting the metric convention $g_{\mu\nu} = \text{diag}(-a^2, a^2, a^2, a^2)$ where a(t) is the cos-

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mological scale factor. To lowest order the non-canonical negative kinetic term in the Lagrangian produces the following expressions for the pressure and energy density of this dark energy:

$$\rho_{\phi} = -\frac{1}{2a^2} \dot{\phi}^2 + V(\phi)$$
 (2)

$$p_{\phi} = -\frac{1}{2a^2} \dot{\phi}^2 - V(\phi)$$
 (3)

where over-dots represent derivatives with respect to conformal time η , and $dt = ad\eta$. Negating the kinetic term also alters the equation of motion of the field from that of a field with a canonical Lagrangian by switching the sign of the $\partial V/\partial \phi$ term,

$$\ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\frac{\partial V}{\partial \phi} = 0.$$
(4)

Clearly the evolution of the scalar field is coupled into the evolution of the background Friedmann equations:

$$\mathcal{H}^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} a^2 \sum_i \rho_i.$$
 (5)

Here $\mathcal{H}=aH=a(a^{-1}da/dt)$, taking into account that our derivatives are with respect to conformal time η . The effect of ϕ on the evolution of the background cosmology can be implemented with only a slight modification to the CMBFAST integration code [8] to include the new component's pressure and energy density.

Once the expressions for ρ_{ϕ} and p_{ϕ} are obtained, it is simple to invert them to obtain the potential V in terms of the constant w and derivatives of the scalar field. In order to prevent tachyonic modes of the scalar field from developing, we address only models with constant w [4]:

$$V = -\frac{\phi^2}{2a^2} \left(\frac{1-w}{1+w}\right). \tag{6}$$

Following [4], by treating the dark energy as a perfect fluid, we can obtain an explicit expression for $\dot{\phi}$:

$$\dot{\phi} = \sqrt{-(1+w)\rho_{\text{crit}}\Omega_{\phi}} a^{-(1+3w)/2} \tag{7}$$

where $\rho_{\rm crit}=3\mathcal{H}^2/8\pi Ga^2$, and $\Omega_{\phi}=\rho_{\phi}/\rho_{\rm crit}$. Taking its derivatives we observe

$$\frac{\partial V}{\partial \phi} = (1 - w) \frac{3\mathcal{H}\dot{\phi}}{2a^2} \tag{8}$$

$$\frac{\partial^2 V}{\partial \phi^2} = (1 - w) \frac{3}{2a^2} \left[\dot{\mathcal{H}} - \mathcal{H}^2 \left(\frac{5}{2} - \frac{3w}{2} \right) \right].$$
(9)

Thus V and its derivatives will be functions of a and the parameters w, Ω_{ϕ} , and ρ_0 only.

To calculate the anisotropy in the CMB we must include the perturbations of ϕ to first order (for an alternative treatment, see [9]). We work throughout in the synchronous gauge. Since we are solving linearized equations it is useful to Fourier transform the perturbations so that we can solve the differential equations for each *k*-mode independently. The equation of motion for a particular *k*-mode is

$$\ddot{\delta\phi_k} + 2\frac{\dot{a}}{a}\dot{\delta\phi_k} + a^2 \left(k^2 - \frac{\partial^2 V}{\partial\phi^2}\right)\delta\phi_k = -\frac{1}{2}\dot{h}\dot{\phi} \qquad (10)$$

where *h* is the trace of the metric perturbation in the synchronous gauge. The scalar field perturbation, being of spin-0, does not depend on the other metric perturbation directly. The $\delta \phi_k$'s produce the fluctuations in the energy density and pressure of the ϕ field:

$$\delta \rho_{\phi} = \left(\frac{2}{1-w}\right) \frac{\partial V}{\partial \phi} \,\delta \phi_k \tag{11}$$

$$\delta p_{\phi} = \left(\frac{2w}{1-w}\right) \frac{\partial V}{\partial \phi} \,\delta \phi_k \quad . \tag{12}$$

These stress-energy perturbations must be included in the evolution of the metric perturbations for a self-consistent solution. The Φ CDM component contributes a source term to the right hand side of the Einstein equations. To make these calculations, we have modified CMBFAST by including the Φ CDM contribution to the evolution of the background cosmology in the Friedmann equations. We have also added the contribution of density and pressure perturbations of the ϕ field to the total curvature perturbation, using the results derived above. Even though the background pressure is negative, we are certain that these perturbations are stable, because $\partial^2 V/\partial \phi^2$ is a negative definite quantity in this case. Examining Eq. (10) we see that the square of the effective mass will never become negative, and exponentially growing tachyonic modes will not occur.

Note that $\delta\phi$ cannot source gravitational wave modes because ϕ is a scalar field, while a source of gravitational waves must have spin 2. Therefore, the only effect of ϕ on the gravitational wave anisotropies is through the change in the evolution of the scale factor [10].

III. CMB ANISOTROPY SPECTRA

Figures 1 and 2 display the scalar and tensor components of the temperature anisotropy for several values of the equation of state parameter w. In each case we assumed that the cosmological parameters were h=.67, $\Omega_m=0.3=1-\Omega_{\phi}$, $\Omega_b=0.04$ and that the scalar spectral index $n_s=1$, and the tensor spectral index $n_t=0$. It is standard practice [11] to write the temperature field on the sky as a sum of spherical harmonics with coefficients a_{lm} . The multipole moments, C_l , are then defined as $C_l \equiv |a_{lm}|^2$ and the power spectrum, $l(l+1)C_l/(2\pi)$, is approximately the power per logarithmic interval in l.

Since we are working to linear order in perturbation theory both the scalar and tensor spectra have one free over-



FIG. 1. The scalar CMB temperature anisotropy spectra, $l(l + 1)C_l/2(\pi)$ vs *l* for several values of *w*. The solid lines are w = -1.1, dashed are w = -3, dot-dashed are w = -10, and dashed-dashed are w = -0.4. The spectra have been normalized at C_{10} to fit the COBE data.

all normalization (ignoring for a moment any constraints imposed by, e.g., inflation). In our calculations we fixed the scalar normalization by requiring that the initial spatial curvature in the total matter gauge [12] $\zeta = 1$. For the tensor normalization² we have chosen h=1 initially. This choice makes it easy to interpret the dependence on w, and also to calculate the cosmology dependent fitting function presented in Sec. IV. However with such a normalization the curves are very similar beyond $l \sim 100$ for all values of w so for plotting purposes we have normalized the curves at l=10 in Figs. 1 and 2. In Fig. 3, the E-E and B-B (tensors only) polarization spectra are presented with $\zeta = 1$ or h=1 normalization, which illustrates clearly that the height of the features is uniform with w.

Consider first the anisotropy spectrum in Fig. 1. There are three features exhibiting w dependence: the height of the first peak, the height of the tail at low multipoles, and the location of the peaks. The change in the height of the first peak traces back to our decision to re-normalize the spectra to fit the Cosmic Background Explorer (COBE) data. In fact the first two effects are manifestations of the same physical property: the suppression of the late integrated Sachs-Wolfe (ISW) effect for lower w. In models such as Φ CDM, a period of accelerated expansion begins in the universe when the scalar field ϕ becomes the dominant form of energy density. For a fixed matter density Ω_m (measured relative to the critical density) the value of the parameter w determines how early matter- ϕ equality occurs, and therefore how long the universe has been experiencing the current period of accelerated expansion. When an accelerated expansion is occurring, potential wells which had formed earlier as a result of gravitational collapse begin to decay. If the well is decaying, photons which fall into the well gain more energy than they lose climbing out, which generates an anisotropy. This is the dominant effect at low *l*. For w < -1 models, equality happens even later than in the case of the cosmological constant, while for w > -1, it happens earlier. For very negative val-



FIG. 2. The tensor CMB temperature anisotropy spectra as a function of w. As in Fig. 1, the solid lines are w = -1.1, dashed are w = -3, dot-dashed are w = -10, and dashed-dashed are w = -0.4. Notice that the curves are almost degenerate for the first 40 multipoles. There appears to be negligible dependence of the tensor normalization on the equation of state w.

ues of w, there has been almost no acceleration in the recent history, so the potential wells have not decayed as dramatically as they do in Λ CDM models. As a result, the late integrated Sachs-Wolfe (ISW) effect has been almost entirely turned off [4].

When normalized to $\zeta = 1$, the effect on the spectrum is to enhance the amplitude at low multipoles as *w* decreases. Renormalizing to fit the COBE data at l = 10, the enhanced low*l* tail translates into a change in the peak height. For energy density perturbations, the dependence of the spectrum normalization on the equation of state is principally due to the lack of late ISW effect as *w* decreases.



FIG. 3. The polarization anisotropy spectra. For the scalar modes, the E-E polarization anisotropies are shown (top), and for the tensor modes the E-E and B-B polarization anisotropies are presented (middle and bottom). The scale is arbitrary, depending on our particular choice of normalization.

²The tensor perturbation is gauge independent.

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The normalization of the tensor spectrum appears not to be affected by w in any significant way (Fig. 2). This can be seen by noting that the peaks are all roughly at the same height, and the spectra between 2 < l < 10 agree closely (compare with Fig. 1). The small w dependence that can be seen occurs for the lowest multipoles, corresponding to largest angular scales, and is again due to the ISW effect on the tensor spectrum at late times.

For all the anisotropy spectra (both temperature and polarization anisotropies) there is a universal shift to the right for lower values of w. The reason for the shift is the same in all cases. The l position of any feature in a spectrum depends upon the (comoving) angular diameter distance to the last scattering surface. This distance depends on w as follows:

$$d_{\rm lss} = \int_{a_{\rm lss}}^{1} \frac{da}{a^2 H} \tag{13}$$

where

$$H^{2}(a) = \Omega_{r}a^{-4} + \Omega_{m}a^{-3} + \Omega_{\phi}a^{-3(1+w)}.$$
 (14)

One notes that if w is more negative, this integral is larger, which implies that the last scattering surface is further away. A feature of a given physical size thus subtends a smaller angle, shifting the anisotropy spectra to higher values of l. In the case of the polarization anisotropies, this is the only dependence on w (Fig. 3).

IV. DEPENDENCE OF T/S ON w AND n_S

For $\Omega_b = .04$ and assuming a flat universe, we have explicitly calculated the evolution of the normalization of the spectrum from an initial curvature perturbation $\zeta = 1$ and from h = 1. The normalization at the tenth multipole is equal to the value just after primordial inflation, multiplied by a cosmology dependent "transfer function," which we will call f_s and f_T . Using the method described in [13] we have numerically determined this function for the ranges -2.5 $< w < -0.5, 0.6 < \Omega_{\phi} < 0.8, \text{ and } -0.3 < n_T < 0.3$ and provide fitting functions below. The tensor spectral index n_T is related to the scalar spectral index $n_T = n_S - 1$ for power law models of primordial inflation. We have deliberately left a w term in the evolution of the tensor normalization to illustrate how insignificant it is compared to the terms proportional to the spectral index and Ω_{ϕ} . A fit is presented below, where a = 0.6156, b = 1.6, and c = -.2. The scalar transfer function has a mean accuracy of 2% over the range indicated above, and at worst is off by 10% near the edges of the parameter domains. This would correspond to a worst case 5% error in the temperature. Since the fit begins to go bad near the edges of the parameter space, we do not recommend extrapolating these functions further. The tensor function has a mean accuracy of .8%, with a worst case of 6% for a few points at the very edge of the fit:



FIG. 4. The normalization of the tensor and scalar spectra at the 10th multipole for an initial perturbation $\zeta = 1$ and h = 1.

$$f_{S}(w,n_{T},\Omega_{\phi}) = -1.341 + 6.031\Omega^{a}n^{b}(-w)^{c} + 1.936\Omega^{a}$$
$$+ 3.312n^{b} + 4.584(-w)^{c} - 5.148\Omega^{a}n^{b}$$
$$- 9.375\Omega^{a}(-w)^{c} - 1.946n^{b}(-w)^{c}$$
$$+ 5.172\Omega(-w)^{c} - 2.692n(-w)^{c}$$
$$- .072n^{b}w - .211(n-1)^{2}$$
(15)

$$f_T(w, n_T, \Omega_{\phi}) = 0.358 + 0.001w + 0.822n_T - 0.046\Omega_{\phi} + 0.337n_T^2 + 0.029\Omega_{\phi}^2 - 0.487n_t\Omega_{\phi}.$$
(16)

The tensor to scalar ratio is given by

$$\frac{C_{10}^{T}}{C_{10}^{S}} = \frac{f_{T}(w, n_{T}, \Omega_{\phi})}{f_{S}(w, n_{T}, \Omega_{\phi})} \frac{A_{T}^{2}}{A_{S}^{2}}.$$
(17)

Figure 4 shows the dependence of both the scalar and tensor normalizations at C_{10} on the equation of state *w* for an even broader domain than was used in the fitting formulas above. Observe that even over this huge range in the equation of state, the tensor normalization changes by less than 1%.

V. DISCUSSION

We have extended the work of [4] on the Φ CDM model by calculating the tensor component of the CMB anisotropies. We have shown that the normalization of the tensor component is essentially independent of the equation of state w in this model. For a fixed value of the initial curvature perturbation after inflation $\zeta = 1$ and a fixed initial gravitywave amplitude, we have quantified the w, Ω_{ϕ} , and n_T dependence of both the tensor and scalar anisotropy normalizations at C_{10} for a broad domain of w. We displayed the anisotropy spectra for several values of *w* and discussed the features.

Most of the results we present for w < -1 are only weakly sensitive to the particular model chosen [4]. While truly general statements are not possible, we can make several points about model dependence. The fact that our acceleration is driven by a field of spin zero implies there is no source of gravitational waves as ϕ begins to dominate. This would remain true for other scalar (or vector) driven theories, while more complicated assumptions (e.g., higher order gravity theories) may have explicit source terms. However, we would expect only the longest wavelength gravitational waves to differ due to this in any significant way. This is because the acceleration takes place very late in the history of the universe for w < -1, so only the longest modes are affected. The longest wavelength modes contribute primarily to the very low-l moments of the spectrum. A similar argument can be made for the scalar modes, where the overall growth rate should be independent of the detailed model, but the longest wavelength modes will depend on any fluctua-

- L.M. Krauss and M.S. Turner, Gen. Relativ. Gravit. 27, 1137 (1995); A.R. Liddle, D.H. Lyth, P.T.P. Viana, and M. White, Mon. Not. R. Astron. Soc. 282, 281 (1996); J.P. Ostriker and P.J. Steinhardt, Nature (London) 377, 600 (1995).
- H. Kodama and M. Sasaki, Prog. Theor. Phys. 78, 1 (1984); B.
 Ratra and P.J.E. Peebles, Phys. Rev. D 37, 3406 (1988).
- [3] K. Coble, S. Dodelson, and J. Frieman, Phys. Rev. D 55, 1851 (1997); M.S. Turner and M. White, *ibid.* 56, 4439 (1997); P.G. Ferreira and M. Joyce, *ibid.* 58, 023503 (1998); R.R. Caldwell, R. Dave, and P.J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998); P.J.E. Peebles and B. Ratra, Astrophys. J. Lett. 325, L17 (1988).
- [4] R.R. Caldwell, astro-ph/9908168.
- [5] L. Parker and A. Raval, Phys. Rev. D 62, 083503 (2000).

tions in the "phantom" component through their impact on the evolution of the metric perturbations. Finally, the angular shift with w of features in the spectrum is robust.

We have quantified the dependence of the tensor to scalar ratio C_{10}^T/C_{10}^S on w, Ω_{ϕ} , and n_T . For the model of Ref. [4], Eq. (17) allows us to relate the (potentially) observable C_{10}^T/C_{10}^S to the energy scale of inflation, in the event that a gravitational wave signal is actually observed from the CMB.

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- [6] C. Armendariz-Picon, T. Damour, and V. Mukhanov, Phys. Lett. B 458, 209 (1999).
- [7] A.R. Liddle and D.H. Lyth, *Cosmological Inflation and Large Scale Structure* (Cambridge University Press, Cambridge, England, 2000).
- [8] U. Seljak and M. Zaldarriaga, Astrophys. J. 469, 437 (1996).
- [9] Jaume Garriga and V.F. Mukhanov, Phys. Lett. B 458, 219 (1999).
- [10] M. White, Phys. Rev. D 46, 4198 (1992).
- [11] M. White, D. Scott, and J. Silk, Annu. Rev. Astron. Astrophys. 32, 319 (1994).
- [12] W. Hu, D.N. Spergel, and M. White, Phys. Rev. D 55, 3298 (1997).
- [13] M.S. Turner and M. White, Phys. Rev. D 53, 6822 (1996).