Global conformal anomaly in $\mathcal{N}=2$ string

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We show the existence of a global anomaly in the one-loop graphs of $\mathcal{N}=2$ string theory, defined by sewing tree amplitudes, unless spacetime supersymmetry is imposed. The anomaly is responsible for the nonvanishing maximally helicity violating amplitudes. The supersymmetric completion of the $\mathcal{N}=2$ string spectrum is formulated by extending the previous cohomological analysis with an external spin factor; the target space-time spin-statistics of these individual fields in a self-dual background are compatible with previous cohomological analysis as fields of arbitrary spin may be bosonized into one another. We further analyze duality relations between the open and closed string amplitudes and demonstrate this in the supersymmetric extension of the target space-time theory through the insertion of zero-momentum operators.

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I. INTRODUCTION

The $\mathcal{N}=2$ string is unique among string theories, as its critical dimension is four and it contains only massless states in its spectrum. It has two local supersymmetries on the world sheet, and its local world-sheet action consists of the $\mathcal{N}=2$ gravitational multiplet coupled to two complex chiral supermultiplets, and was originally proposed and examined in [1]. This string theory has topological space-time properties intrinsic to the world sheet, linked to a vector in the gravitational multiplet.

A. Classical $\mathcal{N}=2$ strings

Ooguri and Vafa [2] (see also Marcus [3]), in the correct critical dimension of four (space-time $4+0$ or $2+2$) found the classical theory to describe in the target space-time classical self-dual gravity in the closed $\mathcal{N}=2$ string and classical self-dual Yang-Mills (YM) theory in the open formulation. The target space-time light-cone action $S=\mathrm{Tr}\int d^4x\mathcal{L}$ for the closed string is given by

$$\mathcal{L}=\phi(\Box\phi+g\partial^{\beta}{}_{\dot{+}}\partial^{\alpha}{}_{\dot{+}}\phi\partial_{\beta\dot{+}}\partial_{\alpha\dot{+}}\phi), \tag{1.1}$$

and the action they proposed, related by gauge fixing the self-duality equations differently to contain the mixed derivatives $\partial_{\alpha\dot{+}}\partial_{\beta\dot{-}}$, produce vanishing tree-level amplitudes in accord with those of the $\mathcal{N}=2$ string; however, they lack Lorentz invariance and require a dimensionful coupling constant. The quantization is problematic for these reasons. Different self-dual gravity and self-dual YM actions have been proposed and quantized which agree with this previous analysis at the tree level: Covariant versions for gravity are $\mathcal{L}=e^{\alpha\dot{\gamma}}\wedge e^{\beta}{}_{\dot{\gamma}}\wedge d\omega_{\alpha\beta}$ in terms of the vierbein and selfdual Lagrange multiplier one-forms $e^{\alpha\dot{\alpha}}$ and $\omega_{\alpha\beta}$ [4], and a higher-derivative version $\mathcal{L}=\rho^{\alpha\beta}\wedge R_{\alpha\beta}$ in terms of the self-dual curvature and Lagrange multiplier two-forms $R_{\alpha\beta}(e)$ and $\rho^{\alpha\beta}$ [5,6]. Either has the light-cone gauge-fixed form containing two fields:

$$\mathcal{L}=\tilde{\phi}(\Box\phi+g\partial^{\beta}{}_{\dot{+}}\partial^{\alpha}{}_{\dot{+}}\phi\partial_{\beta\dot{+}}\partial_{\alpha\dot{+}}\phi). \tag{1.2}$$

Siegel [7,4] demonstrated how both Lorentz invariance and dimensional analysis could be restored in the target space-time description by incorporating space-time supersymmetry, although the bosonic truncation of this proposal is also Lorentz invariant and possesses the dimensional analysis to be conformal. This action was quantized in [8], and the *S*-matrix shown to agree with one-loop maximal helicity violating (MHV) amplitudes in gauge theory and gravity at one loop. The fact that vertices in self-dual field theories are independent of helicity allows spin to be introduced as an internal symmetry. This is implemented through superselection sectors analogous to Chan-Paton factors, as in [9]. The helicity independence in the target space-time is due to spectral flow in the $\mathcal{N}=2$ string description; additional states may be incorporated in the $\mathcal{N}=2$ string that carry fermionic statistics. The latter are distinguished from the naive cohomology analysis by only a line factor. In fact, a simple doubling of fields to helicities of both signs is sufficient to restore Lorentz invariance and dimensional analysis, but maximal supersymmetry puts all fields of both signs in the same multiplet.

B. Loops

These target space-time actions force all diagrams at more than one loop to vanish [8], in agreement with the $\mathcal{N}=2$ string theory higher-genus scattering amplitudes: The negative-helicity field appears linearly in the two-field self-dual action (1.2), and thus it counts loops. It appears as an external field once in each connected tree graph, never in one-loop graphs, and no higher-loop diagrams exist [8]. Furthermore, in any supersymmetric extension the one-loop

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field theory graphs also vanish, because helicity independence of the couplings implies exact cancelation between fermions and bosons in the loop.

Turning to $\mathcal{N}=2$ string calculations, cancelations of scattering amplitudes at all genera have been shown through an $\mathcal{N}=4$ reformulation [10] as well as through Ward identities [11], modulo subtleties associated with contact term ambiguities, *if* modular invariance is assumed.[1] The three-point function does not vanish due to kinematics in $d=2+2$ and is infrared divergent [14]. (The ambiguity in defining the three-point function due to regularization [6] necessitates higher-point genus calculations to identify the quantum target space-time theory.) Furthermore, the genus-one four-point function in the closed $\mathcal{N}=2$ string has been examined in detail in [6] together with a mapping at *n*-point to this genus order and an implementation of line factors for covariance of the scattering; agreement with the vanishing theorems of [10] is found, again assuming modular invariance. This cancellation is most naturally explained in terms of a four-dimensional supersymmetric self-dual target space-time theory. At genus one the vanishing of the $\mathcal{N}=2$ string amplitudes was demonstrated directly by a calculation of the amplitudes in the RNS $\mathcal{N}=2$ formulation [6] (including only a single massless scalar degree of freedom in the spectrum), which demonstrated at the integrand level that the reason was an additional factor of τ_2 associated with the ghost system of the world-sheet gauge field [3,6].[2] The quantum four-point amplitude has been examined at the level of ordering of limits (α' small and spin structure summation) as well as contact term interactions on the world sheet in [6] and interpretations of the quantum target theory in the $\mathcal{N}=2$ string are given, the one-loop amplitudes being identical to the dimensionally reduced MHV amplitudes.[3] In this work we reexamine the quantum scattering and its consistency in different orders of perturbation theory via the inclusion of a supersymmetric multiplet of states in the massless spectrum through the construction of the superselection sectors. (A previous attempt towards a supersymmetric extension based on a Z_2 twisting is presented in [16].)

On the other hand, loop amplitudes can be calculated directly, without any assumption of modular invariance. The genus one amplitudes are conveniently evaluated (since the days of lightcone path integrals, and even earlier with operator methods) by sewing, and this procedure is equivalent to the field theory one. The result is known from field theory methods, and will be re-derived here by string theory methods: The bosonic $\mathcal{N}=2$ string has *nonvanishing* one-loop amplitudes through this approach, while the super $\mathcal{N}=2$ string has vanishing ones. Since modular invariance requires vanishing amplitudes at all loops (for $n\geqslant 4$ *n*-point amplitudes), this implies that the bosonic $\mathcal{N}=2$ string has a conformal anomaly. In other words, supersymmetry is required to cancel the anomaly, just as SO(32) is required to cancel anomalies in the open $\mathcal{N}=1$ superstring. [Also, a trivial gauge group choice of $SO(2)$ cancels the interactions in the target space-time theory in the open string case and thus also the anomaly.] Thus, in this case world-sheet conformal invariance requires space-time supersymmetry. Hidden supersymmetry has previously been analyzed in [17] in the context of two-dimensional models. Global anomalies were originally considered in [18].

C. Self-duality in field theory

Several exact sequences of one-loop gauge theory amplitudes, for example the MHV ones [19] (constructed recursively through analyticity requirements and then derived for an internal quark in [20]) and the gravity analog [21] have allowed for explicit comparisons between the $\mathcal{N}=2$ string theory quantum amplitudes and those in field theory. Furthermore, self-duality poses an interesting structure and reformulation of gauge theory as a perturbative construction around the self-dual point [22]; this translates, as opposed to an expansion in loops, to an expansion in helicity around the maximal helicity configuration. (Both can be formulated as coupling constant expansions.) Amplitudes in non-self-dual Yang-Mills theory and gravity simplify as the number of helicities with the same sign increases. Supersymmetry identities at the tree level enforce the vanishing of the non-supersymmetric MHV amplitudes to this order, and the next simplest tree amplitude, next-to-MHV, was conjectured and proven in [23]. Supersymmetry also forces the one-loop amplitudes with differing internal virtual states to be the same up to a sign.

Self-duality of the field equations implies vanishing of the tree-level amplitudes through a construction of conserved currents in the case of gravity [24], and through a direct map of the classical scattering to vanishing amplitudes in gauge theory [8]: Bardeen demonstrated a relationship of off-shell gauge fields (at the tree level) between self-dual and non-self-dual theories [25]. The amplitudes at the tree level in self-dual theories all vanish. To one loop the MHV *S*-matrices are found to describe the quantum scattering of self-dual field theories [8,26]. (The relation of these different self-dual actions at the quantum level is analyzed in [8], and the Lorentz covariant versions may be found through a truncation of non-self-dual gauge theory to the self-dual limit.)

D. Outline

In Sec. II we analyze the spectrum of this string, after including internal degrees of freedom for the single massless state, as the supersymmetric gauged extension of self-dual gravity, and analyze its corresponding Becchi-Rouet-Stora-Tyutin (BRST) cohomology. We show that there is a supersymmetric completion of the $\mathcal{N}=2$ string through the addi-

[1]The Weyl-Petersson integration measure $d^2\tau/\tau_2^2$ on the torus is analyzed in many works, for example in Sec. IV A of [12]. The integration measure for the critical $\mathcal{N}=2$ string has been examined in [13].

[2]The path integral quantization at arbitrary genus is analyzed in [15].

[3]An explicit relation between the $\mathcal{N}=2$ string amplitude at genus one and the ultra-violet portion of the type IIB supergravity non-MHV amplitude in $d=10$ was also found through a dimension shifting relation.

tion of the factors labeling spin of external states. In Sec. III we analyze duality in the context of $\mathcal{N}=2$ string scattering. In Sec. IV we review earlier results for genus-one amplitudes. In Sec. V we perform sewing and unitarity constructions of the genus-one amplitude and compare with known results, obtaining the conformal anomaly. The last section contains concluding remarks.

II. SPECTRUM

The minimal characterization of the spectrum contains a single massless degree of freedom. However, as is the case in open string theories, superselection sectors and the line factors of the external legs may be incorporated which represent the internal symmetries of the string. In the $\mathcal{N}=2$ string a graded internal Lorentz symmetry may be introduced with representatives labeling the spin states of fields with self-dual couplings. In a background field formalism, at quadratic order in the quantum fields, for only maximum-helicity background fields and lower-helicity quantum fields, the action has the form of a non-self-dual action in a self-dual background. It is known that half-integral spin fields may be "bosonized" in the target space-time around such a self-dual field configuration. In this section we examine the target space-time supersymmetrization of the $\mathcal{N}=2$ string.

A. Bosonization and second-order formulation

In this subsection we summarize briefly the spin statistics of particles in a self-dual background. The action $S=\mathrm{Tr}\int d^4x\mathcal{L}$ for a minimally coupled fermion ψ^α and its gauge conjugate ξ_α is

$$\mathcal{L}=\psi^\alpha\nabla_{\alpha\dot\alpha}\bar{\xi}^{\dot\alpha}+\text{c.c.}+m(\psi^\alpha\xi_\alpha+\bar{\psi}^{\dot\alpha}\bar{\xi}_{\dot\alpha}), \tag{2.1}$$

with $\nabla_{\alpha\dot\beta}=i\partial_{\alpha\dot\beta}+A_{\alpha\dot\beta}$, the covariant derivative. Functionally integrating out the dotted fields gives rise to a second-order form for the fermionic couplings [27],

$$\mathcal{L}=-\frac{1}{m}\psi^\alpha(\nabla^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}-m^2)\xi_\alpha+\frac{1}{m}\psi^\alpha\xi^\beta F_{\alpha\beta}, \tag{2.2}$$

where $F_{\alpha\beta}$ is the self-dual projection of the field strength. In an anti-self-dual background the fermionic coupling becomes that of a scalar. In the massless limit (after appropriately scaling the fields with the mass), the fermionic coupling in such backgrounds is

$$\mathcal{L}=\psi^\alpha\Box\xi_\alpha, \tag{2.3}$$

and the index on the field only enters into amplitude calculations through an external line factor. In this formulation the fermion can be said to be "bosonized," and there is a direct analog in the $\mathcal{N}=2$ string. In the supersymmetric theory, the single spinor index may be represented as an internal symmetry factor (i.e., a line factor associated with the Lorentz group representations) attached to the single creation operator arising in the cohomology analysis.

B. Supersymmetric self-dual gauged supergravity

The supersymmetric extension of self-dual systems in the context of N-extended self-dual supergravity theory [4] is described in the following. The target space-time supersymmetric extension of the $\mathcal{N}=2$ string has the particle content of this supersymmetric system. This theory is described by the local $OSp(N|2)$ algebra tensored with the $SL(2)'$ half of the Lorentz algebra:

$$[\nabla_{A\dot\alpha},\nabla_{B\dot\beta}\}=\frac{1}{2}C_{\dot\alpha\dot\beta}F_{ABCD}M^{CD}. \tag{2.4}$$

The supercoordinate is $z^{M\dot\mu}=(x^{\mu\mu'},\theta^{m\dot\mu})$, and the $OSp(N|2)$ superindex $A=(a,\alpha)$ contains the $SL(2)$ index "α" and a vector $SO(N)$ one "a." The covariant derivatives are defined by

$$\nabla_{A\dot\alpha}=E_{A\dot\alpha}{}^{M\dot\mu}\partial_{M\dot\mu}+\frac{1}{2}\Omega_{A\dot\alpha BC}M^{CB}, \tag{2.5}$$

where letters from the beginning of the alphabet denote flat local tangent space indices and those from the middle the coordinates. The coupling constants are defined as part of the $OSp(N|2)$ metric: $\eta^{ab}=g\delta^{ab}$, $\eta^{\alpha\beta}=\kappa C^{\alpha\beta}$, $\eta^{a\beta}=0$. The gravitational coupling constant κ has dimensions of inverse mass.

The light-cone gauge $\nabla_{A\dot+}=\partial_{A\dot+}$ solves the $(\dot\alpha,\dot\beta)=(\dot+,\dot+)$ constraint. The $(\dot+,\dot-)$ and $(\dot-,\dot+)$ constraints may be solved through the introduction of a scalar superfield Φ leading to the covariant derivative $\nabla_{A\dot-}$ given by

$$E_{A\dot-}{}^{M\dot\mu}\partial_{M\dot\mu}=\partial_{A\dot-}+(\partial_{A\dot+}\partial_{B\dot+}\Phi)\eta^{CB}\partial_{C\dot+} \tag{2.6}$$

$$\Omega_{A\dot-BC}=\partial_{A\dot+}\partial_{B\dot+}\partial_{C\dot+}\Phi, \tag{2.7}$$

and the non-vanishing field strength

$$F_{ABCD}=-\partial_{A\dot+}\partial_{B\dot+}\partial_{C\dot+}\partial_{D\dot+}\Phi. \tag{2.8}$$

The remaining constraint $[\nabla_{A\dot-},\nabla_{B\dot-}\}=0$ gives rise to the field equation

$$\partial_A{}^{\dot\alpha}\partial_{B\dot\alpha}\Phi+\eta^{DC}(\partial_{A\dot+}\partial_{C\dot+}\Phi)(\partial_{D\dot+}\partial_{B\dot+}\Phi)=0. \tag{2.9}$$

As further noted in [4], within the field equation (2.9) we may solve for all of the $\theta^{a\dot-}$ dependence explicitly by examining the $A=a$ and $B=b$ components: In the superfield Φ we may take $\theta^{a\dot-}=0$.

The scalar superfield Φ expanded in components is then

$$\Phi=\phi+\theta^a\psi_a+\theta^a\theta^b\phi_{ab}+\cdots. \tag{2.10}$$

Because only $\theta^{a\dot+}$ appears in the explicit theta expansion we will drop the index $\dot+$ in the paper from the supercoordinate and have it implied. The lowest component in the theta expansion of Eq. (2.10), ϕ, is the -2 helicity graviton state, which is a singlet under the gauged $SO(N)$. The components ϕ_{ab} are the -1 helicity $SO(N)$ gluons in the adjoint representation ($\phi_{ab}=-\phi_{ba}$). The remaining states in the superfield expansion (2.10) are in representations labeled by antisymmetric products of $SO(N)$ fundamental indices. There are a total of 2^{N-1} bosonic and 2^{N-1} fermionic degrees of freedom contained in the superfield (2.10).

The two-field light-cone superspace action is found from the field equation (2.9) by incorporating a Lagrange multiplier field $\tilde{\Phi}$ conjugate to Φ and contracting the components $A=\alpha$ and $B=\beta$ with $C^{\alpha\beta}$,

$$S=\mathrm{Tr}\int d^4x\, d^N\theta[\tilde{\Phi}\Box\Phi+\eta^{BA}\tilde{\Phi}(\partial^{\alpha}{}_{\dot{+}}\partial_{A\dot{+}}\Phi)(\partial_{B\dot{+}}\partial_{\alpha\dot{+}}\Phi)], \tag{2.11}$$

with the superspace measure $d^N\theta=\Pi_{j=1}^{N}d\theta^j$ (or an action related by exchanging $\partial^{\alpha}{}_{\dot{+}}$ with $\partial^{\alpha}{}_{\dot{-}}$). For N=8 the two superfields are identified. The interactions in Eq. (2.11) have two types of derivative structures for the types of couplings from $\eta^{AB}=(\kappa C^{\alpha\beta},g\delta^{ab})$. All of the interactions proportional separately to g in Eq. (2.11) are of the same form and are distinguished only by their respective group theory factors, and similarly for those proportional to κ. Via spectral flow and the field theory bosonization described previously the interactions of the different spin fields may be incorporated in the $\mathcal{N}=2$ supersymmetric string through the addition of line factors attached to the insertion of local vertex operators.

C. Supersymmetry and spectrum

We next analyze the superconformal algebra of the $\mathcal{N}=2$ system and demonstrate the supersymmetric spectrum. The generators of the algebra are spanned by T, $G^{\pm}$ and a $U(1)$ current J that may be constructed by the usual Noether method on the gauge-fixed $\mathcal{N}=2$ world-sheet action, including the ghost systems (b,c), $(\beta^{\pm},\gamma^{\pm})$, and (b',c'). The bosonized form of the ghost systems are

$$c=e^{\sigma},\quad b=e^{-\sigma} \tag{2.12}$$

$$c'=e^{\sigma'},\quad b'=e^{-\sigma'} \tag{2.13}$$

for the reparametrizations and $U(1)$ gauge invariance, and

$$\beta^{\pm}(z)=\partial_z\chi^{\pm}e^{-\phi^{\mp}},\quad \gamma^{\pm}=\eta^{\pm}e^{\phi^{\pm}} \tag{2.14}$$

for the doubled holomorphic supersymmetries. Further bosonization of the fermionic $\chi^{\pm}$ and $\eta^{\pm}$ are possible, but unnecessary for this work.

Also, the system possesses a gauged $U_A(1)$ symmetry. Its holomorphic representative is generated by the operator

$$\Sigma(\theta)=\exp\left[2\pi\theta\oint\frac{dz}{2\pi i}\ln(z)J(z)\right]=e^{2\pi i\theta\hat{\phi}(z)}, \tag{2.15}$$

where $\hat{\phi}(z)=\phi^{+}-\phi^{-}+\psi^{+}-\psi^{-}+cb'$, the bosonized form of the complete $U(1)$ current $J=\partial\hat{\phi}$. The operator in Eq. (2.15) is essentially a continuous modification of a spin field in the usual $\mathcal{N}=1$ string. It possesses an inverse, $\Sigma(-\theta)$. The insertion in a string amplitude of $\Sigma(\theta)\Sigma(-\theta)=1$, by contour deformation, leads to a branch cut of monodromy $e^{2\pi i\theta}$ between two chosen points z_1 and z_2 on the Riemann surface.[4] The spectral flow operator in Eq. (2.15) at $\theta=1/2$ connects the Neveu-Schwarz vacuum to the Ramond vacuum and generates target space-time fermionic statistics when placed at the point of emission of a bosonic state. Furthermore, an automorphism of the superconformal algebra[5] on the holomorphic currents is generated through

$$\Sigma(\theta)T(z)\Sigma(-\theta)=T(z)+\frac{\theta}{z}J(z)+\frac{c}{6}\frac{\theta^2}{z^2} \tag{2.16}$$

$$\Sigma(\theta)G^{\pm}\Sigma(-\theta)=z^{\mp\theta}G^{\pm}(z) \tag{2.17}$$

and

$$\Sigma(\theta)J(z)\Sigma(-\theta)=J(z)+\frac{c}{3}\frac{\theta}{z}. \tag{2.18}$$

At criticality $c=c_m+c_{gh}=6-6=0$ and we see the $U(1)$ current to be invariant and that the revolution by θ twists the supercurrents $G^{\pm}$ with the same monodromy due to a θ moded spin field.

The holomorphic BRST current on the super world sheet (and anti-holomorphic relative) is given by

$$\begin{aligned}J_{BRST}=&cT+\gamma^{+}G^{-}+\gamma^{-}G^{+}+c'J+c\partial cb+c\partial c'b'\\&-4\gamma^{+}\gamma^{-}b+2\partial\gamma^{-}\gamma^{+}b'-2\partial\gamma^{+}\gamma^{-}b'\\&+\tfrac{3}{4}\partial c(\gamma^{+}\beta^{-}+\gamma^{-}\beta^{+})-\tfrac{3}{4}c(\partial\gamma^{+}\beta^{-}+\partial\gamma^{-}\beta^{+})\\&+\tfrac{1}{4}c(\gamma^{+}\partial\beta^{-}+\gamma^{-}\partial\beta^{+})+c'(\gamma^{+}\beta^{-}-\partial\gamma^{-}\beta^{+}),\end{aligned} \tag{2.19}$$

together with ghost number current

$$J_{gh}=-bc-b'c'+\eta^{+}\xi^{-}+\eta^{-}\xi^{+}, \tag{2.20}$$

and a conserved BRST charge of

$$Q=\oint\frac{dz}{2\pi i}J_{BRST}. \tag{2.21}$$

We shall not list all of the various transformations due to BRST of the currents but point out the relations

$$[Q,\Sigma(\theta)]=0,\quad \partial_z\Sigma(\theta)=-\theta\{Q,b'(z)\Sigma(\theta)\}, \tag{2.22}$$

which illustrates that $\Sigma(\theta)$ is BRST-closed and that $\partial_z\Sigma(z)$ is exact.

Together from the ghost system and the BRST charge we define the picture changing operators

[4] Such a branch may be resolved by a $1/\theta$ covering of the Riemann surface, which is itself a higher genus surface found via a multiple covering of the original punctured surface. Thus spectral flow also is compatible with the triviality of the S-matrix in the $\mathcal{N}=2$ theory.

[5] We denote currents in the critical case with a hat, e.g. $\hat{G}^{\pm}$.

$$PCO^{\pm}=\{Q,\xi^{\pm}\} \tag{2.23}$$

required at higher-genus for modular invariance [28], which are also represented on the world sheet by insertions of the rebosonized operators

$$PCO^{\pm}(z)=\delta(\beta^{\pm}(z))\hat{G}^{\pm}(z). \tag{2.24}$$

The spectral flow operator $\Sigma(\theta)$ commutes with $PCO^{\pm}$ and shifts the picture number via

$$(\pi^{+},\pi^{-})\rightarrow(\pi^{+}+\theta,\pi^{-}-\theta). \tag{2.25}$$

We next re-examine the states of the $\mathcal{N}=2$ string in the ''NS-sector'' and the ''R-sector,'' related to each other by spectrol flow operations of $\Sigma(\frac{1}{2})$.

In the canonical formulation we denote ground states of picture number (of two supercurrents) with

$$|\pi^{+},\pi^{-};k\rangle, \tag{2.26}$$

which are related to each other through actions of $PCO^{\pm}$, shifting $\pi^{\pm}\rightarrow\pi^{\pm}+1$, respectively. The single state, without internal line factors attached, is found in [29] as

$$c_1|-1,-1;k\rangle \tag{2.27}$$

with the on-shell condition $k^2=0$, which is the unique state at ghost number equal to one. The (holomorphic) vertex operator constructing this state is

$$V^{(1)}_{-1,-1}=ce^{-\phi^{+}-\phi^{-}}e^{ik\cdot x}, \tag{2.28}$$

and may be transformed into a graviton vertex operator of polarization $++$ found in the following after normalizing the line factor [6].

The vertex operators produce from the (first-quantized) vacuum state the asymptotic string states in the scattering amplitude under consideration. They correspond to the physical states of the $\mathcal{N}=2$ closed string and carry their quantum numbers. Being representatives of the (semi-chiral and semi-relative) BRST cohomology, they are unique up to BRST-trivial terms and normalization due to internal symmetry factors. The physical subspace of the $\mathcal{N}=2$ string Fock space through the covariant quantization scheme contains the ground state $|k\rangle$, a scalar on the massless level, i.e., for center-of-mass momentum $k^{\pm a}$ with $k\cdot k=0$. This is not including internal symmetry factors and states related to the ground state through spectral flow. The canonical massless scalar field is denoted by

$$\Phi(x)=\int d^4k\, e^{-ik\cdot x}\,\tilde{\Phi}(k), \tag{2.29}$$

which allows for a supersymmetric completion. We may relate the higher states in the supermultiplet to the ground state through spectral flow and the isospin labeling of the Lorentz group through Eq. (2.10).

The field representative of the vertex operator for a self-dual graviton, with polarization $\epsilon^{++}_{\mu\nu}$, is in the (0,0) pictures

$$V(k)=\int d^2z\sqrt{g}\,\epsilon^{++}_{\mu\nu}\cdot(\partial x^{\mu}+i\psi^{\mu}k\cdot\psi)\times(\overline{\partial x}^{\nu}+i\overline{\psi}^{\nu}k\cdot\overline{\psi})e^{ik\cdot x}, \tag{2.30}$$

and that for a holomorphic space-time fermion in the $(\frac{1}{2})$ pictures,

$$V_F(k)=\int dz\,L_{\mu}\left(\partial x^{\mu}+\frac{i}{4}k\cdot\psi\psi^{\mu}\right)e^{ik\cdot x}+\frac{1}{2}u^{\alpha}e^{3\phi/2}b\,\eta S_{\alpha}e^{ik\cdot x} \tag{2.31}$$

with L a spin operator that connects the Neveu-Schwarz (NS) to the Ramond (R) vacua (the latter term is not relevant for the calculations but necessary for BRST invariance). In the $\mathcal{N}=1$ superstring this line factor is composed, after the bosonization, as

$$L_{\mu}=u_{\alpha}\Gamma^{\alpha\beta}_{\mu}\Sigma_{-1/2}S_{\beta}, \tag{2.32}$$

with spin fields

$$\Sigma_{\pm 1/2}=e^{\pm i\sigma(z)/2},\quad \beta=\partial_z\chi\, e^{i\sigma},\quad \gamma=\eta e^{-i\sigma} \tag{2.33}$$

(note also that $\{Q_{\alpha},Q_{\beta}\}=\Gamma^{\alpha\beta}_{\mu}P^{\mu}$) and the spin operator from the internal bosonized world-sheet fermions,

$$S_{\alpha}=e^{\pm i\phi_1/2\pm i\phi_2/2+\cdots}. \tag{2.34}$$

Furthermore, on the world-sheet supersymmetry generators are found by

$$Q_{\alpha}=\oint\frac{dz}{2\pi i}V_{\alpha}=\oint\frac{dz}{2\pi i}\Sigma_{-1/2}S_{\alpha}, \tag{2.35}$$

at zero momentum. The index α denotes the different $\pm$ signs in the exponential ($2^{d/2}$ for d even) and the world-sheet fermions bosonized into $\psi^a_{\pm}=e^{\pm i\phi^a}$ after complexifying pairs of them into $\psi^a_{\pm}=\psi^a\pm i\psi^{d/2+a}$. Currents are $J_a=-i\partial_z\phi_a$.[6] The gauge-fixed $\mathcal{N}=1$ string has a global $\mathcal{N}=2$ supersymmetry that allows a direct connection with the vertex operator construction in the $\mathcal{N}=2$ superstring.

There is an additional $\beta\gamma$ ghost system in the $\mathcal{N}=2$ string because there are twice as many local supersymmetries (denoted by $\beta^{\pm}\gamma^{\pm}$). The fact that fermions can be bosonized in the self-dual background means that the square-root monodromies inserted into the string world sheet transfer into normalizations after doing the same for the vertex operator in the $\mathcal{N}=1$ string. This can be made manifest in the $\mathcal{N}=2$ supersymmetric string due to spectral flow, i.e., the presence of an additional gauge current on the world sheet at criticality.

In the $\mathcal{N}=2$ superstring the vertex operator for the emission of a space-time fermion may be found by taking every

[6] In the $(-1/2)$ picture the fermionic vertex operator has the form $V_F=\Sigma_{1/2}u^{\alpha}S_{\alpha}e^{ik\cdot x}$.

component of the previous bosonized form of the world-sheet fermionic contributions and gauging the spin operator via the $U(1)$ current. The four-component spin operator in $d=2+2$ target space-time is

$$S_{+}=e^{i\phi_1/2+i\phi_2/2},\quad S_{-}=e^{i\phi_1/2-i\phi_2/2} \tag{2.36}$$

$$S_{\dot{+}}=e^{-i\phi_1/2-i\phi_2/2},\quad S_{\dot{-}}=e^{-i\phi_1/2+i\phi_2/2}. \tag{2.37}$$

Gauging the expanded component form of the above fermionic vertex operator with a $U(1)$ current allows a twist of the vertex operator

$$\Sigma(1/2)u_\alpha(k)\Gamma_\mu^{\alpha\beta}\Sigma_{-1/2}S_\beta=\epsilon_\mu^+(k), \tag{2.38}$$

which amounts to a momentum dependent twist of the single vertex operator into that of a space-time boson. The $\mathcal{N}=2$ string carries this in a straightforward way at the level of the cohomology through spectral flow and allows internal quantum numbers to be assigned to states into *a priori* independent sectors related by the spectral flow. The translation introduces an additional redundancy into the components of the $\epsilon_\mu^+=\epsilon_{\alpha\dot{\alpha}}^+$ and allows this twisted line factor to be reduced further to a two-component spinor v_α. The physical polarization in Eq. (2.38) has the explicit bi-spinor form

$$\epsilon_{\alpha\dot{\alpha}}^+(k;q)=\frac{q_\alpha k_{\dot{\alpha}}}{q^\beta k_\beta}, \tag{2.39}$$

with q_α arbitrary (the reference momenta [30]) and $\epsilon^\pm\cdot\epsilon^\mp=-1$ and $\epsilon^\pm\cdot\epsilon^\pm=0$. Different choices of q_i generate axial forms,

$$\epsilon_{\alpha\dot{\alpha}}^+(k;q_1)=\epsilon_{\alpha\dot{\alpha}}^+(k;q_2)+f(k;q_1,q_2)k_{\alpha\dot{\alpha}}, \tag{2.40}$$

and are the same on shell due to transversality for on-shell amplitudes. An analogous spinor reference momentum is incorporated into the line factor for the spinor in Eq. (2.3) [27].

The two vertex operators are similar, except that in the $\mathcal{N}=1$ string the factor L^μ introduces a branch cut on the world sheet and ϵ^{++} a Lorentz covariant line factor (introduced into the $\mathcal{N}=2$ string following spinor helicity techniques in [6]). In a self-dual background the two are the same up to a line factor, as required by known field theory results of bosonization.

A spectral flow operation in a scattering process allows the identification

$$\begin{aligned}&\langle \ldots V_F(k_1)V_F(k_2)\ldots\rangle\\ &\quad=\langle \ldots V_F(k_1)\Sigma(-\theta)\Sigma(\theta)V_F(k_2)\ldots\rangle\\ &\quad=\langle \ldots V_b(k_1)V_b(k_2)\ldots\rangle|_{\theta=1/2},\end{aligned} \tag{2.41}$$

where in the latter a contour deformation of the spin operator is taken around the two vertex operators to convert them to bosonized statistics (see Fig. 1). A similar procedure may be performed for all correlators involving an even number of

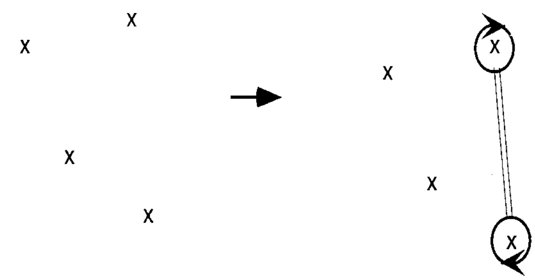

FIG. 1. Contour deformation of the insertion of $\Sigma(\theta)\Sigma(-\theta)$.

fermionic vertex operators. This is the world-sheet derivation of the bosonization of the fermions in a self-dual background.

In the supersymmetric target space-time theory the quantum amplitudes are zero to all orders in the field theory limit due to a supersymmetric identity of MHV amplitudes (originally found in [31]) in both the closed supersymmetric $\mathcal{N}=2$ string as well as the open $\mathcal{N}=2$ supersymmetric string. The $\mathcal{N}=2$ superstring, found by multiplying the Fock space with internal factors labeling spin, agrees to all orders with the vanishing theorems of the $\mathcal{N}=2$ string that assumed modular invariance.

III. DUALITY IMPLIES VANISHING

In the early days of string theory many properties of amplitudes were shown to follow from (Dolen-Horn-Schmid) duality [32] (via ''stretching the world sheet'') and without identification as any particular string theory. For example, duality alone shows that all loop diagrams can be expressed as insertions of tadpoles into trees [33]. We analyze this duality within the gauge theory MHV amplitudes in the following.

For a theory with a finite number of particles of generic mass, duality is a much more stringent constraint: Since a finite sum of poles in one channel cannot equal such a sum in another channel, the tree amplitudes must all vanish (except for the three-point function, which does not have any channels). This analysis can be extended to one-loop diagrams: For example, a planar one-loop open-string graph is related by duality to such a tree graph with the insertion of a closed-string tadpole. The MHV one-loop YM amplitude [19,20], which coincides with the one-loop self-dual *S*-matrix, has the form

$$\mathcal{A}_{n;1}^{[1]}(k_i)=-\frac{i}{48\pi^2}\sum_{1\le i<j<k<l\le n}\frac{\langle ij\rangle[jk]\langle kl\rangle[li]}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \tag{3.1}$$

in a color-ordered form. Relations between this amplitude in d dimensions and a supersymmetric amplitude in $d+4$ dimensions relates scattering between the $\mathcal{N}=1$ and $\mathcal{N}=2$ strings [6]. The inner products are written in terms of twistor variables $\langle ij\rangle=k_i^\alpha k_{j\alpha}$ and $[ij]=k_i^{\dot{\alpha}}k_{j\dot{\alpha}}$, with $s_{ij}=(k_i+k_j)^2=\langle ij\rangle[ji]$. In $d=2+2$ the amplitude becomes real and the inner products $\langle ij\rangle$ and $[ij]$ are not complex conjugates. A similar result holds for the MHV gravitational amplitudes

[21]. The form in Eq. (3.1) is manifestly crossing symmetric $(i\leftrightarrow j)$ after including the color factor

$$C_\sigma=\mathrm{Tr}\,T^{\sigma_1}T^{\sigma_2}\cdots T^{\sigma_n} \tag{3.2}$$

with group generators T_i and summing over all permutations, as all the outgoing helicities are the same.

The analytic structure (in momentum space) of a closed tree diagram with zero-momentum inserted states, and an open loop diagram containing a finite number of particle types, in general do not agree. We next analyze the insertion of a single operator in a field theory tree diagram. The (color ordered) gauge field corresponding to a tree diagram with m out-going helicities the same (labeled from legs 1 to m) and one leg unamputated and off-shell is

$$A_m^{\alpha\dot\beta}(q;k_i^+)=\frac{k^\alpha q^{\gamma\dot\beta}k_\gamma}{\langle k1\rangle\langle 12\rangle\cdots\langle mk\rangle}, \tag{3.3}$$

with k the reference momentum [30] chosen in the off-shell extension of the leg with momentum q $(=-\Sigma_{j=1}^m k_j)$ [23]. The insertion of a zero-momentum state interacting with a vertex associated with j lines requires products of up to j fields. The overall constant c_m is suppressed in the subsequent analysis. These composite operator insertions model local anomaly mechanisms; as an example we list

$$S=\mathrm{Tr}\int d^4x\,\phi F\tilde{F}. \tag{3.4}$$

We denote the product of two fields as

$$A_m^{\alpha\dot\beta}A_{n-m}^{\delta\dot\rho}=\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}H^{\alpha\dot\beta,\delta\dot\rho}, \tag{3.5}$$

and consider the general contraction with two local derivatives with momentum q,

$$\mathcal{A}=q^{\alpha_1\dot\beta_1}q^{\alpha_2\dot\beta_2}H^{\alpha_3\dot\beta_3,\alpha_4\dot\beta_4}T_{\alpha_1\dot\beta_1,\alpha_2\dot\beta_2,\alpha_3\dot\beta_3,\alpha_4\dot\beta_4}, \tag{3.6}$$

with T a general tensor. As the momentum p flowing into the product of two fields is taken to zero, the field is evaluated with momentum q and $-q$. The unamputated and off-shell field is conserved via $q^{\alpha\dot\beta}A_{\alpha\dot\beta}=0$, and the most general tensor contractions are obtained from the tensors,

$$T_{(1)}^{\alpha_1\dot\beta_1,\alpha_2\dot\beta_2,\alpha_3\dot\beta_3,\alpha_4\dot\beta_4}=\epsilon^{\alpha_1\alpha_3}\epsilon^{\alpha_2\alpha_4}\epsilon^{\dot\beta_1\dot\beta_3}\epsilon^{\dot\beta_2\dot\beta_4}, \tag{3.7}$$

$$T_{(2)}^{\alpha_1\dot\beta_1,\alpha_2\dot\beta_2,\alpha_3\dot\beta_3,\alpha_4\dot\beta_4}=\epsilon^{\alpha_1\alpha_4}\epsilon^{\alpha_2\alpha_3}\epsilon^{\dot\beta_1\dot\beta_4}\epsilon^{\dot\beta_2\dot\beta_3}, \tag{3.8}$$

$$T_{(3)}^{\alpha_1\dot\beta_1,\alpha_2\dot\beta_2,\alpha_3\dot\beta_3,\alpha_4\dot\beta_4}=\epsilon^{\alpha_1\alpha_3}\epsilon^{\alpha_2\alpha_4}\epsilon^{\dot\beta_2\dot\beta_3}\epsilon^{\dot\beta_1\dot\beta_4}, \tag{3.9}$$

$$T_{(4)}^{\alpha_1\dot\beta_1,\alpha_2\dot\beta_2,\alpha_3\dot\beta_3,\alpha_4\dot\beta_4}=\epsilon^{\alpha_1\alpha_4}\epsilon^{\alpha_2\alpha_3}\epsilon^{\dot\beta_1\dot\beta_3}\epsilon^{\dot\beta_2\dot\beta_4}, \tag{3.10}$$

together with those that trivially contribute zero due to momentum conservation, such as $\epsilon^{\alpha_1\alpha_2}\epsilon_{\alpha_3\alpha_4}\epsilon^{\dot\beta_1\dot\beta_3}\epsilon^{\dot\beta_2\dot\beta_4}$ and permutations. The contractions with the various tensors after summing over all possible m and $n-m$ point fields generates

$$\mathcal{A}_1=\sum_{i=1}^{m}\sum_{j=m+1}^{n}\sum_{a=1}^{m}\sum_{b=m+1}^{n}\frac{\langle ka\rangle\langle kb\rangle[ia][jb]\langle ik\rangle\langle jk\rangle\langle m,m+1\rangle\langle n1\rangle}{\langle k1\rangle\langle mk\rangle\langle k,m+1\rangle\langle nk\rangle}, \tag{3.11}$$

$$\mathcal{A}_2=\sum_{i=1}^{m}\sum_{j=m+1}^{n}\sum_{a=1}^{m}\sum_{b=m+1}^{n}\frac{\langle ka\rangle\langle kb\rangle[ib][ja]\langle ik\rangle\langle jk\rangle\langle m,m+1\rangle\langle n1\rangle}{\langle k1\rangle\langle mk\rangle\langle k,m+1\rangle\langle nk\rangle}, \tag{3.12}$$

$$\mathcal{A}_3=\sum_{i=1}^{m}\sum_{j=m+1}^{n}\sum_{a=1}^{m}\sum_{b=m+1}^{n}\frac{\langle ka\rangle\langle kb\rangle[ib][ja]\langle ij\rangle\langle jk\rangle\langle m,m+1\rangle\langle n1\rangle}{\langle k1\rangle\langle mk\rangle\langle k,m+1\rangle\langle nk\rangle}, \tag{3.13}$$

and

$$\mathcal{A}_4=0. \tag{3.14}$$

In deriving Eqs. (3.11) and (3.12) the null vector q has been written as $q^{\alpha\dot\beta}=\Sigma_{j=1}^m k_j^\alpha k_j^{\dot\beta}$ together with momentum conservation. The first sum in Eq. (3.11) is separately odd under both $i\leftrightarrow a$ and $j\leftrightarrow b$, and the second is odd under $i\leftrightarrow b$ and $j\leftrightarrow a$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal to zero. The series in $\mathcal{A}_3$ is odd under $i\leftrightarrow b$ and $j\leftrightarrow a$ separately [and even under simultaneous $(i,j)\leftrightarrow(a,b)$].

We next consider the zero-momentum insertion associated with a three-point vertex. Summing over all possible products of gauge fields associated with a general tensor gives

$$\mathcal{A}^{\alpha_1\dot\beta_1,\alpha_2\dot\beta_2,\alpha_3\dot\beta_3}=\sum_{i=1}^{m}\sum_{j=m+1}^{p}\sum_{a=p+1}^{n}k^{\alpha_1}k_i^{\dot\beta_1}k^{\alpha_2}k_j^{\dot\beta_2}k^{\alpha_3}k_a^{\dot\beta_3}\times\frac{\langle ik\rangle\langle jk\rangle\langle ak\rangle}{\langle k1\rangle\langle m,m+1\rangle\langle p,p+1\rangle\langle n1\rangle\langle mk\rangle\langle k,m+1\rangle\langle pk\rangle\langle k,p+1\rangle\langle qk\rangle}. \tag{3.15}$$

Any contraction of the indices α_j associated with k_α or a $\epsilon^{\alpha\beta}$ generates $\langle kk\rangle$ and is identically equal to zero.

The last possibility contains a four-point vertex,

$$\mathcal{A}^{\alpha_1\dot\beta_1,\alpha_2\dot\beta_2,\alpha_3\dot\beta_3,\alpha_4\dot\beta_4}=\sum_{i=1}^{m}\sum_{j=m+1}^{p}\sum_{a=p+1}^{q}\sum_{b=q+1}^{n}k^{\alpha_1}k_i^{\dot\beta_1}k^{\alpha_2}k_j^{\dot\beta_2}k^{\alpha_3}k_a^{\dot\beta_3}k^{\alpha_4}k_b^{\dot\beta_4}\times\frac{\langle ik\rangle\langle jk\rangle\langle ak\rangle\langle bk\rangle\langle m,m+1\rangle\langle p,p+1\rangle\langle q,q+1\rangle\langle n1\rangle}{\langle k1\rangle\langle mk\rangle\langle k,m+1\rangle\langle pk\rangle\langle k,p+1\rangle\langle qk\rangle\langle k,q+1\rangle\langle nk\rangle}, \tag{3.16}$$

the form of which also generates zero for any tensor contraction.

The above analysis shows that the non-supersymmetric gauge theory MHV amplitudes cannot be generated by a single zero-momentum operator insertion into a tree amplitude. We conclude that the string duality between open and closed (in the $\mathcal{N}=2$ system) requires that the amplitudes vanish. Self-dual *super* Yang-Mills and self-dual *super*gravity are examples of such theories, but the corresponding non-supersymmetric theories are not.

IV. MODULAR INVARIANCE VS FIELD THEORY

The evaluation of scattering amplitudes for the string, in either the path integral or operator approach, involves sums over the inequivalent geometries associated with the world sheet topology. There are two steps to this procedure, finding (1) the Green functions on that space and (2) the corresponding measure. The second step is the hardest; in earlier evaluations of one-loop quantities in $\mathcal{N}=2$ string theory, it was *assumed* that the measure was modular invariant, to enforce conformal invariance in the critical dimension of four. Calculations with this integration measure led directly to vanishing genus-one diagrams. This result directly contradicted explicit evaluations of the corresponding graphs by field theoretic methods. In this section we will review these results; a more rigorous analysis in the following section, based on sewing or unitarity constructions, will reveal an anomaly in conformal invariance in the non-supersymmetric string.

We first examine the consistency of the zero-point function in the supersymmetric context. At genus one it is given by

$$Z=\int\frac{d^2\tau}{\tau_2^2}=\frac{\pi}{3} \tag{4.1}$$

and corresponds in the target space-time field theory [on a $2d$-real dimensional Kähler manifold of signature (4,0) or (2,2)] to the zero-point function [2] derived from

$$\int_{\mathcal{M}}d^dx\sqrt{g}=\frac{1}{(d/2)!}\int_{\mathcal{M}}\omega\wedge\omega\ldots\wedge\omega, \tag{4.2}$$

with a product of $d/2$ factors of the Kähler form ω. The integrand in Eq. (4.2) is locally a total derivative, as $\omega=d\bar{d}K$ (e.g. $d\omega=\bar{d}\omega=0$), and the integrand is totally antisymmetric,

$$\int_{\mathcal{M}}d^dx\sqrt{g}\sim\int_{\partial M}\bar{d}K\wedge\omega\wedge\omega\ldots=\int_{\overline{\partial M}}dK\wedge\omega\wedge\omega\ldots. \tag{4.3}$$

The fact that the cosmological term in Eq. (4.2) is a total derivative (locally) means that it does not contribute to the field equations. The absence is in agreement with the vanishing of the cosmological term in the self-dual gravity theory and in the supersymmetric extension.

The four-point one-loop MHV gravitational amplitude in four-dimensions (in $d=3+1$ dimensions) has the form

$$\mathcal{A}_4^{[2]}(k_i)=-i\left(\frac{\kappa}{2}\right)^4\frac{1}{120(4\pi)^2}\left(\frac{s_{12}s_{23}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right)^2\times(s_{12}^2+s_{23}^2+s_{13}^2), \tag{4.4}$$

derived in [34]. Its n-point form is presented in [21] and generates the S-matrix for the self-dual gravitational field theory [8]. The supersymmetric completion (integrating out a virtual supersymmetric multiplet) is identically zero, and the amplitude satisfies a relation $\mathcal{A}^{[2]}=\mathcal{A}^{[0]}=-\mathcal{A}^{[1/2]}$ for an internal graviton, complex scalar and Weyl fermion respectively. The d-dimensional form, found by analytically continuing the internal momenta or by inserting factors of $\tau_2^{-d/2+2}$ into the first quantized integral form, is equal to zero in $d=2$ [6].

The all-genus amplitudes have been shown to be equal to zero through calculations in the $\mathcal{N}=4$ topological reformulation of the $\mathcal{N}=2$ string (modulo contact term ambiguities analyzed in [6]), based on the assumption of modular invariance. This has been verified directly at genus one in the RNS $\mathcal{N}=2$ formulation, taking into account both contact terms as well as the different ordering of limits in the zero-slope limit [6] (integrating spin structure first or evaluating $\alpha'\to 0$ first), generating the continued form of the result in Eq. (4.4) to $d=2$. In the latter calculation it was shown that an additional factor of τ_2 in the closed string calculation arising from the $b'c'$ ghost system associated with the world-sheet gauge field maps the integral representations of the amplitude into two internal space-time dimensions, and the result for the amplitude equals zero. However, this additional factor in the integration measure does not follow directly from sewing trees or from unitarity considerations (in the Wick-rotated sense to $d=3+1$ dimensions); these latter techniques would generate the non-vanishing gravitational MHV amplitudes (4.4) at n-point.

The chiral $\mathcal{N}=2$ matter multiplets $X=(x,\psi)$ and ghost systems at genus one contributes the following determinant factors to the evaluation of the amplitude:

$$Z_d\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau,\bar{\tau})=Z_x(\tau,\bar{\tau})\,Z_\psi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau,\bar{\tau})\,Z_{bc}(\tau,\bar{\tau})\,Z_{\beta\gamma}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\times(\tau,\bar{\tau})\,Z_{b'c'}(\tau,\bar{\tau}), \tag{4.5}$$

with $d=4$ in the critical string and where the respective factors are

$$Z_x(\tau,\bar{\tau})=\tau_2^{-d/2}|\eta(\tau)|^{-2d},$$

$$Z_\psi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau,\bar{\tau})=\left|\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0,\tau)\right|^{d}|\eta(\tau)|^{-d}, \tag{4.6}$$

$$Z_{bc}(\tau,\bar{\tau})=\tau_2|\eta(\tau)|^4,$$

$$Z_{\beta\gamma}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau,\bar{\tau})=\left|\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0,\tau)\right|^{-4}|\eta(\tau)|^4. \tag{4.7}$$

The ghost determinant associated with the local $U(1)$ symmetry is

$$Z_{b'c'}(\tau,\bar{\tau})=\tau_2|\eta(\tau)|^4. \tag{4.8}$$

The Dedekind eta and theta functions with continuous characteristic $[{}^{\alpha}_{\beta}]$ comprise the determinants,

$$\eta(\tau)=q^{1/24}\prod_{n\neq 0}(1-q^n),$$

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\tau)=\sum_{n\in Z}e^{\pi i\tau(n+\alpha)^2+2\pi i(n+\alpha)(z+\beta)}, \tag{4.9}$$

where $q=e^{2\pi i\tau}$ and τ denoting the modular parameter of the torus. The product of these factors generates

$$Z_d\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau,\bar{\tau})=\tau_2^{-(d-4)/2}\left|\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0,\tau)\right|^{d-4}|\eta(\tau)|^{-3(d-4)}, \tag{4.10}$$

and equals unity in four real dimensions. The moduli associated with the path integral quantization of the $\mathcal{N}=2$ string are (1) the parameter τ labeling the inequivalent tori, and (2) the parameters $[{}^{\alpha}_{\beta}]$ labeling the continuous spin structures [or the $U(1)$ gauge bundle of the torus]. The modular invariant integration measure is (up to an arbitrary constant)

$$\frac{d^2\tau}{\tau_2^2},\quad \frac{du\,d\bar{u}}{\tau_2}=d\alpha\, d\beta, \tag{4.11}$$

where $u=(1/2-\alpha)+(1/2-\beta)\tau$. (The volume integral is $\int du\,d\bar{u}=\tau_2$.) However, this is not the measure found from path integral (or operator) quantization, as we show in the following section.

V. ANOMALY IN $\mathcal{N}=2$ STRING

A *rigorous* alternative to invariance arguments is ‘‘sewing,’’ which *unambiguously* determines loop amplitudes inductively in loop order. This property is an automatic consequence of the path integral because of the locality of the world-sheet action: For example, the genus-one world sheet can be ‘‘cut’’ into trees, and integration over variables at the cut are performed after integration over those inside the tree. (Similar remarks apply in the operator formalism.) This is equivalent to the Feynman tree theorem [35], and is the original method for evaluating one-loop graphs in string theory (see, e.g., [36,37]). The basic idea is that the one-loop graph is given by the trace of the string propagator (with additional external states).

For example, this method was applied to the type I superstring to show the existence of anomalies (and divergences) for groups other than SO(32). Although the integration measure is determined directly from the Teichmüller variations, conformal invariance is not enough to fix the normalization of these one-loop diagrams: In fact, it gives an inconsistent result, since the theories for groups other than SO(32) are anomalous; assuming no anomaly would imply different relative normalizations to enforce cancelation. In particular, ignoring the possibility of Chan-Paton factors would give the U(1) or SO(2) (if symmetrized) superstring, which does violate conformal invariance at one loop. Of course, conformal invariance does not fix overall normalization of loop diagrams in any string theory, since any constant overall factor is invariant; rather unitarity determines the normalization factor.

The simplest case to consider is the open string planar loop, since the moduli space of the resulting integral is simpler (at least for the part coming from the world sheet metric). Specifically, the propagator for an open string in external fields is simply

$$\Delta=\frac{1}{L_0+V}=\frac{1}{L_0}-\frac{1}{L_0}V\frac{1}{L_0}+\cdots$$

where L_0 is the free kinetic operator and V is the vertex operator(s). The loop amplitude, for some fixed number of external lines, is then given by sewing a term in this sum, which is represented in this operator language as a trace:

$$\mathcal{A}=\mathrm{Tr}\left(\frac{1}{L_0}V(1)\frac{1}{L_0}V(2)\ldots\right).$$

At this starting point there are no moduli whatsoever, since the tree-level path integral has already been performed. In general, moduli appear only as integration variables, and do not specify external states. However, it is convenient to reintroduce one modulus for purpose of evaluating the sewing: Exponentiating all the free propagators $1/L_0$ with the usual Schwinger parameters, this modulus appears as their sum.

We will not review all the details here, just those that differ from the $\mathcal{N}=0$ and $\mathcal{N}=1$ strings. (See, e.g., [38] for a discussion for the bosonic open string.) For simplicity, we can consider Neveu-Schwarz or Ramond boundary conditions for the world sheet spinors: This allows direct comparison to the $\mathcal{N}=1$ case. (By spectral flow, also known in the maximally helicity violating sector as spacetime supersymmetry, we know the result is independent of this choice.) The evaluation of the Green function part of the path integral (or operator evaluation) is then the same as for $\mathcal{N}=1$, since the physical $\mathcal{N}=2$ variables are the same as those for $\mathcal{N}=1$ in $d=4$. (However, unlike the $\mathcal{N}=1$ case, spectral flow allows us to generalize to arbitrary boundary conditions: see below.) The ghosts do not appear in the vertex operators, and thus the Green functions, if we restrict the external states to be on shell and gauge fixed.

Of course, the evaluation of the Green functions is not affected by the anomaly, as they are the classical part of the JWKB expansion. The evaluation of the measure in the $\mathcal{N}=2$ case is actually simpler. As in the $\mathcal{N}=1$ and 2 cases, it depends on only the modulus ''τ'' related to the sum of the Schwinger parameters: Thus we can forget the external lines and look at just the partition function. We first examine the contribution of the nonzero modes (oscillators). In $d=4$ these contributions cancel identically by counting: The world sheet variables of half-integral and integral worldsheet spin (or conformal weight) each satisfy the same boundary conditions, independent of statistics. (This is true for arbitrary choice of boundary conditions.) But each of the two sets has equal numbers of opposite statistics: four x's vs (b,c,b',c'), and four ψ's vs $(\beta,\gamma,\beta',\gamma')$. Thus all the contributions of the nonzero-modes cancel, leaving a partition function $f=1$. (For other dimensions, f is a d-independent function to the power $d-4$.)

All these functional integration results so far agree with those found previously by invariance arguments (see previous section). The final step is the integration of the zero modes. The integration over x zero modes is identical to that in ordinary field theory, so we leave it for last, and do not discuss it. That leaves only the integration over the zero modes of the fermionic ghosts. Since L_0 has no dependence on these zero modes, these integrals are trivial: If $1/L_0$ were the complete propagator, they would give zero, since, e.g., $\int db_0 1=0$. This problem is fixed in the same way as for $\mathcal{N}=0$: We include in the full propagator a numerator of these zero modes:

$$\Delta' = \frac{b_0 b_0'}{L_0}. \tag{5.1}$$

In fact, this numerator is already required for evaluating tree graphs, when ghosts are included. The result is that the measure is identical to that obtained by field theory methods. (Earlier evaluations based on invariance arguments did not directly address the problem of integration over these zero modes.) The fact that it disagrees with the result obtained from modular invariance arguments (though only by a factor of τ) is the anomaly in this invariance.

To expand on the above we derive the scattering amplitude for a specified set of (unintegrated) spin structures of the world sheet fermions that specify an arbitrary spin. This generalizes the calculation above, although in a trivial way because of the spin independence (spectral flow). (Thus, in the supersymmetric theory the various contributions cancel.) The propagator discussed above is defined as

$$\langle X_1,\psi_1|_\theta \int d\tau e^{-\tau L_0}|X_2,\psi_2\rangle_\theta = \langle X_1,\psi_1|_\theta \frac{1}{L_0}|X_2,\psi_2\rangle_\theta, \tag{5.2}$$

with

$$L_0=\frac{1}{2}\sum_{n=-\infty}^{\infty} :a^\mu_{-n}a_{n,\mu}: + \frac{1}{2}\sum_{n=-\infty}^{\infty}(n+\theta):d^{\theta,\mu}_{-n}d^\theta_{n,\mu}:, \tag{5.3}$$

and

$$d^\theta_n = d_{n+\theta}, \quad d^\theta_{-n}=d_{-n-\theta}, \quad n>0. \tag{5.4}$$

Here the oscillators are labeled by the twist angle θ that interpolates between the Ramond and Neveu-Schwarz sectors, induced by the spectral flow. (There is a zero mode associated with the bosonic ghosts at $\theta=0$ that requires special treatment, which we do not discuss here.) At every θ there is a complete set of states, and the sectors are related by the spectral flow automorphism (2.18); the spectral flow also twists the boundary conditions at pairs of points where vertex operators are located, as described in Sec. III.

The tree amplitudes are constructed via

$$A_n(k_i)=\langle k_1|^{\theta_1}\left(\frac{1}{L_0}\right)V^{\theta_2}(k_2)\ldots\left(\frac{1}{L_0}\right)|k_n\rangle^{\theta_n}, \tag{5.5}$$

with twist parameters θ_i at each vertex which are required to specify the boundary conditions of the worldspinors and thus the spin as described in previous sections. (We have not explicitly inserted the line factor labeling the spin in this equation.) The sewing relation found by inserting a complete set of states in the loop generates

$$A^{\rm loop}_n = \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}_{\{\theta_i\}}\left(\frac{1}{L_0}V_1^{\theta_1}\frac{1}{L_0}V_2^{\theta_2}\ldots V_n^{\theta_n}\right). \tag{5.6}$$

We note that the factor inside the trace in this equation is identical to the $\mathcal{N}=1$ string amplitude evaluated in four-dimensions (for θ's $=0$ or 1/2), after normalizing the $\mathcal{N}=2$ string vertex operators with covariant line factors (2.30): The gauge-fixed action for the matter components of the $\mathcal{N}=1$ string has a global $\mathcal{N}=2$ supersymmetry on the world sheet.

The complete expression following from the above is

$$A_n(k_j)=\int_0^\infty \frac{d\tau}{\tau^{3-n}}\int\prod_{j=1}^n dz_i K^{\theta_j}_{KN}(z_i,\tau). \tag{5.7}$$

where the oscillators associated with the modes are inserted, including the ghost terms but without the zero mode associated with the latter. The integration generates in the zero-slope limit the integrand for the maximally helicity violating amplitudes in four dimensions; the kinematic factor in Eq. (5.7) is

$$K^{\theta_j}_{KN}(z_i,\tau_2)=\int\prod_{j=1}^n d^2\tilde\theta_j\prod_{i<j}e^G \tag{5.8}$$

$$G=\left[-\frac{1}{2}k_i\cdot k_j G_{ij}+\epsilon_{[i}\cdot k_{j]}D_i^+ G_{ij}\right.$$
$$\left.+\epsilon_i\cdot\epsilon_j D_i^+ D_j^+ G_{ij}\right]_{\text{multi-linear}}$$

where $D=\partial_{\tilde{\theta}^-}+\tilde{\theta}^-\partial_z$ and G_{ij} the two-point function between the points $(z_i,\tilde{\theta}_i^{\pm})$ and $(z_j,\tilde{\theta}_j^{\pm})$ for fermions of spin structure θ. The expansion in this equation is in products which are linear in each polarization vector ϵ_j. This expansion may be found in [6] for general choices of reference momenta of the polarization vectors ϵ_j^{++}. The complete integration over the modes in the q-expansion beyond the massless sector is non-trivial, but the fermionic pieces at leading order in q integrate to zero by a supersymmetric Ward identity [6].

However, the contribution from the world sheet Green functions is not the same as from the field theory, although they agree in the zero-slope limit. This indicates that this anomaly is more serious, since contributions from unphysical massive modes to the Green functions do not cancel, in conflict with BRST invariance. On the other hand, in the spacetime supersymmetric case, contributions from spacetime fields of opposite spacetime statistics cancel. Thus, just as anomaly cancelation requires SO(32) for the open superstring, anomaly cancelation for the $\mathcal{N}=2$ strings requires spacetime supersymmetry.

At one loop, sewing can be replaced with a unitarity construction based on dimensional continuation. In the remainder of this section we examine this construction on the tree-amplitudes obtained through the field theory describing the classical dynamics. (The self-dual Lagrangians reproduce the classical scattering of the $\mathcal{N}=2$ string.) Then two-particle unitarity cuts are sufficient to determine the complete amplitudes through an integral reduction of one-loop integrals onto a finite set of integral functions together with their unique cut structures. The unitarity cut in a two-particle channel containing momenta $k_1+\cdots+k_m$ (with $\Sigma_{j=1}^n k_j=0$) is

$$\mathcal{I}\mathrm{m}\mathcal{A}_n^{[2]}(k_i)=\sum_{\lambda_i=\pm}\int d^dp\,d^dq\,\delta^{(d)}(p^2)\,\delta^{(d)}(q^2)\Theta(p_0)\Theta(q_0)$$
$$\times\mathcal{A}^{\star}_{m+2}(p^{\lambda_1},k_1,\ldots,k_m,q^{\lambda_2})$$
$$\times\mathcal{A}_{n-m+2}(q^{-\lambda_2},k_{n-m},\ldots,k_1,p^{-\lambda_1}),\quad(5.9)$$

with helicity states denoted by λ for an internal state of spin two. (The identical helicity configuration generates the one-loop amplitude to the self-dual gravity theories, exact in the two-field formulation.) Scattering of external states of identical out-going helicity has tree amplitudes on either side of the unitarity cut that equal zero: The tree amplitudes in four-dimensions with helicity configuration $\mathcal{A}(\pm,+,\ldots,+)$ equal zero at n-point. The absence of unitarity cuts is manifest in the functional form (3.1); continuation to $d=2+2$ dimensions results in the reality of the inner products $\langle ij\rangle$ and $[ij]$ and the amplitude is then purely real for all kinematic configurations.

Although the imaginary part of the amplitude in Eq. (5.9) is equal to zero in four dimensions, upon continuing to $d=4-2\epsilon$ dimensions it picks up a non-vanishing result (see, for example, [39,21]). This unitarity construction leads to a direct evaluation of the MHV amplitude in different dimensions, and the integration measure over the cut loop momenta with the integration d^dl. However, this result is in contradiction to the assumption of conformal invariance in the bosonic $\mathcal{N}=2$ string.

The sewing relation in string theory (5.9) precludes the moduli associated with the $U(1)$ gauge field and the ghost determinant, i.e., the integration over the boundary conditions $(\alpha\beta)$ in Eq. (4.11). In the relation (5.9) we project onto a finite number of massless states, found by summing discretely over the spin structures. The target space-time implementation in the sewing of integrating out these fields (and the induced spectral flow) requires a summation over intermediate massless states with spin ranging continuously from zero to two. In this construction of the loop the absence of the τ_2 factor from the $(b'c')$ ghost determinant in Eq. (4.8) forces the amplitude to be evaluated in four dimensions, and a choice of the spin structure in the anti-periodic–anti-periodic, $\alpha=\beta=0$, sector maps the result to the non-vanishing gravitational MHV amplitudes. (The periodic spin structures generate a vanishing contribution holomorphically to the amplitude [6] point-wise in the integration over τ.) Indeed, the Koba-Nielsen representation of the amplitude from the $\mathcal{N}=1$ string is identical to that from the $\mathcal{N}=2$ string without this factor; the $\mathcal{N}=1$ string has a manifest global $\mathcal{N}=2$ supersymmetry. The sum over the intermediate states in Eq. (5.9) with the appropriate supersymmetric spectrum makes agreement with the direct path integral quantization including the complete set of determinant factors in Eq. (4.10).

The tree amplitudes of the string theory agree with those of self-dual theories: nonvanishing three-point amplitudes and vanishing higher-point. Sewing these trees, in either a field theory or string description, straightforwardly produces vanishing one-loop amplitudes for the space-time supersymmetric theories, but nonvanishing amplitudes in the bosonic theories. Since the assumption of modular invariance of the one-loop amplitudes implies vanishing amplitudes, the one-loop amplitudes violate conformal invariance unless the theory is spacetime supersymmetric: The one-loop amplitudes as derived by sewing are not modular invariant in the nonsupersymmetric theories.

VI. CONCLUSIONS

In this work we demonstrated the global conformal anomaly within the $\mathcal{N}=2$ closed string. This conformal anomaly is potentially related to an index in the $\mathcal{N}=2$ string theory. Cancelation of the anomaly requires, through unitarity or sewing, that the theory be space-time supersymmetric. The supersymmetric extension of the $\mathcal{N}=2$ closed string is accomplished via the attachment of line factors on the vertex operators labeling spin on the scalar state in the spectrum. The bosonization of states of different spin in the field theory, or the spectral flow of the $\mathcal{N}=2$ system, allows the incorporation of supersymmetry through these factors. A similar analysis may be performed in the open string.

We further analyzed duality between open and closed string world sheets in the context of self-dual Yang-Mills theory and gravity and its supersymmetric completion. Duality in this context and in a theory containing only a finite number of fields requires supersymmetry and a vanishing *S*-matrix. This is analyzed directly in the field theory by showing that a general zero-momentum insertion into the maximally helicity violating tree amplitude in Yang-Mills theory generates zero, in accord with the supersymmetric closed self-dual gravity at the quantum level to all orders in the loop expansion.

The fact that the nonvanishing *n*-point maximally helicity violating loop amplitudes are due to such a simple anomaly suggests that an even simpler derivation of these amplitudes might be possible, analogous to the way the effective action of the Schwinger model follows simply from its anomaly. The Liouville multiplet offers an avenue to compensate the factor of τ_2 in the closed string calculation to obtain these amplitudes. Furthermore, the unintegrated zero momentum operator insertion noted in [6], i.e., $\sqrt{g}\partial X^\mu \bar{\partial} X_\mu$, representing the insertion of a tadpole would produce a factor of τ_2 necessary to obtain these amplitudes. A simple way to see this is to note that the one-loop amplitude can be written in terms of the trace of the Schwinger parametrized (τ_2) propagator, and an extra τ_2 corresponds to squaring the propagator,

$$\mathrm{tr}(K^{-1}) = \mathrm{tr}\left(\int_0^\infty d\tau\, e^{-\tau K} \right), \quad \mathrm{tr}(K^{-2}) = \mathrm{tr}\left(\int_0^\infty d\tau\, \tau e^{-\tau K} \right), \tag{6.1}$$

which is the same as inserting a zero-momentum state between the two propagators, i.e., a tadpole insertion.

Note added. The integration measure containing the spin structure summation that generates the following results is

$$\frac{d^2\tau}{\tau_2^2}, \quad \frac{du\,d\bar{u}}{\tau_2^2} = \frac{d\alpha\, d\beta}{\tau_2}. \tag{6.2}$$

It may be found by inserting into the *torus* amplitude (with the Jacobian torus associated with the gauge fields, or equivalently the spin structures, fibred on it) the ghost modes associated with the Beltrami differentials, $c\bar{c}b\bar{b}\tilde{c}\bar{\tilde{c}}\tilde{b}\bar{\tilde{b}}$; in this manner the zero modes of both the (bc) and $(b'c')$ system are treated on an equal footing.[7]

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[1] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, and R. Pettorino, Phys. Lett. **62B**, 105 (1976); M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, R.P. Pettorino, and J. Schwarz, Nucl. Phys. **B111**, 77 (1976); D.J. Bruce, D.B. Fairlie, and R.G. Yates, *ibid.* **B108**, 310 (1976); E.S. Fradkin and A.A. Tseytlin, Phys. Lett. **106B**, 63 (1981); A. D'Adda and F. Lizzi, Phys. Lett. B **191**, 85 (1987).

[2] H. Ooguri and C. Vafa, Nucl. Phys. **B361**, 469 (1991).

[3] N. Marcus, Nucl. Phys. **B387**, 263 (1992).

[4] W. Siegel, Phys. Rev. D **47**, 2504 (1993).

[5] S.J. Gates, Jr., H. Nishino, and S.V. Ketov, Phys. Lett. B **297**, 99 (1992).

[6] G. Chalmers, O. Lechtenfeld, and B. Niemeyer, Nucl. Phys. **B591**, 39 (2000).

[7] W. Siegel, Phys. Rev. D **46**, R3235 (1992).

[8] G. Chalmers and W. Siegel, Phys. Rev. D **54**, 7628 (1996).

[9] W. Siegel, Phys. Rev. D **47**, 2512 (1993).

[10] N. Berkovits and C. Vafa, Nucl. Phys. **B433**, 123 (1995); N. Berkovits, Phys. Lett. B **350**, 28 (1995).

[11] K. Jünemann and O. Lechtenfeld, Phys. Lett. B **474**, 58 (2000).

[12] E. D'Hoker and D.H. Phong, Rev. Mod. Phys. **60**, 917 (1988).

[13] S.V. Ketov and O. Lechtenfeld, Phys. Lett. B **353**, 463 (1995).

[14] M. Bonini, E. Gava, and R. Iengo, Mod. Phys. Lett. A **6**, 795 (1991).

[15] S.D. Mathur and S. Mukhi, Nucl. Phys. **B302**, 130 (1988); J. Bischoff and O. Lechtenfeld, Int. J. Mod. Phys. A **12**, 4933 (1997).

[16] J. Bischoff, S.V. Ketov, and O. Lechtenfeld, Nucl. Phys. **B438**, 373 (1995).

[17] E. Witten, Nucl. Phys. **B142**, 285 (1978).

[18] E. Witten, Phys. Lett. **117B**, 324 (1982).

[19] Z. Bern, G. Chalmers, L. Dixon, and D.A. Kosower, Phys. Rev. Lett. **72**, 2134 (1994).

[20] G. Mahlon, Phys. Rev. D **49**, 4438 (1994).

[21] Z. Bern, L. Dixon, M. Perelstein, and J.S. Rozowsky, Phys. Lett. B **444**, 273 (1998).

[22] G. Chalmers and W. Siegel, hep-th/9712191; Phys. Rev. D **59**, 045013 (1999).

[23] S.J. Parke and T.R. Taylor, Phys. Rev. Lett. **56**, 2459 (1986); F.A. Berends and W.T. Giele, Nucl. Phys. **B306**, 759 (1988).

[24] J.F. Plebanski, J. Math. Phys. **16**, 2395 (1975); M.J. Duff, in *Proceedings of 1979 Supergravity Workshop*, edited by P. van Nieuwenhuizen and D.Z. Freedman (North-Holland, Amsterdam, 1979); J.F. Plebanski and M. Przanowski, Phys. Lett. A **212**, 22 (1996); A. Jevicki, M. Mihailescu, and J.P. Nunes, Chaos, Solitons Fractals **10**, 385 (1999).

[25] W.A. Bardeen, Prog. Theor. Phys. Suppl. **123**, 1 (1996).

[26] D. Cangemi, Nucl. Phys. **B484**, 521 (1997).

[27] R. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); L.M. Brown, *ibid.* **111**, 957 (1958); M. Tonin, Nuovo Cimento **14**, 1108 (1959); A.G. Morgan, Phys. Lett. B **351**, 249 (1995);

G. Chalmers and W. Siegel, Phys. Rev. D **59**, 045012 (1999); M. Veltman, Acta Phys. Pol. B **29**, 783 (1998).

[28] D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. **B271**, 93 (1986).

[29] K. Junemann and O. Lechtenfeld, Commun. Math. Phys. **203**, 53 (1999).

[30] F.A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and T.T. Wu, Phys. Lett. **103B**, 124 (1981); P. De Causmaecker, R. Gastmans, W. Troost, and T.T. Wu, Nucl. Phys. **B206**, 53 (1982); R. Kleiss and W.J. Stirling, *ibid.* **B262**, 235 (1985); J.F. Gunion and Z. Kunszt, Phys. Lett. **161B**, 333 (1985); Z. Xu, D.-H. Zhang, and L. Chang, Nucl. Phys. **B291**, 392 (1987).

[31] M.T. Grisaru, H.N. Pendleton, and P. van Nieuwenhuizen, Phys. Rev. D **15**, 996 (1977); M.T. Grisaru and H.N. Pendleton, Nucl. Phys. **B124**, 81 (1977); S. Mandelstam, *ibid.* **B213**, 149 (1983); S.J. Parke and T.R. Taylor, Phys. Lett. **157B**, 81 (1985); Z. Kunszt, Nucl. Phys. **B271**, 333 (1986).

[32] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

[33] D.J. Gross, A. Neveu, J. Scherk, and J.H. Schwarz, Phys. Lett. **31B**, 592 (1970); C. Lovelace, *ibid.* **34B**, 500 (1971); E. Cremmer and J. Scherk, Nucl. Phys. **B50**, 222 (1972).

[34] M.T. Grisaru and J. Zak, Phys. Lett. **90B**, 237 (1980); Z. Bern, D.C. Dunbar, and T. Shimada, Phys. Lett. B **312**, 277 (1993); D.C. Dunbar and P.S. Norridge, Nucl. Phys. **B433**, 181 (1995).

[35] R.P. Feynman, Acta Phys. Pol. **24**, 697 (1963).

[36] S. Mandelstam, Phys. Rep., Phys. Lett. **13**, 259 (1974).

[37] K. Kikkawa, B. Sakita, and M.A. Virasoro, Phys. Rev. **184**, 1701 (1969).

[38] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987), Vol. 2, Sec. 8.1.1.

[39] Z. Bern, L. Dixon, D.C. Dunbar, and D.A. Kosower, Phys. Lett. B **394**, 105 (1997).