Stationary dark energy: The present universe as a global attractor

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We propose a cosmological model that makes a significant step toward solving the coincidence problem of the near similarity at the present of the dark energy and dark matter components. Our cosmology has the following properties: (a) among flat and homogeneous spaces, the present universe is a global attractor; *all* the possible initial conditions lead to the observed proportion of dark energy and dark matter; once reached, it remains fixed forever; (b) the expansion is accelerated at the present; (c) the model is consistent with the large-scale structure and microwave background data; (d) the dark energy and the dark matter densities scale similarly after equivalence and are close to within one order of magnitude.

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Since the introduction of inflationary models the notion of attractor cosmological solutions has been regarded as a desirable property of any successful model. Unfortunately, inflation itself has never completely solved the problem of the initial conditions, since the subsequent decelerated era is no longer an attractor, and any fluctuation away from flatness will be amplified in the future, unless a new accelerated era prevents it.

The search for cosmological attractors has been revived by the recent findings [1] according to which the dominant component of the universe medium is in a form of energy density possessing peculiar characteristics: negative pressure and weak clustering. This energy, dubbed dark energy or quintessence [2–4], should fill roughly 70% of the critical energy density and, along with another 30% in ordinary dark matter (and a minor component of baryons), explains the supernova type Ia (SNIa) observations, is consistent with the cosmic microwave background (CMB) data (see e.g., [5]), and other evidence such as the cluster masses. The fact that the energy densities of the dark energy and the dark matter are comparable at the present time is indeed an enigma, since we have no reason to expect that the dark energy and the dark matter components, which have always given a very different contribution to the total density in the past and will again give a different one in the future, are almost equal right now. In terms of the phase-space view of the cosmological equations, the problem is that the mixture of dark energy and dark matter we observe today is not a global attractor; a different initial condition or, equivalently, a different instant of observation, gives a different sharing of the total density. The problem lies in the fact that the two energy forms scale differently with time because they are assumed to be completely unrelated. To explain the coincidence we propose to couple dark energy to dark matter.

The model we propose in this paper, denoted *stationary dark energy*, is based on a non-linear coupling of dark energy to dark matter. The resulting cosmological solution has the following properties: (a) among flat and homogeneous spaces, the present universe is a global attractor; *all* the possible initial conditions lead to the observed percentages of dark energy and dark matter; once reached, they remain fixed forever; (b) the expansion is accelerated at the present, as requested by the SNIa observations; (c) the model is consistent with the large-scale structure and CMB data; (d) the dark energy and the dark matter densities always scale similarly after equivalence and are close to within one order of magnitude. Based on the literature known to us, no other cosmological model satisfies all four requirements. For instance, the quintessence models of Refs. [4,6–8] are accelerated and consistent with observational data but the present universe is not a global attractor: the observed percentages of energy density will change in the future until the cosmic medium will be dominated by quintessence alone (except in [8], which is an oscillatory model). Notice that in all these models there is a ''tracking'' solution, that can be defined as an attractor in a subspace of the phase space; in contrast, a true attractor as we have in our model is an attractor in the full phase space. The model proposed in [9], based on an exponential potential, satisfies (a) only if is not accelerated. With the inclusion of a linear coupling between dark energy and dark matter, as in Refs. [10–12], it can satisfy (a), (b), and (d) but still not (c) or, alternatively, can be accelerated and consistent with observations but then the present universe is not a global attractor. In Ref. [12] a model that can satisfy all criteria is proposed, but it requires the introduction of two different forms of dark matter, only one of which is coupled to dark energy. In Ref. [13] a dark matter with an effective anti-friction can satisfy (a), (b), and (d) but the effects on structure formation and microwave background have not been tested.

In our model the final state of the universe is an accelerated expansion with a fixed ratio of dark matter and dark energy. We can remark that since the accelerated expansion flattens a curved space, the global attractor in our model attracts also all open universes and all those closed universes which have not already collapsed to a singularity by the time the final stationary state sets in (see [14] for a discussion on attractor solutions in curved spaces). It is to be noticed, however, that although in our model the universe will always reach a final state that may represent our present world, there is no guarantee that this state has already been reached, nor that it had done so late enough to grow sufficient structure formation. This two requirements do limit the range of acceptable initial conditions.

In this work we introduce the concept and derive the main constraints imposed by primordial nucleosynthesis, age of the universe, large scale structure and SNIa results. More precise constraints from CMB observations will be derived in another paper. The dark energy scalar field in our model is defined by two functions: the coupling to dark matter and the potential. The coupling generalizes the linear one introduced in Ref. [15] (see also [16,12,17]), which is in fact the Einstein frame version of a Brans-Dicke theory: now we will adopt a non-linear coupling, as detailed below. To avoid the strong constraints on such a coupling we adopt the species-dependent coupling proposed in Ref. [18], leaving the baryons uncoupled (see also [21,11]). To introduce our model we first recall the main properties of the coupled quintessence [11]: a scalar field with exponential potential and linear coupling. Consider three components, a scalar field ϕ, baryons and cold dark matter (CDM), described by the energy-momentum tensors $T_{\mu\nu(\phi)}$, $T_{\mu\nu(b)}$, and $T_{\mu\nu(m)}$, respectively. General covariance requires the conservation of their sum so that it is possible to consider a coupling such that

$$\begin{aligned} T^{\mu}_{\nu(\phi);\mu} &= (C_m T_{(m)} + C_b T_{(b)})\phi_{;\nu}, \\ T^{\mu}_{\nu(m);\mu} &= -C_m T_{(m)}\phi_{;\nu}, \\ T^{\mu}_{\nu(b);\mu} &= -C_b T_{(b)}\phi_{;\nu}. \end{aligned} \tag{1}$$

A similar coupling is obtained by conformally transforming a non-minimally coupled gravity theory. The radiation field (subscript γ) remains uncoupled, since $T_{(\gamma)}=0$. We derive the background equations in the flat Friedmann-Robertson-Walker (FRW) metric, assuming the exponential potential

$$U = Ae^{s\phi} \tag{2}$$

as proposed, e.g., in [3] and suggested by supergravity and superstring theories (see discussion in [9]). As shown in [15], all Brans-Dicke theories with a potential that can be expressed as a power of the coupling function can be recast by a conformal transformation into a coupled theory with exponential potential. As anticipated, we will couple the dark energy scalar field to the dark matter only, putting $C_b=0$. We call this choice dark-dark coupling. Introducing the variables [22] $x=(\kappa/H)(\dot{\phi}/\sqrt{6})$, $y=(\kappa/H)\sqrt{U/3}$, $z=(\kappa/H)\sqrt{\rho_\gamma/3}$, (where $H=\dot{a}/a$, $G=c=1$, and $\kappa^2=8\pi$), and adopting the e-folding time $\alpha=\log a$, we can write the field and radiation conservation equation as a system in the variables x,y,z that depends on the parameters μ,β,

$$\begin{aligned} x' &= (z'/z-1)x - \mu y^2 + \beta(1-x^2-y^2-z^2), \\ y' &= \mu xy + y(2+z'/z), \\ z' &= -z(1-3x^2+3y^2-z^2)/2, \end{aligned} \tag{3}$$

where the prime denotes derivation with respect to α, and where $\beta=C_m\sqrt{3/(2\kappa^2)}$, $\mu=s\sqrt{3/(2\kappa^2)}$. The baryons are here neglected, since they act on the dynamical system only as a minor perturbation. The dimensionless constant β sets the ratio of the strength of the dark-dark interaction with respect to the gravitational interaction; β is clearly not constrained by local experiments or by $\dot{G}/G$ measurements [18]. Here we restrict the attention to $\beta\geqslant 0$, $\mu\geqslant 0$, without loss of generality (see [11]).

The system (3) has several different global attractors, depending on the values of the parameters β and μ, but only two can be accelerated. One, to be denoted attractor a, exists for $\mu<3$, and is accelerated for $\mu<\sqrt{3}$. On this attractor, the energy density is entirely in the dark energy component, and as such it cannot represent our universe. Of course, our universe could be described by this solution if the attractor has not been reached yet (see e.g., [5]). The other, the attractor b, exists for $\mu+\beta>3/2$ and is accelerated for $\mu<2\beta$. On this attractor, contrary to the case a, the energy density is shared by the dark energy and the dark matter in the following constant proportions:

$$\Omega_\phi = \frac{4\beta^2+4\beta\mu+18}{4(\beta+\mu)^2}, \quad \Omega_m = 1-\Omega_\phi. \tag{4}$$

The universe expands as a power law $a\sim t^p$ with exponent $p=2(1+\beta/\mu)/3$. Choosing the parameters, we can ensure that the final state is, for instance, $\Omega_\phi=0.7$ and $\Omega_m=0.3$. Once reached, these values will remain fixed forever. The problem with the model above is that when the radiation epoch ends, the system rapidly reaches the attractor b and a matter dominated epoch never sets in: the inhomogeneities never grow and the model fails completely to explain the large scale structure. What is lacking is an intermediate phase of matter domination and structure formation. Such a phase would be present if Ω_ϕ were small, say less than 0.1, so that Ω_m dominates. However, as can be seen from Eq. (4), to get a small Ω_ϕ we need a large ratio μ/β, but then this attractor would not be accelerated since $p<1$. In other words, if we want acceleration, we need a large coupling, $\beta\gg\mu$; if we want structure formation, we need on the contrary a small coupling, $\beta\ll\mu$. To have both, we need two couplings.

To realize this we start by observing that the simplest Brans-Dicke Lagrangian coupling $f(\psi)R$ where $f=1-\xi\kappa^2\psi^2$ and R is the Ricci scalar, proposed for instance in Refs. [20], already contains a mechanism that switches from no coupling at small ψ to a finite coupling at large ψ. Denoting with ϕ the conformally transformed field, if there is no potential or the potential is of the chaotic-inflation type, $V\sim\phi^n$ with $n>0$, then the dynamics is such that the coupling decreases with time (general relativity is a global attractor, see [19]). But if the potential is as suggested by quintessence models, i.e. a potential with no minimum that pushes the field to ever growing values, as e.g. an inverse power law or exponential, then the dynamics is reversed, and one realizes a coupling switch of the kind we need. In other words, *a quintessence potential naturally switches on the coupling* as the universe evolves. For instance, Futamase and Maeda [20] have shown that putting $\xi=1/6$ the conformally transformed field ϕ is related to ψ by the relation $\kappa\psi=\sqrt{6}\tanh(\phi/\Delta)$ where $\Delta=1/\sqrt{\kappa^2\xi}$; this gives the non-linear coupling [15]

$$\beta(\phi)=\frac{1}{2}\frac{f'}{\sqrt{2\kappa^2 f/3+f'^2}}\bigg|_{\psi=\psi(\phi)}=\frac{1}{\sqrt{6}}\tanh\left(-\frac{\phi}{\Delta}\right) \quad (5)$$

where $f'=df/d\psi$. (For $\xi\neq 1/6$ one obtains expressions qualitatively similar but much more complicated.)

However, we need a $\beta(\phi)$ function that interpolates from zero to some arbitrarily large value (to be specified later). We generalize therefore the non-linear coupling (5) so that it switches between zero (or a small coupling β_1) and β_2 when ϕ rolls down the potential. Although this generalization is straightforward in our Einstein frame, we found that there is no easy way to derive the corresponding coupling function in the Jordan (Brans-Dicke) frame. For the numerical calculations of this paper we assume

$$\beta(\phi)=\frac{1}{2}\beta_2\left[\tanh\left(\frac{\phi_*-\phi}{\Delta}\right)+1\right]. \quad (6)$$

The precise form of the coupling function $\beta(\phi)$ is not really important; any step-like function that switches on the coupling after structure formation will give a qualitatively similar behavior. We choose the constant β_2 and the slope μ so that

$$1\ll\mu\ll\beta_2. \quad (7)$$

With the coupling (6) we obtain the same equations (3) where now the constant β becomes a function $\beta(y,H)$ and where an extra equation for H is needed, $H'=-(3H/2)(1+x^2-y^2+z^2/3)$. In Fig. 1 we present a numerical integration of the full set of equations; here and in the following we assume $\mu=8$ and $\beta_2=18$. As expected, there exist three distinct phases of constant energy density ratios among the various components. First, the model passes through a radiation dominated epoch with a vanishing contribution of matter and a small contribution of the scalar field, $\Omega_\phi=6/\mu^2$. After equivalence, it falls upon the saddle (4) where matter dominates but there is also a finite contribution from the dark energy ($\Omega_\phi=9/2\mu^2$), this stage is denoted plateau I. So far, the model is similar to the uncoupled exponential model of Ferreira and Joyce [9] generalized in [11] in the presence of a small coupling. Finally, when ϕ rolls below ϕ_*, the coupling becomes stronger and the final global accelerated attractor (4) with $\beta=\beta_2$ is reached. The model enters the present epoch of dark energy domination, with a 30% contribution from the dark matter: this is the plateau II. The condition $\beta_2/\mu\gg 1$ (actually $\beta_2>\mu/2$ is sufficient) ensures that the expansion is accelerated. The universe will inflate forever with a constant ratio of dark energy to dark matter. The value ϕ_* sets the instant at which the coupling changes strength and the universe crosses from the dark matter epoch to the dark energy epoch, while Δ modulates the rapidity of the transition. Notice that for any given trajectory we could as well rescale the field so that $\phi_*=0$. For the model to explain the large scale structure, the only crucial requirement is that the last transition occurs late enough for the inhomogeneities to grow. In the same Fig. 1 we plot also the effective parameter of state $w_{eff}=1+p_{tot}/\rho_{tot}$: values $w_{eff}<2/3$ imply acceleration.

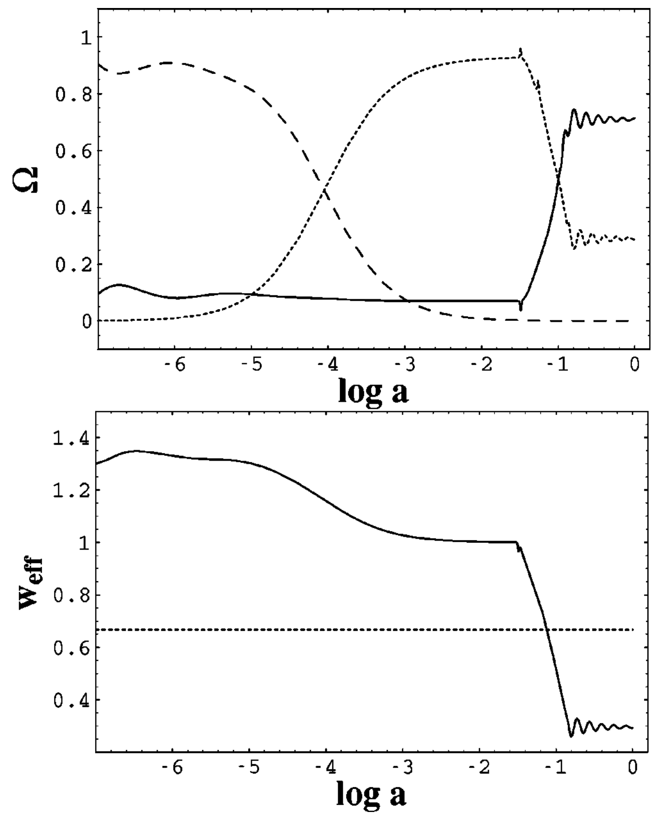

FIG. 1. Top panel. Trends of Ω_γ (dashed line), Ω_m (dotted), and Ω_ϕ (continuous) versus $\log a$. The three regimes mentioned in the text are evident: first, radiation dominates, then matter dominates (plateau I), and then finally the system falls on the final accelerated attractor (plateau II) with 30% of dark matter and 70% of dark energy. The constants have been chosen here as $\mu=8$ and $\beta_2=18$. Bottom panel. The effective parameter of state w_{eff} during the three regimes: first equals 4/3, then goes down to 1, and finally becomes accelerated, $w_{eff}=0.3$.

An obvious objection to our model is that we are trading the coincidence between dark energy and dark matter for a coincidence with the instant when the strong coupling is switched on. However, it is to be remarked that in our case dark energy and dark matter have been similar to within one order of magnitude even after equivalence: Ω_ϕ goes from $9/2\mu^2$ (equal to 0.07 with our choice of μ) in plateau I to 0.7 in plateau II. The present coincidence is therefore no longer particularly striking: in other words, after equivalence, the ratio of dark matter to dark energy is never far from unity, while in all the other models it is so only at one particular instant, and extremely large or small at any other time. In Fig. 2 we contrast the behavior of the ratio ρ_ϕ/ρ_m in our model (for different initial conditions) with that in a inverse power law as in [6]. As can be seen, while in our case the ratio remains relatively close to unity at all times after equivalence, in the inverse power-law case the ratio spans several orders of magnitude, and the coincidence occurs only today. In addition, the fact that dark energy and dark matter

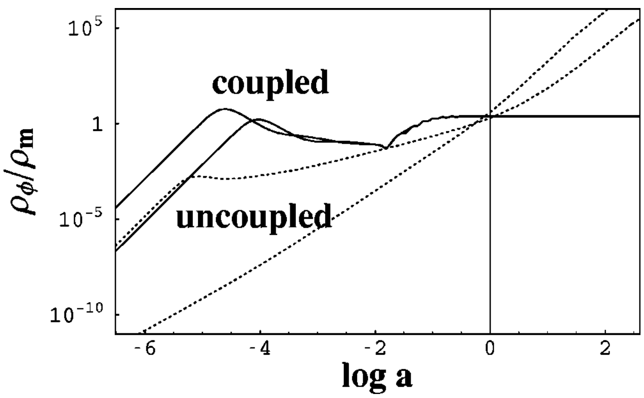

FIG. 2. The plot shows the behavior of ρ_ϕ/ρ_m in our model (continuous lines) for two different initial conditions, and in two inverse power-law models without coupling (dotted line), with potentials ϕ^{-1} and ϕ^{-6}. The vertical line marks the present time. In the coupled model the ratio is close to unity ever after equivalence, while in the power-law models the ratio spans several orders of magnitude and crosses unity only today.

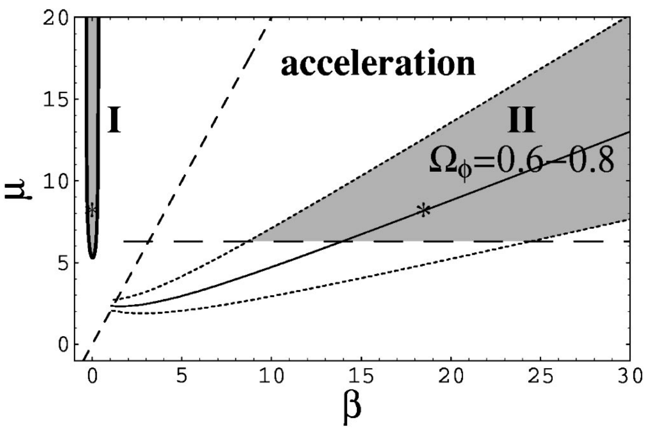

FIG. 3. Parameter space of the model. To the right of the short-dashed line the expansion is accelerated; above the long-dashed line the nucleosynthesis constraint is passed. The parameters within the gray region I on the left produce enough structure formation. Those inside the gray region II on the right yield an accelerated expansion with Ω_ϕ between 0.6 and 0.8 (the continuous line is $\Omega_\phi=0.7$). Any coupling function that switches from the first region to the second after structure formation gives an acceptable model. The two asterisks mark the effective parameters we employed in the numerical calculations.

are allowed to reach a constant proportionality only after equivalence explains also the "triple coincidence" noticed by Arkani-Hamed *et al.* [23] among radiation and the other components: in our model, it is the end of radiation dominance that triggers dark energy and dark matter to equalize.

Let us consider now the main constraints on the model. First of all, we fix $\Omega_\phi=0.7\pm0.1$, and $w_{eff}|_0\in(0,0.6)$ as required by SNIa observations along with the condition of flatness. Then, we impose that the universe age be sufficiently large. Neglecting the radiation epoch and assuming instantaneous transition from plateau I to plateau II at a redshift z_c we obtain

$$T=\frac{2}{3H_0}\left[\frac{1-(1+z_c)^{-(3/2)w_2}}{w_2}+\frac{(1+z_c)^{-(3/2)w_1}}{w_1}\right], \quad (8)$$

where $w_1=1$ and $w_2=\mu/(\mu+\beta_2)$. Because of the inequalities (7) we may approximate, if $z_c\gg1$ (but also $z_c\ll1000$ in order to ensure structure formation; $z_c\simeq5$ would be acceptable) $T=(2/3H_0)(\frac{3}{2}\log z_c+z_c{}^{-3/2})$, which is always larger than the matter dominated age $2/(3H_0)$, so that we pass easily the age test.

We come now to the condition of sufficient structure formation. The growth of perturbations in plateau I has been considered in Ref. [10]. In the limit of vanishing coupling, the dark matter inhomogeneity δ during plateau I grows as a^m with $m=(-1+\sqrt{25-24\Omega_{\phi1}})/4$, $\Omega_{\phi1}=9/2\mu^2$ being the field energy density parameter during plateau I. This reduces to the usual linear growth $m=1$ for $\Omega_{\phi1}=0$, that is in the standard case without scalar field. In order for the perturbations to grow not much less than in the usual matter dominated era, we need to be close to $m=1$. For instance, if $m=0.9$, then the fluctuations grow from $z\simeq1000$ down to $z\simeq1$ by half the standard case. Considering the present uncertainty on the amplitude of fluctuations at the present, we may take this as the lower limit for the fluctuation growth. Then, for m close to unity, we have $\mu>\sqrt{2.7/(1-m)}$, or $\mu>5.3$ (increasing the threshold m to 0.99 requires $\mu>16$). For as concerns the CMB, to be analyzed in detail in another paper, let us only remark that the present observations are mostly sensitive to the total amount of matter, rather than to its equation of state (see e.g., [15]), so that our model has no problem in fitting the present data. Finally, the nucleosynthesis constraint reduces to the request that during the radiation era the contribution of the dark energy is sufficiently low, e.g. less than 15% (otherwise the change in the relativistic degrees of freedom would alter the abundances, see e.g. [9,24]). Since during the radiation era the constant contribution from the scalar field amounts to $\Omega_\phi=6/\mu^2$, nucleosynthesis requires $\mu>6.3$. In Fig. 3 we summarize the constraints derived so far. Every coupling function, or potential, that moves the effective parameters μ,β from region I (the gray region on the left) to region II (the gray region on the right) after structure formation produces an acceptable model. Notice also that, as an extra positive feature, the dimensional parameters C_m and s of Eqs. (1) and (2) are not far from unity in Planck units.

In conclusion, we have shown that it is possible to construct a relatively simple model in which the present universe has *already* reached the global attractor. This offers two advantages over the previous dark energy models with tracking solutions. The first is that the presently observed ratio of dark matter to dark energy density has been close to within one order of magnitude even after equivalence, thereby reducing radically the impact of the cosmic coincidence problem. The second is that *all* the initial conditions will lead sooner or later to this state, while in all the other models only a finite fraction of the phase space lead to our universe (see, e.g., the discussion in [6]). The near coincidence of dark energy and dark matter energy densities no longer depends on the initial conditions but only on the coupling constants, and will be the same at any future epoch. We believe this is a significant step toward the solution of the cosmic coincidence problem. Of course, current observations do delimit the range of acceptable initial conditions; in fact, many initial conditions will give trajectories that fall onto the final attractor either too soon, so that not enough structure forms, or too late, so that we are still short of the attractor. In other words, although the phase-space trajectory that the universe follows from some point onward is unique and independent of the initial conditions, the current position on the trajectory does depend on them, as in all cosmological models.

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