Dark matter phase space densities

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The low velocity part of a kinetic equilibrium dark matter distribution has a higher phase space density and is more easily incorporated in the formation of a low mass galaxy than the high velocity part. For relativistically decoupling fermions (bosons), this explains one (two) orders of magnitude of the observed trend, that phase space densities in dark matter halo cores are highest in the smallest systems, and loosens constraints on particle masses significantly. For nonrelativistic decoupling and/or finite chemical potentials even larger effects may occur. It is therefore premature to dismiss dissipationless particle distributions as dark matter on the basis of phase space arguments.

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It has recently become clear that the otherwise successful cold dark matter (CDM) model for cosmic structure formation has several severe problems, such as predictions of cusps in the central density profiles of galaxies, too many low-mass subclumps within dark matter halos, and a lack of angular momentum in galaxy disks compared to observations. In one way or the other these problems are all related to the fact that CDM has no initial velocity spread, and therefore has an infinite initial density in phase space. Self-interactions among the dark matter particles have been suggested as a possible solution [1]. Another suggested solution has been the reintroduction of warm dark matter (WDM) consisting of particles with a moderate primordial velocity spread [2]. Contrary to CDM, thermal WDM particles lead to finite core density but apparently require a high mass and therefore extremely early decoupling from the primordial plasma to account for the observed core phase space density of $10^{-4}M_{\odot}/\mathrm{pc}^3(\mathrm{km/s})^{-3}$ in dwarf spheroidal galaxies. There may also be difficulties explaining the decrease in phase space density by a factor 10–100 for dwarf spirals and low surface brightness galaxies, and a further factor of 10–100 for normal spirals [3,4]. A single particle mass for WDM would appear to lead to a definite prediction for the central halo phase space density because of conservation of fine-grained phase space density (Liouville's theorem).

However, whereas such arguments are correct in terms of the average phase space density, they do not take into account that, while originally almost uniform in real space, the fine-grained density is an (exponentially) varying function of position in momentum space. Low momentum particles are in denser parts of phase space than high momentum particles, and depending on the actual distribution function (fermion or boson, zero or nonzero chemical potential, relativistic or nonrelativistic decoupling), the densest part of the distribution may have a phase space density significantly above average. Furthermore, halo formation should typically include particles from the low momentum end first. A system with low gravitational potential like a dwarf spheroidal will effectively probe only the densest part of the phase space distribution, whereas a large spiral galaxy probably contains something close to the average phase density (possibly diluted by mergers, etc.).

Other effects (baryons, mergers, phase space dilution during gravitational collapse, particle self-interactions, etc.) can further enhance the diversity in observed phase space densities in galaxy cores, and as demonstrated in [4] a merging hierarchy where larger systems are gradually formed by merging of smaller units can explain many of the features observed. But as demonstrated in the following, even in the absence of mergers, a significant spread in primordial phase space densities in dark matter cores is predicted, with core densities decreasing with increasing escape velocity of the system in question. The magnitude of the effect ranges from one order of magnitude for relativistically decoupling, nondegenerate fermions, to several orders of magnitude for nonrelativistically decoupling bosons. *Ad hoc* additions to these simplest distribution functions would allow a further range. Thus, there may still be room for a single dissipationless elementary particle explanation of dark matter in dwarf spheroidal as well as dwarf spiral galaxies [5]. Furthermore, the natural selection of the highest phase density particles in the smallest systems leads to a significant reduction in the minimum mass for the particle responsible, loosening the rather strong requirements on the epoch of dark matter decoupling in the early universe.

An isotropic gas of particles in kinetic equilibrium has a spatial number density

$$n=(g/h^3)\int 4\pi f(p)p^2dp, \tag{1}$$

where h is Planck's constant, g is the number of helicity states, and

$$f(p)=\{\exp[(E-\mu)/kT]\pm 1\}^{-1} \tag{2}$$

is a Fermi-Dirac $(+)$ or Bose-Einstein $(-)$ distribution with chemical potential μ. Energy E is related to momentum p and particle mass m via $E^2=(pc)^2+m^2c^4$, where c is the speed of light.

A particle species that decouples from the remaining plasma in the early universe at temperature T_D redshifts its momenta in proportion to the expansion of the Universe, $p=p_DR_D/R$, where $R(t)$ is the cosmic scale factor, and its number density (particle number is conserved) evolves like

$n \propto R^{-3}$. From this it follows that the distribution function f at a time after decoupling is related to the distribution at decoupling f_D [from Eq. (2)],

$$f(p) = f_D(pR/R_D). \tag{3}$$

$f(p)$ even keeps an equilibrium shape after decoupling in two regimes [6], namely, ultrarelativistic decoupling ($E \approx pc$, $T = T_D R_D/R$, $\mu = \mu_D R_D/R$), with

$$f_R(p) = \{\exp[(pc-\mu)/kT] \pm 1\}^{-1}, \tag{4}$$

and nonrelativistic decoupling [$E-\mu \approx p^2/2m - \mu_{\rm kin}$, $\mu_{\rm kin} \equiv \mu - mc^2 = \mu_{\rm kin,D}(R_D/R)^2$, $T = T_D(R_D/R)^2$], so

$$f_N(p) = \{\exp[(p^2/(2m) - \mu_{\rm kin})/kT] \pm 1\}^{-1}. \tag{5}$$

The distribution of fine-grained phase space density is conserved in time (Liouville's theorem). This fact is expressed by Eq. (3), and it means that the phase space distribution at decoupling can be directly related to measurements of dark matter phase densities today. Applied to conservation of the maximum phase space density this was the basis for the Tremaine-Gunn limit on dark matter fermion masses [7], later generalized to bosons (where no maximum exists) by means of the average phase space density [8].

Recently Hogan and Dalcanton [9] reconsidered the issue of the primordial average phase space density compared with observations of dark matter phase space densities in halo cores. From the distribution function they calculated the mass density $\rho = mn$, and pressure

$$P = (g/h^3)\int p^2/(3E) f d^3p \approx (g/h^3)\int p^4/(3mc^2)4\pi dp$$
$$= mn\langle v^2\rangle/(3c^2),$$

performing the calculation in the nonrelativistic regime, where $E \approx mc^2$ and $p = mv$. This leads to the following expression for the phase space density Q defined by

$$Q \equiv \frac{\rho}{\langle v^2\rangle^{3/2}} = \frac{m^4 g}{h^3} 4\pi \frac{\left[\int f(p)p^2 dp\right]^{5/2}}{\left[\int f(p)p^4 dp\right]^{3/2}}. \tag{6}$$

Hogan and Dalcanton [9] use this expression in the form (units with $\hbar = c = 1$)

$$Q_X = q_X g_X m_X^4 \tag{7}$$

for particle type X, where $q_X = 0.0019625$ for a relativistically decoupling, $\mu = 0$ fermion, and $q_X = 0.036335$ for a relativistically decoupling, $T = 0$, $\mu \gg m$ degenerate fermion, where the q values come from taking the complete integrals over the distribution function.

For the remainder of this investigation we return to dimensional units and keep factors of $\hbar$ and c. In these units

$$q_X \equiv Q_X h^3 m^{-4} g^{-1} = 4\pi \frac{\left[\int f(p)p^2 dp\right]^{5/2}}{\left[\int f(p)p^4 dp\right]^{3/2}}. \tag{8}$$

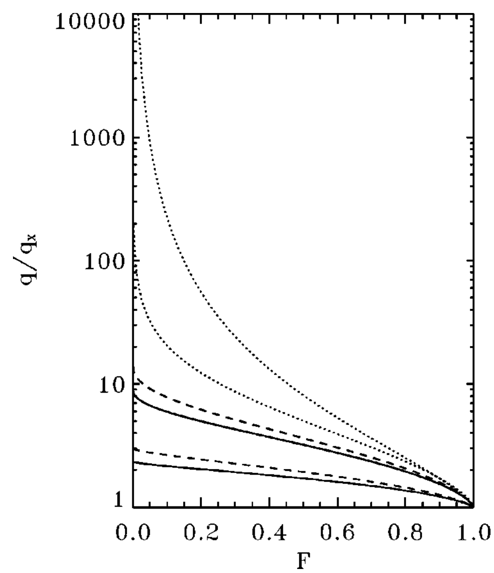

FIG. 1. Phase space density of dark matter particles in units of the mean density as a function of fraction of particles, F. Upper (lower) solid curves are fermions with $(m-\mu_D)/T_D = 0$ and $m/T_D \to 0$ (∞). Lower (upper) dotted curves are for bosons in the same limits. Dashed curves are for fermions and bosons alike in the limits $(m-\mu_D)/T_D \to \infty$ for $m/T_D \to 0$ (upper) and $m/T_D \to \infty$ (lower). Fully degenerate fermions ($(m-\mu_D)/T_D \to -\infty$) have no amplification factor, i.e., $q/q_X \equiv 1$.

For a degenerate fermion with relativistic decoupling [limit $(m-\mu)/kT_D \to -\infty$] this gives $q_{RFdeg} = 4\pi 5^{3/2}/3^{5/2} = 9.0128415$, so $\overline{f} = q/9.0128415$ is, in general, a measure of the average occupation number in a phase space distribution (in units of g/h^3). For a zero chemical potential fermion the corresponding average q values for relativistic and nonrelativistic decoupling (in the latter case a zero chemical potential means $\mu_{\rm kin} = 0$) are $q_{RF0} = 0.4868039$ and $q_{NF0} = 1.9223$, whereas the similar numbers for bosons are $q_{RB0} = 0.9071055$ and $q_{NB0} = 21.521$. Notice the larger values for bosons that express the fact that bosons have a higher fraction of low momentum, high phase space density particles than fermions.

So far only average characteristics of the distributions have been discussed, but it turns out to be quite interesting to study the whole distribution of phase densities. Figure 1 shows these distributions in a plot of $q(p)/q_X$ as a function of the fraction of particles with momentum less than p (calculated at decoupling, but the distribution is conserved in time by Liouvilles theorem). Here q_X is given in Eq. (8) integrating from 0 to ∞, whereas $q(p)$ is defined by the same equation, but integrating only from 0 to p [10]. The ratio therefore illustrates the amplification of phase space density relative to the average for a given dark matter distribution function if only the densest parts of phase space are utilized, for instance, in formation of a galaxy halo.

Notice that relativistically decoupling fermions show an order of magnitude amplification, whereas two orders of magnitude can be gained for relativistically decoupling bosons and several orders of magnitude for nonrelativistically decoupling bosons.

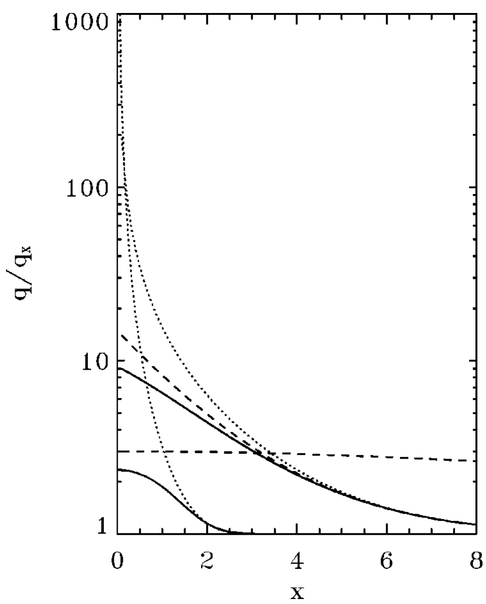

FIG. 2. As in Fig. 1, but as a function of the dimensionless momentum x.

Another way of illustrating the amplification effect is shown in Fig. 2, where amplification is plotted as a function of the dimensionless momentum $x \equiv p_D c/kT_D$, which is the natural integration variable in the calculations. For particles that are nonrelativistic at the epoch of galaxy formation R_g x is related to particle speed v at that time by

$$x = \frac{v}{c}\frac{mc^2}{kT_D}\frac{R_g}{R_D}. \tag{9}$$

The occurrence of a phase space amplification factor at low momenta is a natural consequence of Eq. (2). The fine-grained occupation number (and therefore also the coarse-grained phase space occupation) has a maximum at $p=0$ equal to $f_{\rm max} = \{\exp[(mc^2-\mu)/kT] \pm 1\}^{-1}$, which is 1 for degenerate fermions $((mc^2-\mu_D)/kT_D \to -\infty)$, 0.5 for fermions decoupling when $(mc^2-\mu_D)/kT_D = 0$, diverges for bosons in the same limit, and equals $\exp[(\mu_D - mc^2)/kT_D]$ for fermions and bosons in the limit $(mc^2-\mu_D)/kT_D \to \infty$. Quantitatively, the amplification factor for a relativistically decoupling fermion behaves like $9.26(1-5x/16)$ to first order in x (Fig. 2), or $9.26(1-0.691F_F^{1/3})$ expressed in terms of the fraction of fermions, F_F (Fig. 1). The similar limits for bosons are $19.59x^{-1}(1-7x/30)$, or $8.93F_B^{-1/2} - 4.57$ (notice that these factors diverge for small x or F_B).

Using entropy conservation in the cosmic expansion, R_g and R_D entering the equation for x are related by $g_{*g}T_{\gamma g}^3 R_g^3 = g_{*D} T_{\gamma D}^3 R_D^3$, where g_* counts the total number of effective particle degrees of freedom at the given epoch. Today and at galaxy formation, $g_{*g} = 43/11$. Introducing the redshift of galaxy formation z_g via $T_{\gamma g} = (1+z_g)T_{\gamma 0}$, where the present photon temperature is $T_{\gamma 0} = 2.726$ K, x can be expressed as

$$x = 0.0223\left(\frac{mc^2}{1\ \mathrm{eV}}\right)\left(\frac{10}{1+z_g}\right)\left(\frac{v}{10\ \mathrm{km/s}}\right) g_{*D}^{-1/3}. \tag{10}$$

The value of g_{*D} depends on the epoch of decoupling. For standard neutrino decoupling at $kT_D \approx 1$ MeV, $g_{*D} = 43/4$. But much higher values of g_{*D} are possible for earlier decoupling, with $g_{*D} \approx 50$ above the quark-hadron phase transition temperature, $kT \approx 100$ MeV, $g_{*D} \approx 100$ above $kT \approx 200$ GeV, and even higher values possible earlier on.

The value of g_{*D} not only determines the x-v relation, but also crucially impacts on the total mass density contribution of the particle in question and thereby determines its potential as a dark matter candidate. For the case of relativistic decoupling $((m-\mu_D)/T_D = 0,\ m/T_D \to 0)$ fermions contribute to the cosmic density,

$$\Omega_X h^2 = 0.0572\left(\frac{mc^2}{1\ \mathrm{eV}}\right) g/g_{*D}, \tag{11}$$

where h is now the Hubble parameter in units of 100 km s^{-1} Mpc^{-1}, and a similar expression (multiplied by 4/3) applies for bosons.

It is interesting to note that an explanation of the highest phase space densities measured, those in dwarf spheroidals of order $10^{-4} M_\odot \mathrm{pc}^{-3} (\mathrm{km/s})^{-3}$, for a $g=1$ boson requires a particle mass of only 224 eV (309 eV) if the densest 1% (10%) of the bosons are used, increasing to 681 eV for the average occupation number (masses are reduced by a factor $2^{-1/4}$ if $g=2$). Such masses are not in conflict with estimates of cosmic density for reasonably high values of g_{*D}, and x can indeed be low enough for reasonable values of z_g to select only the densest part of phase space for dwarf spheroidals with typical velocity dispersions below 10 km/s.

For fermions with $g=2$, the high-phase space density selection expected for formation of dwarf spheroidals reduces warm dark matter particle mass limits from 669 eV to 383 eV, again loosening constraints on g_{*D} [11].

Even stronger effects may occur in a nonrelativistic decoupling regime (upper dotted curve in Fig. 1), or one might consider adding extra features to the dark matter distribution functions such as a small amount of Bose-Einstein condensation in the zero momentum, the infinite phase density part of a boson distribution.

Such ‘‘fine-tuning’’ may further loosen constraints on dissipationless dark matter, but even without it, part of the observed trend for the core phase space density to decrease with increasing gravitational potential when going from dwarf spheroidal to dwarf spiral galaxies is naturally explained by the selection of low-momentum dark matter particles described here [5]. At the same time constraints on WDM particle masses and decoupling epochs are significantly reduced. These considerations should be taken into account and tested in detailed numerical simulations of dark matter halo formation, which are needed to settle the question of whether dissipationless particles may after all account for the dark matter in galaxies.

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[11] Too small a particle mass may lead to a free-streaming cutoff scale in the power spectrum that is too big, but this will depend on details of the scenario, such as $f_D(p)$. For example, bosons keep somewhat more small-scale power than fermions; cf. J. Madsen, Astrophys. J. Lett. **371**, L47 (1991). Alternatively, future observations may find objects with even higher core phase space densities Q than the present record holders, in which case the mass limits (increasing like $Q^{1/4}$) will be easier to reconcile with the preferred cutoff scale without breaking concordance with $\Omega_X h^2$.