Supersymmetric solutions to topologically massive gravity and black holes in three dimensions

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We obtain a general class of exact solutions to topologically massive gravity with or without a negative cosmological constant. In the first case, we show that the solution is supersymmetric and asymptotically approaches the extremal BTZ black hole solution, while in the latter case it goes to flat space-time.

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The discovery of Bañados-Teitelboim-Zanelli (BTZ) black holes [1] enhanced the interest in (1+2)-dimensional gravity models considerably. However, it is well known that general relativity in 1+2 dimensions has no propagating degrees of freedom and no Newtonian limit (see, e.g., Ref. [2], and the references therein). A physically interesting modification of the (1+2)-dimensional general relativity that cures at least some of these deficiencies is provided by the addition of the gravitational Chern-Simons term to the usual Einstein-Hilbert term in the action. This theory is usually called topologically massive gravity (TMG) [3], whose field equations include the Cotton tensor, which is the analogue of the Weyl tensor in three dimensions, in addition to the usual Einstein tensor. With this addition new degrees of freedom are introduced and one now has a dynamical theory with a massive graviton. The BTZ metric satisfies the TMG field equations in a trivial way as the Cotton tensor vanishes identically. Other known solutions include the ''Gödel-like'' Vuorio solution [4] and its generalization to solutions with a constant twist [5]. Another class of these cosmological-type solutions is given by the finite action exact solutions of the TMG field equations [6] that also provide a classification of homogeneous solutions for Euclidean and Lorentz signatures. Exact static solutions are known to exist for spinning point sources when the spin and the mass of the sources obey a certain relation [7]. Even though they are not asymptotically anti–de Sitter (AdS) and it is not known how to define mass and angular momentum in this case, there also exist solutions with event horizons [8]. There is another class of solutions (that can be obtained from the more general solution we present here by a certain choice of parameters and by making a coordinate transformation) which asymptotically approach extremal BTZ black holes but are geodesically complete and have no event horizons (unlike what we find for our solutions) [9].

In fact a solution to the linearized version of TMG for a stationary rotationally symmetric source was found and it was conjectured that there are no asymptotically flat stationary solutions in the absence of any sources [10]. Here we exhibit a general class of exact supersymmetric solutions that, with an appropriate choice of integration constants, may have event horizons and asymptotically approach the extremal BTZ black hole solution. We do emphasize that our solution is, to our knowledge, the first nontrivial example of a supersymmetric solution to TMG.

We consider the action $I[e,\omega]=\int_M L$ where the Lagrangian 3-form

$$L=\frac{1}{\mu}\left(\omega^a{}_b\wedge d\omega^b{}_a+\frac{2}{3}\omega^a{}_c\wedge\omega^c{}_b\wedge\omega^b{}_a\right)+\frac{1}{2}\mathcal{R}*1-\lambda*1, \tag{1}$$

contains the Einstein-Hilbert term, a negative cosmological constant $\lambda=-1/l^2<0$ and the gravitational Chern-Simons term with the coupling constant μ, written in terms of Levi-Civita connection 1-forms $\omega^a{}_b$. Thus the variation of I with respect to orthonormal coframes e^a yields

$$\frac{1}{\mu}C_a+G_a+\lambda*e_a=0, \tag{2}$$

where the Einstein 2-forms $G_a\equiv G_{ab}*e^b=-\frac{1}{2}R^{bc}*e_{abc}$ and the Cotton 2-forms $C_a\equiv DY_a=dY_a+\omega^b{}_a\wedge Y_b$. We defined $Y_a\equiv(Ric)_a-\frac{1}{4}\mathcal{R}e_a$, in terms of the Ricci 1-forms $(Ric)_b\equiv\iota_a R^a{}_b$, and the curvature scalar $\mathcal{R}\equiv\iota_a(Ric)^a$ where ι_a denotes the interior product operator with respect to a frame vector that acts on the space of forms and creates a $(p-1)$-form out of a p-form. Here $R^a{}_b=d\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b$ are the curvature 2-forms of the Levi-Civita connection 1-forms that satisfy Cartan structure equations $de^a+\omega^a{}_b\wedge e^b=0$. Hodge duality is specified by the oriented volume element $*1=e^0\wedge e^1\wedge e^2$.

The solutions will be given in terms of the local coordinates (t,ρ,ϕ) by the metric tensor

$$g=-e^0\otimes e^0+e^1\otimes e^1+e^2\otimes e^2, \tag{3}$$

where we choose

$$e^0=f(\rho)dt,\quad e^1=d\rho,\quad e^2=h(\rho)[d\phi+a(\rho)dt]. \tag{4}$$

Denoting the derivatives with respect to ρ by a prime, the connection 1-forms are found to be

$$\omega^0{}_1=\alpha e^0-\frac{1}{2}\beta e^2,\quad \omega^0{}_2=-\frac{1}{2}\beta e^1,\quad \omega^1{}_2=-\frac{1}{2}\beta e^0-\gamma e^2, \tag{5}$$

where we set the connection coefficients

$$\alpha\equiv\frac{f'}{f},\quad \beta\equiv\frac{a'h}{f},\quad \gamma\equiv\frac{h'}{h}. \tag{6}$$

The corresponding curvature 2-forms turn out to be

$$R^0{}_1=Ae^1\wedge e^0+Be^2\wedge e^1,\quad R^0{}_2=Ce^2\wedge e^0,$$

$$R^1{}_2 = Be^0 \wedge e^1 + De^2 \wedge e^1, \tag{7}$$

where we defined

$$A \equiv \alpha' + \alpha^2 - \frac{3}{4}\beta^2, \quad B \equiv \frac{1}{2}\beta' + \gamma\beta, \quad C \equiv \alpha\gamma + \frac{1}{4}\beta^2,$$

$$D \equiv \gamma' + \gamma^2 + \frac{1}{4}\beta^2.$$

After some algebra the field equations can be reduced to

$$-D + \frac{1}{l^2} + \frac{1}{\mu}\left[B' + B\gamma + \frac{1}{2}\beta(C-A)\right] = 0, \tag{8}$$

$$-B + \frac{1}{\mu}\left[\frac{1}{2}(D-A-C)' + \alpha(D-C) + \frac{3}{2}\beta B\right] = 0, \tag{9}$$

$$C - \frac{1}{l^2} + \frac{1}{\mu}\left[(\gamma - \alpha)B + \frac{1}{2}\beta(A-D)\right] = 0, \tag{10}$$

$$A - \frac{1}{l^2} + \frac{1}{\mu}\left[B' + \alpha B + \frac{1}{2}\beta(C+D-2A)\right] = 0. \tag{11}$$

We found it remarkable that the following conditions on the connection coefficients

$$\alpha = \frac{k}{2}\beta + \frac{1}{l}, \quad \gamma = -\frac{k}{2}\beta + \frac{1}{l} \tag{12}$$

(with $k^2 = 1$), that follow from the field equations and were essential for finding the general self-dual solutions of the Einstein-Maxwell-Chern-Simons theory in 1+2 dimensions [11], turn out to yield solutions in the present case as well. In fact, the conditions (12) have significance in the following sense: Any solution of topologically massive gravity is said to be supersymmetry preserving provided there exists a nontrivial real 2-spinor ϵ satisfying [12]

$$\left(2\mathcal{D} + \frac{1}{l}\Gamma\right)\epsilon = 0 \tag{13}$$

where $\Gamma = \Gamma_a e^a$ and $\mathcal{D} = d + \frac{1}{2}\omega^{ab}\sigma_{ab}$ with $\sigma_{ab} = \frac{1}{4}[\Gamma_a, \Gamma_b]$. To see that Eq. (13) in fact implies Eq. (12), start by taking ϵ to be $\epsilon = \hat{\epsilon}N(\rho)$ where $\hat{\epsilon}$ is a constant spinor and the form of ϵ is in accord with the fact that our metric functions are functions of the variable ρ only. (Hence ϵ is defined locally.) Then, Eq. (13) yields

$$e^0 N\left[-\alpha\Gamma_2\hat{\epsilon} + \left(\frac{1}{l} - \frac{\beta}{2}\right)\Gamma_0\hat{\epsilon}\right] + e^1\left[2N'\hat{\epsilon} + \left(\frac{1}{l} - \frac{\beta}{2}\right)N\Gamma_1\hat{\epsilon}\right] + e^2 N\left[-\gamma\Gamma_0\hat{\epsilon} + \left(\frac{1}{l} + \frac{\beta}{2}\right)\Gamma_2\hat{\epsilon}\right] = 0. \tag{14}$$

For this to be satisfied for a nontrivial N, one finds that

$$-\alpha\Gamma_2\hat{\epsilon} + \left(\frac{1}{l} - \frac{\beta}{2}\right)\Gamma_0\hat{\epsilon} = 0, \tag{15}$$

$$2N'\hat{\epsilon} + \left(\frac{1}{l} - \frac{\beta}{2}\right)N\Gamma_1\hat{\epsilon} = 0, \tag{16}$$

$$-\gamma\Gamma_0\hat{\epsilon} + \left(\frac{1}{l} + \frac{\beta}{2}\right)\Gamma_2\hat{\epsilon} = 0, \tag{17}$$

have to be satisfied simultaneously. Now a suitable set of real Γ matrices can be chosen as

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2 = \Gamma_0\Gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18}$$

and the constant spinor $\hat{\epsilon}$ can be taken as

$$\hat{\epsilon} = \begin{pmatrix} \epsilon_L \\ \epsilon_R \end{pmatrix}. \tag{19}$$

Substituting these in Eq. (17) one gets

$$\epsilon_L = \gamma\left(\frac{1}{l} + \frac{\beta}{2}\right)^{-1}\epsilon_R = \left(\frac{1}{l} + \frac{\beta}{2}\right)\gamma^{-1}\epsilon_R \tag{20}$$

which gives

$$\gamma = \pm\left(\frac{1}{l} + \frac{\beta}{2}\right) = -\frac{k}{2}\beta \pm \frac{1}{l} \tag{21}$$

($k^2 = 1$). Similarly using Eq. (15), one gets

$$\epsilon_L = \alpha\left(\frac{1}{l} - \frac{\beta}{2}\right)^{-1}\epsilon_R = \left(\frac{1}{l} - \frac{\beta}{2}\right)\alpha^{-1}\epsilon_R \tag{22}$$

and

$$\alpha = \pm\left(\frac{1}{l} - \frac{\beta}{2}\right) = \frac{k}{2}\beta \pm \frac{1}{l}. \tag{23}$$

Hence, choosing the + sign for the term $1/l$ in the expressions (21) and (23), Eq. (12) is obtained. Finally, Eq. (16) implies

$$\epsilon_L = 2\frac{N'}{N}\left(\frac{\beta}{2} - \frac{1}{l}\right)^{-1}\epsilon_R = \frac{N}{2N'}\left(\frac{\beta}{2} - \frac{1}{l}\right)\epsilon_R \tag{24}$$

or

$$2\frac{N'}{N} = \pm\left(\frac{\beta}{2} - \frac{1}{l}\right) \tag{25}$$

which determines the spinor function N in terms of β to be

$$N(\rho) = \exp\left[-\frac{k}{2}\int^{\rho} d\rho\left(\frac{\beta}{2} - \frac{1}{l}\right)\right]. \tag{26}$$

Hence any solution satisfying Eq. (12) will also preserve supersymmetry. It should be noted, however, that with the above assumption only the extremal BTZ solution can be recovered as $|\mu|\to\infty$.

The curvature components are now greatly simplified and they are given by

$$A=U+\frac{1}{l^2},\quad B=kU,\quad C=\frac{1}{l^2},\quad D=\frac{1}{l^2}-U, \tag{27}$$

where

$$U=-\frac{1}{2}\beta^2+\frac{k}{l}\beta+\frac{k}{2}\beta'. \tag{28}$$

Equations (8)–(11) are satisfied simultaneously provided U satisfies

$$U'-k\beta U+\left(\frac{1}{l}+k\mu\right)U=0. \tag{29}$$

By setting $V=k(\beta/U)$ in Eq. (29), we arrive at the linear first order ordinary differential equation

$$V'+\left(\frac{1}{l}-k\mu\right)V=2. \tag{30}$$

This is easily integrated and

$$V=\frac{2}{1/l-k\mu}[1+\beta_0 e^{-(1/l-k\mu)\rho}]$$

for some integration constant β_0. Hence, going back to the definition of U (28) and substituting for V, we obtain a differential equation for β as

$$\beta'+\beta\left(\frac{2}{l}-\frac{(1/l-k\mu)}{1+\beta_0 e^{-(1/l-k\mu)\rho}}\right)-k\beta^2=0. \tag{31}$$

Setting $\omega=1/\beta$, one finds

$$\omega'+\left(\frac{(1/l-k\mu)}{1+\beta_0 e^{-(1/l-k\mu)\rho}}-\frac{2}{l}\right)\omega+k=0. \tag{32}$$

When integrated, this yields

$$\beta=\frac{1}{\omega}=k\frac{2/l+\beta_2(1/l+k\mu)e^{(1/l-k\mu)\rho}}{1+\beta_1 e^{2\rho/l}+\beta_2 e^{(1/l-k\mu)\rho}} \tag{33}$$

for integration constants $\beta_2\equiv 2/[l\beta_0(1/l+k\mu)]$ and β_1. Finally the metric functions are found to be

$$f=f_0 e^{2\rho/l}[1+\beta_1 e^{2\rho/l}+\beta_2 e^{(1/l-k\mu)\rho}]^{-1/2}, \tag{34}$$

$$h=h_0[1+\beta_1 e^{2\rho/l}+\beta_2 e^{(1/l-k\mu)\rho}]^{1/2}, \tag{35}$$

$$a=-a_0+k\frac{f_0}{h_0}e^{2\rho/l}[1+\beta_1 e^{2\rho/l}+\beta_2 e^{(1/l-k\mu)\rho}]^{-1}, \tag{36}$$

where a_0, f_0, and h_0 are some new integration constants.

Depending on the values of the integration constants β_1 and β_2 (of course as well as on l and μ), one might have singularities in these metric functions. It is not difficult to verify that for $1+\beta_2>\beta_1>0$, the metric function g_{tt} changes sign for some $\rho_0\in[0,\infty)$. However, an analysis as the one given in Ref. [1] cannot be given here since, at the very starting point, it is impossible for one to invert the functional relation $r=h(\rho)$ and to rewrite the metric in terms of r. That step is crucial for one to convert the metric into the well studied form of the BTZ (and hence the AdS) metric and make use of the vast literature on that subject. Then we ask what else can be done and for that we go back to the full solution and analyze the quasilocal mass and the angular momentum. We refer the reader to Ref. [11] for a discussion of how these quantities can be found in this AdS background. For the quasilocal angular momentum, we have

$$j(r)=k{h^2}_0\varphi(r), \tag{37}$$

where $\varphi(\rho)\equiv 2/l+\beta_2(1/l+k\mu)e^{(1/l-k\mu)\rho}$ and again one finds that one has to invert $r=h(\rho)$ so that φ can be written as a function of r. Similarly the quasilocal energy turns out to be

$$E(r)=\frac{{h^2}_0}{2r}\varphi(r)=\frac{k}{2r}j(r), \tag{38}$$

whereas the quasilocal mass is

$$m(r)=a_0 j(r)=ka_0{h^2}_0\varphi(r). \tag{39}$$

The total angular momentum J and the total mass M are defined by the limits $J\equiv j(r)|_{r\to\infty}$ and $M\equiv m(r)|_{r\to\infty}$, respectively. To see what can be said about J and M, we first start by examining $a(r)$. Depending on the values of l and μ, a either goes to $-a_0$ or $-a_0+kf_0/(h_0\beta_1)$ as $r\to\infty$. Hence for a to vanish asymptotically as $r\to\infty$, a_0 should be chosen either as 0 or as $kf_0/(h_0\beta_1)$. When $1/l>k\mu$, $a_0=0$ and hence $M=0$ whereas $J\to\infty$. For $1/l<k\mu$, J is finite and $J=2k{h^2}_0/l$. Then $a_0=kf_0/(h_0\beta_1)$ and $M=a_0J$ is finite as well. As $k\mu\to\infty$, this solution approaches the extremal BTZ solution. To see this, use the freedom to choose the radial coordinate and replace ρ by $r=h(\rho)$. So now $e^1=g(r)dr$ for some function g and

$$g\equiv\frac{d\rho}{dr}=\frac{lr}{(r^2-{h^2}_0)},\quad f=\frac{f_0h_0}{\beta_1}\left(\frac{r}{{h^2}_0}-\frac{1}{r}\right),$$

$$a=-a_0+\frac{kf_0}{h_0\beta_1}\left(1-\frac{{h^2}_0}{r^2}\right).$$

Comparing with the BTZ solution [1,2], it is easily seen that choosing $J=2{h^2}_0/l$, $M=J/l$, $k=1$, $a_0=1/l$ and $(f_0/\beta_1)=h_0/l$, one gets the extremal BTZ solution. It is well known that the BTZ solution is quite similar to the Kerr solution in 3+1 dimensions [1]. Since both our solution and the extremal BTZ solution are supersymmetric [12] (just like the extremal Kerr solution in 3+1 dimensions), it is perhaps not surprising that one gets the extremal BTZ solution in the limit.

In the absence of a cosmological constant, the solution has to be reanalyzed since simply setting $1/l=0$ in the above expressions does not give the desired limiting solution. In this case, V is now

$$V=\frac{-2}{k\mu}(1+\beta_0 e^{k\mu\rho}), \tag{40}$$

whereas the equation for ω becomes

$$\omega' - \frac{k\mu}{1+\beta_0 e^{k\mu\rho}}\omega + k = 0. \tag{41}$$

Integrating this, one finds

$$\beta = \frac{1}{\omega} = \frac{\mu(1+\beta_0 e^{k\mu\rho})}{1-\mu\beta_0(\omega_0+k\rho)e^{k\mu\rho}} \tag{42}$$

for some integration constant ω_0. The new metric functions are finally found to be

$$f=f_0[e^{-k\mu\rho}-\mu\beta_0(\omega_0+k\rho)]^{-1/2}, \tag{43}$$

$$h=h_0[e^{-k\mu\rho}-\mu\beta_0(\omega_0+k\rho)]^{1/2}, \tag{44}$$

$$a=-a_0+k\frac{f_0}{h_0}[e^{-k\mu\rho}-\mu\beta_0(\omega_0+k\rho)]^{-1}, \tag{45}$$

where f_0, h_0, and a_0 again denote some integration constants. For this case, all the nontrivial components of the curvature 2-forms $R^a{}_b$ are proportional to U, as can be easily seen by examining Eqs. (7) and (27). By using Eqs. (40) and (42) in the definition of $U=k(\beta/V)$, we find that

$$U=-\frac{\mu^2}{2}\frac{1}{1-\mu\beta_0(\omega_0+k\rho)e^{k\mu\rho}}. \tag{46}$$

If $k\mu>0$, then as $\rho\to\infty$, $U\to 0$ and hence $R^a{}_b\to 0$, which implies that this solution asymptotically approaches flat space. However, we checked that it is not accessible to linearization about flat space in the sense of Ref. [4].

In summary, we have obtained a solution to the topologically massive gravity model with a negative cosmological constant. We have shown that it asymptotically approaches the extremal BTZ solution, and depending on the integration constants, has event horizons. Moreover it does go to flat space as one sets the cosmological constant to zero.

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[1] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
[2] S. Carlip, *Quantum Gravity in 2+1 Dimensions* (Cambridge University Press, Cambridge, England, 1998).
[3] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982); **185**, 406(E) (1988).
[4] I. Vuorio, Phys. Lett. **163B**, 91 (1985).
[5] R. Percacci, P. Sodano, and I. Vuorio, Ann. Phys. (N.Y.) **176**, 344 (1987).
[6] Y. Nutku and P. Baekler, Ann. Phys. (N.Y.) **195**, 16 (1989).
[7] M. E. Ortiz, Class. Quantum Grav. **7**, L9 (1990).
[8] Y. Nutku, Class. Quantum Grav. **10**, 2657 (1993).
[9] G. Clément, Class. Quantum Grav. **11**, L115 (1994).
[10] S. Deser, Phys. Rev. Lett. **64**, 611 (1990).
[11] T. Dereli and Yu. N. Obukhov, Phys. Rev. D **62**, 024013 (2000).
[12] T. Dereli and S. Deser, J. Phys. A **11**, L27 (1978); S. Deser and J. H. Kay, Phys. Lett. **120B**, 97 (1983); S. Deser, in *Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S. deWitt,* edited by S. M. Christensen (Adam Hilger, London, 1984), pp. 374–381.