Multidomain walls in massive supersymmetric sigma models

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Massive maximally supersymmetric sigma models are shown to exhibit multiple static kink-domain wall solutions that preserve 1/2 of the supersymmetry. The kink moduli space admits a natural Kähler metric. We examine in some detail the case when the target of the sigma model is given by the co tangent bundle of $\mathbb{CP}^n$ equipped with the Calabi metric, and we show that there exist BPS solutions corresponding to n kinks at arbitrary separation. We also describe how 1/4-BPS charged and intersecting domain walls are described in the low-energy dynamics on the kink moduli space. We comment on the similarity of these results to monopole dynamics.

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I. INTRODUCTION

Models admitting vortices, lumps, monopoles or instantons typically have Bogomol'nyi-Prasad-Sommerfield (BPS) limits in which the forces between the objects cancel, resulting in a moduli space of static multisoliton solutions. The structure of these moduli spaces carries important kinematical and dynamical information about the solitons. Moreover, they have interesting mathematical properties and appear ubiquitously in string theory. It is thus natural to enquire about the possibility of scalar field theories that might exhibit multikink solutions with similarly interesting moduli spaces.

Consider models with BPS kink solutions with energy $E=|Z|$, where E is the energy per unit area of the wall and Z is a real central charge appearing in the supersymmetry algebra. The BPS energy bound for two parallel domain walls is obviously saturated when they are infinitely separated, so reducing the separation cannot decrease the energy. It follows that the force between the walls at large separation is either repulsive or zero. This force can be calculated [1] and for models with only a single scalar field it is always repulsive. Thus, while there may exist time dependent multidomain wall solutions (such as the kink-antikink breather of the sine-Gordon model), these models contain no static multidomain wall solutions in which the separation may be chosen arbitrarily.

If we consider kinks carrying a complex, or vectorial, central charge then two kinks with non-parallel charges may exert an attractive force on each other, in which case they will eventually fuse into a third kink carrying a central charge that is the vector sum of the charges of the initial two kinks. However, it is also possible that two kinks with non-parallel charges will repel each other. Which possibility is realized depends on the details of the model; a Wess-Zumino model in which walls repel or attract according to the choice of parameters in the superpotential was studied in [2].

The above comments indicate that the simplest models admitting multi BPS kink solitons should have several scalar fields. Multi-kink solutions have been found in generalized Wess-Zumino models [3,4]. However, these theories have four supersymmetries which is not sufficient to endow the resulting kink moduli spaces with a great deal of geometric structure. The only field theories with *eight* supersymmetries that admit static kink solutions are the ''massive'' supersymmetric hyper-Kähler sigma models, so it is to these models that we turn our attention. These typically admit not only kinks, and their charged counterparts, the Q-kinks [5], but also a variety of other BPS solutions, such as Q-lumps [6], intersecting domain walls [7] and D-branes [8]. The purpose of this paper is to exhibit and study a class of massive hyper-Kähler sigma models that admit *multi-kink* (and multi-Q-kink) solutions, for which the moduli space is Kähler.

One might suspect that the cancellation of inter-kink forces that is needed for static multi-kink solutions to exist is a direct consequence of the 8 supersymmetries, but this is certainly not the case. To see why, consider the sigma model with a target space metric given by the multicenter asymptotically locally Euclidean (ALE) 4-metric

$$ds^2=U d\mathbf{X}\cdot d\mathbf{X}+U^{-1}(d\psi+\boldsymbol{\omega}\cdot d\mathbf{X})^2 \tag{1}$$

where $\nabla\times\boldsymbol{\omega}=\nabla U$. This metric has a tri-holomorphic isometry associated to the Killing-vector field ∂_ψ, and the ''centers'' of the metric are the isolated fixed points of this vector field. The norm of ∂_ψ is, up to a multiplicative factor, the unique scalar potential term (for this model) that is compatible with all 8 supersymmetries [9]. The choice of multiplicative factor corresponds to a choice of mass units, so we may take the potential to be

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$$V = \tfrac{1}{2} U^{-1}. \tag{2}$$

The addition of this potential to the action yields a ''massive'' sigma model with isolated vacua at the centers of the metric. For N colinear centers the harmonic function U is given by

$$U = \sum_{i=1}^{N} \frac{1}{|\mathbf{X} - m_i \mathbf{n}|}, \tag{3}$$

where $\mathbf{n}$ is a unit 3-vector and we may order the centers such that $m_i < m_{i+1}$. The N vacua are given by $\mathbf{X} = m_i \mathbf{n}$, and there exist BPS domain walls interpolating between any pair of *adjacent* vacua [10], each of which preserves (the same) half of supersymmetry. However, $\mathbf{X}\cdot\mathbf{n}$ is the only ''active'' sigma-model field of these solutions, so the calculation of the force between two widely separated kinks reduces to a calculation similar to that of [1] for models with only a single scalar field. This force is non-vanishing. Thus finitely separated domain walls interpolating between non-adjacent vacua do not exist in this model.

These considerations suggest that one will need to consider higher-dimensional target spaces to find multi-kink solutions in massive hyper-Kähler sigma models. Here we consider models for which the target space metric is a hyper-Kähler Calabi metric on the co-tangent bundle $T^\star(\mathcal{N})$, where $\mathcal{N}$ is a compact Kähler manifold of complex dimension n. If $\mathcal{N}$ admits a holomorphic killing vector then we may construct a supersymmetric massive sigma-model on $T^\star(\mathcal{N})$ with 8 supercharges. In fact, the kink solutions of this model actually lie within the zero-section of the tangent bundle. In other words, they are also solutions to the massive sigma-model with 4 supercharges on $\mathcal{N}$. In the following section, we discuss several features of kink solutions in these models. The BPS equations describing the spatial and temporal variation of a domain wall coincide with the Morse and Hamiltonian flows of the Killing potential on $\mathcal{N}$, respectively. The domain wall moduli space is therefore identified with the space of Morse flows with given fixed points. It is a non-compact manifold with a natural Kähler metric.

The simplest Calabi metric has $\mathcal{N} = \mathbb{CP}^n$. In the remainder of the paper we discuss in detail the domain wall solutions of this model. As we shall see in Sec. III, the potential allowed by supersymmetry generically has $(n+1)$ isolated vacua, and hence we take $n \geqslant 2$. As with the ALE 4-metrics, these vacua have a natural linear ordering. However, in contrast to the ALE case, the domain walls are not restricted to lie between adjacent vacua. Rather, we shall exhibit explicit BPS kink solutions interpolating between *each* pair of vacua. Moreover, we shall show that the solution which interpolates between the I^{th} and J^{th} vacua is part of a moduli space of solutions of dimension $2|I-J|$. We show that the collective coordinates on this space may be thought of as the position, together with an internal degree of freedom, of $|I-J|$ *fundamental* domain walls, each of which interpolates between neighboring vacua.

In Sec. IV, we discuss the dynamics of domain walls in the $T^\star(\mathbb{CP}^n)$ model. We show that the moduli space metric is toric Kähler. We further discuss the dynamics of domain walls in the presence of two or more potentials and argue that it is given by a massive sigma model on the kink moduli space. We explain how this allows one to describe 1/4-BPS Q-kinks and 1/4-BPS intersecting domain wall solutions in these theories.

We end in Sec. V with a discussion. We comment on the similarities of these results to those for monopole dynamics and mention an application to string theory.

II. DOMAIN WALLS AND MORSE FLOWS

Let us first consider a sigma model with 4 supercharges in $D \leqslant 4$ space-time dimensions with compact target space $\mathcal{N}$ of complex dimension n. We endow $\mathcal{N}$ with a Kähler metric, g, and denote the Kähler form by Ω and the complex structure by J. In $D \leqslant 3$ dimensions there exists a deformation of this theory, consistent with supersymmetry, given by the addition of a potential

$$V = \tfrac{1}{2}\mu^2 k^2 \tag{4}$$

where μ is a mass parameter and k is holomorphic Killing vector field, which we assume to have only isolated, non-degenerate, fixed points.

The one-form $i_k\Omega$ (the contraction of k with Ω) is closed because

$$d(i_k\Omega) = (di_k + i_k d)\Omega \equiv \mathcal{L}_k \Omega = 0, \tag{5}$$

where the first equality follows from the closure of Ω and the second equality from the holomorphicity of k; it follows that

$$dH = i_k \Omega, \tag{6}$$

for some locally-defined Killing potential H. The integral of $i_k\Omega$ is a topological charge equal to the difference between the values of H at the two endpoints. This topological charge can support a BPS kink, which also has a dyonic generalization known as a Q-kink, carrying a Noether charge associated to the Killing vector field k. Denoting by ϕ^i the coordinates on $\mathcal{N}$, the energy density is given by

$$\mathcal{E} = \frac{1}{2}\int \mathrm{d}x\, g_{ij}(\dot\phi^i \dot\phi^j + \phi^{i\prime}\phi^{j\prime}) + \mu^2 g_{ij} k^i k^j. \tag{7}$$

This may be rewritten as

$$\mathcal{E} = \int \Big\{ \mathrm{d}x\, \tfrac{1}{2} g_{ij}(\phi^{i\prime} + \mu\cos\alpha J^i{}_k k^k)(\phi^{j\prime} + \mu\cos\alpha J^j{}_l k^l) + \mu\cos\alpha \frac{\partial H}{\partial x} + \tfrac{1}{2} g_{ij}(\dot\phi^i - \mu\sin\alpha k^i)(\dot\phi^j - \mu\sin\alpha k^j) + \mu\sin\alpha\, \dot\phi^i k_i \Big\} \tag{8}$$

for arbitrary angle α. Maximizing the right-hand side with respect to α, we deduce the Bogomol'nyi bound,

$$\mathcal{E} \geqslant \mu\sqrt{T^2 + Q^2} \tag{9}$$

where

$$T=[H]_{-\infty}^{+\infty}, \quad Q=\int \mathrm{d}x\, \dot{X}^i k_i \tag{10}$$

are the topological and Noether charges respectively. The Bogomol'nyi bound is saturated for solutions of the equations

$$\dot{\phi}^i=\mu \sin \alpha k^i \tag{11}$$

$$\phi^{i\prime}=-\mu \cos \alpha J^i{}_k k^k .$$

Both of these equations have a natural geometrical meaning. Up to rescaling, the temporal evolution of the fields is determined by treating H as a Hamiltonian,

$$\dot{\phi}^i=\frac{\partial H}{\partial \phi^j}\Omega^{ij}. \tag{12}$$

The spatial evolution arises by treating H as a Morse function on $\mathcal{N}$. The non-degeneracy of Ω ensures that H is a good Morse function with critical points at the fixed points of k. The Morse flow is

$$\phi^{i\prime}=\frac{\partial H}{\partial \phi^j}g^{ij} \tag{13}$$

which, again up to rescaling, coincides with the spatial Bogomol'nyi equation [11]. Note that the Morse and Hamiltonian flows on $\mathcal{N}$ are orthogonal. In the remainder of this section, we discuss the time independent Morse flows in more detail.

First note that the critical points of H are in one-to-one correspondence with the vacua of the potential (4). At each point some flows will depart while others will terminate. The dimension, p, of the hypersurface formed by the Morse flows departing from a given critical point is known as the Morse index of that point, and it is equal to the number of negative eigenvalues of the covariant Hessian $(D^2H/D\phi^i D\phi^j)$ at that point. As we assumed the fixed points of k to be non-degenerate, this guarantees that the hypersurface formed by the flows terminating at a fixed point of Morse index p will have dimension $(2n-p)$. Since $\mathcal{N}$ is Kähler, p is even and moreover for the function H, the usual Morse inequalities are saturated; it follows that the number of fixed points with Morse index p is equal to the Betti number B_p. In particular, there exists a single critical point with Morse index $2n$, from which flows only depart, and a single critical point with Morse index 0, from which no flows depart.

As a solution to the sigma model with four supercharges and target space $\mathcal{N}$, a kink interpolating between a vacuum of index p and a vacuum of index p' has $|p-p'|$ fermionic zero modes [11]. Of these only two arise from broken supersymmetries. The remainder are ''accidental.'' The unbroken supersymmetries then ensure the existence of $|p-p'|$ real bosonic zero modes, and hence $|p-p'|$ bosonic collective coordinates. The physical interpretation of one of these is as the center-of-mass position of the domain wall; its complex partner is an angle conjugate to the total Noether charge. This pair of collective coordinates partner the two Nambu-Goldstone fermions arising from the two broken supersymmetries.

What is the physical interpretation of the remaining $|p-p'|-2$ bosonic collective coordinates? They could either correspond to further internal degrees of freedom or, alternatively, to the relative positions and internal coordinates of more than one domain wall. Let us see under which circumstances we may expect the latter interpretation. Suppose we have three critical points with indices p, q and p' such that $p>q>p'$. Suppose further that there exists a Morse flow Γ from $p\rightarrow q$ and a second Morse flow Γ' from $q\rightarrow p'$. Then, by continuity, we expect there to exist a flow from $p\rightarrow p'$ which is close to $\Gamma\cup\Gamma'$. The speed of this flow, determined by Eq. (13), reduces in the vicinity of the critical point q, ensuring that the energy density profile of the solution looks like two well separated domain walls sandwiching the vacuum q.

Let u^a, $a=1,\dots,|p-p'|$ be the collective coordinates. These are promoted to fields of the low-energy effective action for the (multi) kink domain wall. This low-energy dynamics is again a sigma model but now with a target space metric supplied by the usual metric on the soliton moduli space,

$$G_{ab}=\int \mathrm{d}x \frac{\partial \phi^i}{\partial u^a}\frac{\partial \phi^j}{\partial u^b}g_{ij}, \tag{14}$$

which may be thought of as a metric on the space of Morse flows. This metric is Kähler. To see this we first note that the low-energy effective action of the multi-kink domain wall is again a supersymmetric sigma model, with the metric (14) as its target space metric. Next, we recall that our sigma model with four supersymmetries and target space $\mathcal{N}$ may be embedded into a sigma model with 8 supercharges and target space $T^\star(\mathcal{N})$. The kink solutions now have $2|p-p'|$ fermionic zero modes and preserve four of the eight supersymmetries, so the effective kink sigma-model with target metric (14) has four supersymmetries. If we choose the maximal spacetime dimension, $D=5$, for the original massive HK sigma-model then we will have an effective $D=4$ supersymmetric sigma-model governing the low energy dynamics of the kink domain walls in this $D=5$ spacetime. The target space of such a sigma model is necessarily Kähler.

III. DOMAIN WALLS IN $T^\star(\mathbb{CP}^n)$

In this section we discuss in detail the domain walls for $\mathcal{N}=\mathbb{CP}^n$, working with the toric HK $4n$-metric on $T^\star(\mathbb{CP}^n)$ with coordinates $(\mathbf{X}^I,\psi^I)$ $(I=1,\dots,n)$. The Calabi metric is

$$ds^2=U_{IJ}d\mathbf{X}^I\cdot d\mathbf{X}^J+(U^{-1})^{IJ}(d\psi_I+A_I)(d\psi_J+A_J) \tag{15}$$

where

$$A_I=d\mathbf{X}^J\cdot \boldsymbol{\omega}_{JI}, \quad \boldsymbol{\nabla}_{(J}\times \boldsymbol{\omega}_{K)I}=\boldsymbol{\nabla}_J U_{KI}. \tag{16}$$

The functions U are given by

$$U_{IJ}=\frac{\delta_{IJ}}{X^I}+\frac{1}{\left|\mathbf{m}-\sum_{K=1}^{N}\mathbf{X}^K\right|} \tag{17}$$

where $\mathbf{m}$ is a constant 3-vector and the lack of I,J indices in the second term implies that it appears in each component of the matrix. The triplet of Kähler forms are

$$\mathbf{\Omega}=(\mathrm{d}\psi_I+A_I)\mathrm{d}\mathbf{X}^I-\tfrac{1}{2}U_{IJ}\mathrm{d}\mathbf{X}^I\times\mathrm{d}\mathbf{X}^J \tag{18}$$

where the wedge product of forms is implicit. This metric appears in physics as the moduli space of a single $U(n+1)$ instanton on non-commutative $\mathbb{R}^4$, where the 3-vector $\mathbf{m}$ is related to the (anti-self-dual) non-commutativity parameter (see, for example, [12]). In particular, the $n=1$ Calabi 4-metric coincides with the Eguchi-Hanson metric on the $N=2$ ALE space (1). The $4n$-metric has $SU(n+1)$ triholomorphic isometry. In the above coordinates only the Cartan sub-algebra is manifest corresponding to the Killing vector fields $k^I=\partial/\partial\psi_I$. These permit the construction of a potential compatible with supersymmetry given by the square of the length of a linear combination of these vectors [9], say $\mu_I k^I$ for constant μ_I,

$$V=\tfrac{1}{2}\mu_I\mu_J(U^{-1})^{IJ}. \tag{19}$$

In fact, as shown in [13,7], this is not the most general potential allowed by supersymmetry. For theories with $D\leqslant 6$ space-time dimensions one may sum the squares of the lengths of $(6-D)$ independent, mutually commuting, triholomorphic Killing vectors. In the following section we will consider this possibility, but for now we restrict ourselves to the simplest potential given in Eq. (19).

It will prove useful to define a $(n+1)^{\text{th}}$ coordinate,

$$\mathbf{X}^{n+1}=\mathbf{m}-\sum_{I=1}^{n}\mathbf{X}^I \tag{20}$$

so that $\Sigma_{I=1}^{n+1}\mathbf{X}^I=\mathbf{m}$. The potential (19) is given explicitly by

$$V=\tfrac{1}{2}\sum_{I=1}^{N}(\mu_I^2X^I)-\tfrac{1}{2}\frac{\left(\sum_{I=1}^{N}\mu_IX^I\right)^2}{\sum_{J=1}^{n+1}X^J}. \tag{21}$$

Note that the denominator of the second term is not given by $\Sigma_{J=1}^{n+1}X^J=m$ unless $\mathbf{X}^I\cdot\mathbf{m}\geqslant 0$ for each I. In fact, this constraint on the coordinates is precisely the restriction to the $\mathbb{CP}^n$ base of the manifold. We shall not impose this constraint for now, although we shall later see that all BPS kinks do in fact lie within this submanifold.

For generic choice of constants, $\mu_I\neq\mu_J$, the potential (21) has $n+1$ isolated vacua, given by

$$\mathbf{X}^I=\mathbf{m}\,\delta^{IJ}\quad\text{for}\quad J=1,\ldots,n+1. \tag{22}$$

For non-generic potentials there is an enlarged moduli space of vacua. Specifically, if l of the constants μ_I coincide, then there is a $3(l-1)$ dimensional sub-manifold of the Calabi metric with vanishing potential. We will consider only generic potentials and examine the kinks that interpolate between the different isolated vacua. The relevant Morse function is $H=\Sigma_{I=1}^{n}\mu_I\mathbf{X}^I\cdot\mathbf{n}$. Setting all time derivatives to zero, the Bogomol'nyi equations are

$$\mathbf{X}^{I\prime}=(U^{-1})^{IJ}\mu_J\mathbf{n} \tag{23}$$

$$\psi^{I\prime}=\boldsymbol{\omega}_{IJ}\cdot\mathbf{X}^{J\prime} \tag{24}$$

where the unit 3-vector $\mathbf{n}=\pm\mathbf{m}/m$ depending on whether we are considering a kink or anti-kink. A BPS kink interpolating between the I^{th} and J^{th} vacua, with $I,J=1,\ldots,n$ has energy,

$$E_{IJ}=m|\mu_I-\mu_J| \tag{25}$$

while a kink which interpolates between the I^{th} vacuum and the $(n+1)^{\text{th}}$ vacuum has mass,

$$E_{I,n+1}=m|\mu_I|. \tag{26}$$

We may write these formulas in a unified form if we introduce the $(n+1)$ quantities ν_I such that

$$\mu_I=\nu_I-\nu_{n+1}\qquad(I=1,\ldots,n) \tag{27}$$

and the mass of a kink interpolating between the I^{th} and J^{th} vacua is now given by

$$E_{IJ}=m|\nu_I-\nu_J|. \tag{28}$$

Importantly, rewriting the energy in this fashion also makes it clear that there is an ordering to the vacua given by the relative values of ν_I, allowing us to talk of neighboring, or adjacent, vacua. We choose the ordering $\nu_I>\nu_{I+1}$. Notice that the form of the energy (28) is already suggestive of the existence of multikink solutions since, assuming $J<I$, we may write

$$E_{IJ}=\sum_{K=J}^{I-1}E_{K+1,K}. \tag{29}$$

Taken at face value, this suggests that the kink may be decomposed into $d=(I-J)$ kinks, each of which interpolates between neighboring vacua. We will refer to the kink that interpolates between the I^{th} and $(I+1)^{\text{th}}$ vacua as the I^{th} *fundamental* kink. An analysis of the supersymmetry transformations [7] reveals that each of these fundamental kinks preserves the same half of supersymmetry, as would be expected if multikink solutions were to exist. However, one must be wary in drawing such conclusions from the Bogomol'nyi energy bound alone. Indeed, all the above statements apply equally well to kinks in the ALE 4-metrics discussed in the introduction but, as we noted there, in this case there simply do not exist BPS domain wall solutions interpolating between non-adjacent vacua. Nevertheless, we shall see that in the present case the above conclusions are in fact correct.

We start our analysis of the Bogomol'nyi equations by presenting explicit solutions between *any* pair of vacua with $J<I$,

$$\mathbf{X}^K \to \mathbf{m}\,\delta^{KI} \quad \text{as} \quad x\to -\infty$$
$$\mathbf{X}^K \to \mathbf{m}\,\delta^{KJ} \quad \text{as} \quad x\to +\infty \tag{30}$$

with $I,J=1,\ldots,n+1$. We make the ansatz $\mathbf{X}^K=0$ for $K\neq I,J$ which, given the constraint (20), requires that the two remaining coordinates sum to $\mathbf{X}^I+\mathbf{X}^J=\mathbf{m}$. Geometrically, this restricts us to a sub-manifold $T^\star(\mathbb{CP}^1)$ where the $n(n+1)$ choices of vacuum pairs reflect the $n(n+1)$ natural embeddings of $\mathbb{CP}^1$ in $\mathbb{CP}^n$. The Bogomol'nyi equations now reduce to those on the Eguchi-Hanson space whose solutions were given in [5],

$$\mathbf{X}^I=\tfrac{1}{2}\mathbf{m}-\tfrac{1}{2}\mathbf{m}\tanh\!\left(\tfrac{1}{2}(\nu_J-\nu_J)(x-x_0)\right)$$
$$\mathbf{X}^J=\tfrac{1}{2}\mathbf{m}+\tfrac{1}{2}\mathbf{m}\tanh\!\left(\tfrac{1}{2}(\nu_J-\nu_I)(x-x_0)\right). \tag{31}$$

Given these solutions, the second Bogomol'nyi equation may be solved by simply choosing a gauge in which $\boldsymbol{\omega}$ vanishes over the trajectory [14] and setting $\psi_I=-\psi_J=\varphi_0$ to constant. Thus this kink solution has 2 collective coordinates given by the position, x_0, and the internal degree of freedom φ_0. We shall now show that the complete solution involves a further $2(d-1)=2(I-J-1)$ collective coordinates, corresponding to the possibility of separating the domain wall (31) into d fundamental kinks.

First we prove that for the domain wall with boundary conditions (30), any solution to the BPS equations necessarily has $\mathbf{X}^K\equiv 0$ for all $K<J$ and for all $K>I$. To see this, note first that the Bogomol'nyi equations require $\mathbf{X}^K\propto\mathbf{n}$ for all K, and so take the form

$$\mathbf{X}^{K\prime}\cdot\mathbf{n}=\left(\nu_K-\frac{\sum_{L=1}^{n+1}\nu_L X^L}{\sum_{M=1}^{n+1}X^M}\right)X^K. \tag{32}$$

Moreover, unlike Eq. (23), this form is also valid for the $(n+1)^{\text{th}}$ coordinate (30). Near the two end points (20) of the domain wall trajectory, these equations approximate to

$$\mathbf{X}^{K\prime}\cdot\mathbf{n}\approx(\nu_K-\nu_I)X^K \quad \text{as} \quad x\to-\infty$$
$$\mathbf{X}^{K\prime}\cdot\mathbf{n}\approx(\nu_K-\nu_J)X^K \quad \text{as} \quad x\to+\infty. \tag{33}$$

Thus we see that for $K<J$ and for $K>I$, the functions X^K must either vanish or have at least two stationary points. Similarly, for $J<K<I$, the functions must have at least one stationary point while for $K=I$ and $K=J$, they may be monotonic. However, from Eq. (32), we see that X^K is stationary at $X^K\neq 0$ only if

$$\sum_{L=1}^{N}(\nu_K-\nu_L)X^L=0. \tag{34}$$

If we first examine $L=1$ (and assume that $J\neq 1$) then $(\nu_1-\nu_L)<0$ for all L and there are no non-trivial solutions to the stationary point equation. Thus $X^1\equiv 0$. By induction, the same is true for all X^K with $K<J$ and $K>I$. Similarly, this analysis implies that X^I and X^J have no stationary points and are therefore monotonic. However, it does not rule out the possibility of stationary points for X^K with $J<K<I$.

The above result allows us to restrict attention to domain walls interpolating between the first and last vacua (i.e. with boundary conditions $J=1$ and $I=n+1$). We will now examine the Bogomol'nyi equations inductively, starting with the simplest model admitting multikink solutions: $T^\star(\mathbb{CP}^2)$.

$n=2$

In the previous section we worked with an over-complete set of variables subject to the constraint (20) in order to elucidate the vacuum structure of the theory. In this subsection, we revert to the original coordinates (15). The ordering of the vacua described in the previous subsection is equivalent to choosing the potential $\mu_1>\mu_2>0$. The BPS equations for $\mathbf{X}^K$ are

$$\mathbf{X}^{1\prime}\cdot\mathbf{n}=\left(\mu_1-\frac{\mu_1}{m}X^1-\frac{\mu_2}{m}X^2\right)X^1$$
$$\mathbf{X}^{2\prime}\cdot\mathbf{n}=\left(\mu_2-\frac{\mu_2}{m}X^2-\frac{\mu_1}{m}X^1\right)X^2. \tag{35}$$

The fixed points of these equations are the vacua (22) of the theory. There are three such points,

$$\text{vacuum 1:}\quad X^1=m,\ X^2=0$$
$$\text{vacuum 2:}\quad X^1=0,\ X^2=m$$
$$\text{vacuum 3:}\quad X^1=0,\ X^2=0. \tag{36}$$

These lie at the three corners of a right-angle isoceles triangle, with the right-angle at fixed point 3. The three BPS kink solutions given in Eq. (31) form the sides of this triangle, with a fixed direction. Specifically, the kinks interpolate between the vacua $2\to 1$, $3\to 2$ and $3\to 1$.

Near the fixed point 3, the trajectories are

$$(X^1,X^2)\approx(e^{\mu_1 x},e^{\mu_2 x}) \tag{37}$$

so, for positive μ_I, all trajectories start with a straight line through the origin into the triangle. Trajectories can only end at fixed points or at infinity. Moreover, they may not cross. Therefore, all those that enter the triangle must end on fixed points. The only one that may end on fixed point 2 is the $X^1=0$ kink. All others must end on fixed point 3, so there is a one-parameter family of trajectories that begin at fixed point 3 and end on fixed point 1.[1] This is sketched in Fig. 1.

Note that the asymmetry between points 1 and 2 arose from the choice $\mu_1>\mu_2$. In one limit of this parameter we

[1] This was also noted by Kimyeong Lee and Piljin Yi [15].

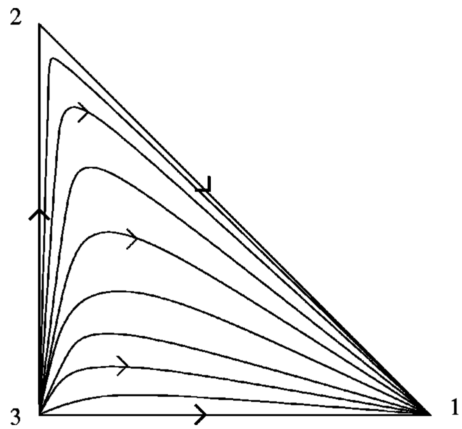

FIG. 1. The BPS flows in the $\mathbb{CP}^2$ massive sigma model. There exists a one-parameter family of kink trajectories corresponding to the separation of two kinks. The two trajectories $3\to2$ and $2\to1$ may be thought of as the limit of infinitely separated kinks. The straight-line trajectory $3\to1$ corresponds to the two kinks with zero separation.

have the straight-line $3\to1$ kink of Eq. (31). In the other limit, we approach arbitrarily close to the union of the trajectories of the $3\to2$ kink and the $2\to1$ kink. This limit itself corresponds to the $3\to2$ and $2\to1$ kinks at infinite separation, but at any point short of this limit the kinks have finite separation. As the separation is decreased, the two kinks eventually merge to form the single $3\to1$ kink. It is natural to call the $3\to2$ and $2\to1$ kinks ''fundamental'' kinks, and the family of $3\to1$ kink solutions as a moduli space of multikink solutions.

The fundamental kinks have a single real relative collective coordinate. Supersymmetry requires that this is paired with a complex partner, such that the relative moduli space is Kähler. This additional collective coordinate comes from the angular coordinates ψ_I, satisfying Eq. (24). The multikink solutions thus have a four-dimensional moduli space of solutions.

All of the kink trajectories lie within the triangle depicted in Fig. 1, ensuring that they may not escape to infinity in field space. This triangle is the toric diagram for $\mathbb{CP}^2$, the zero section of the Calabi bundle (see for example [16]). The two periodic variables ψ_I provide a torus $\mathbf{T}^2$ which fibered over the triangle to reconstruct $\mathbb{CP}^2$. Thus, the kinks described above are equally solutions to the $\mathbb{CP}^2$ sigma model. We will return to this point in Sec. IV.

On each trajectory, there is a unique value of $Y=X^2$ for each value of $X=X^1$. The trajectories can therefore be described by some curve $Y(X)$. To find these curves, we divide the Bogomol'nyi equations (36), to get

$$\mu_1\left(dX^1+\frac{m-X^1}{X^2}dX^2\right)-\mu_2\left(dX^2+\frac{m-X^2}{X^1}dX^1\right)=0. \tag{38}$$

Multiplying by the integrating factor $(m-X-Y)^{-1}$, we deduce that

$$d\log(X^{\mu_2}Y^{-\mu_1}(m-X-Y)^{\mu_1-\mu_2})=0. \tag{39}$$

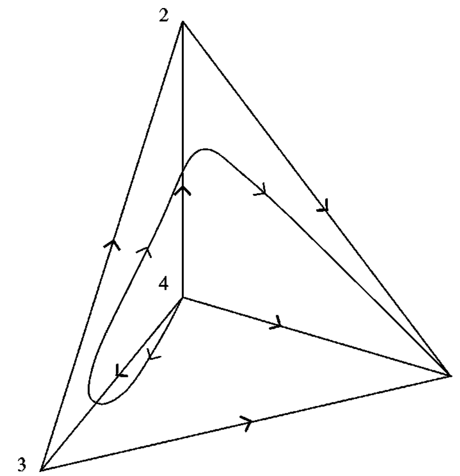

FIG. 2. The BPS flows in the $\mathbb{CP}^3$ massive sigma model. There now exists a two-parameter family of kink trajectories corresponding to the separation of three kinks. The flows on the faces are copies of Fig. 1. A typical trajectory lying within the tetrahedron is drawn.

It follows that the trajectories in Fig. 1 are described by the equation

$$(X^2)^{\mu_1}=c(X^1)^{\mu_2}(m-X^1-X^2)^{\mu_1-\mu_2} \tag{40}$$

where the real modulus $c\geqslant0$ labels the trajectories and is a measure of the separation of the two kinks. The $c=0$ trajectory corresponds to the straight-line $3\to1$ kink, while as $c\to\infty$, the trajectory gets closer and closer to the infinitely separated $3\to2\to1$ trajectories.

$n\geqslant3$

The pattern of kink trajectories described above generalizes simply to the general case. Consider first $n=3$. The vacua now determine the points of a right-angle simplex, with the solutions (31) forming its edges. This is shown in Fig. 2. On each of the four faces of the simplex, the Bogomol'nyi equations reduce to those of the $n=2$ case (36), and the trajectories are therefore restricted to lie in the face, each of which looks like a copy of Fig. 1. An analysis of the Bogomol'nyi equations near the fixed point at the origin (vacuum 4 in the diagram) shows that the trajectories head into the polytope. As each of them cannot escape to infinity without crossing the faces, they must end on a fixed point. Only those trajectories which lie on the $2-3-4$ face will end at vacua 2 and 3 and, of those, only those on the $3-4$ edge will end at vacuum 3. All others end at vacuum 1. A typical trajectory is sketched in Fig. 2. We therefore have a two parameter family of kink solutions. These parameters have the interpretation of the separation between the $4\to3$ kink, the $3\to2$ kink and the $2\to1$ kink. As in the $T^\star(\mathbb{CP}^2)$ case, supersymmetry ensures that these separations are paired with angular collective coordinates arising from the ψ_I.

The generalization of this to $n>3$ is clear. The vacua (22) form the vertices of a n-dimensional simplex, while the solutions (31) form the edges. The trajectories on a face of dimension m are determined by the Bogomol'nyi equations for $T^\star(\mathbb{CP}^m)$ and are restricted to lie on that face. This bounds the trajectories inside the simplex, each of which ends at fixed point 1. Note that in each case the simplex is the toric diagram for $\mathbb{CP}^n$, with the n angular variables ψ_I providing the requisite $\mathbf{T}^n$ fiber. All trajectories lie within the $\mathbb{CP}^n$ base of $T^\star(\mathbb{CP}^n)$ and extend to solutions of the $\mathbb{CP}^n$ sigma-model itself.

One may verify that the functions over the simplex,

$$F(X^I;\alpha_I)=\left(m-\sum_I X^I\right)^{-\Sigma_J\alpha_J}\prod_K (X^K)^{\alpha_K} \tag{41}$$

are constant on BPS trajectories provided that the parameters α_I, $(I=1,\ldots,n)$ satisfy, $\Sigma_I\alpha_I\mu_I=0$. This one constraint on n variables ensures that there is an $(n-1)$ parameter family of $(n-1)$-dimensional hypersurfaces. For each choice of parameters α_I, the family of hypersurfaces parametrized by the value of F fill the n-simplex. Thus, together, F and the $(n-1)$ independent α_I yield an n parameter family of $(n-1)$-dimensional surfaces. Their intersections are the BPS trajectories.

IV. DYNAMICS OF DOMAIN WALLS IN $T^\star(\mathbb{CP}^n)$

In the previous section, we have seen that the Calabi metric on $T^\star(\mathbb{CP}^n)$ admits a $2n$-dimensional moduli space, $\mathcal{M}^n$, of domain wall solutions interpolating between the first and last vacua. Let us denote the collective coordinates parametrizing $\mathcal{M}^n$ by u^a, $a=1,\ldots,2n$. We have argued that, at least asymptotically, these parameters have the interpretation of the position and internal degree of freedom of n fundamental kinks. The low-energy dynamics of these kinks is given by a sigma-model with four supercharges on $\mathcal{M}^n$ with metric given by Eq. (14). Given the smoothness of the domain wall solutions, it seems likely that this metric is complete. On general grounds, we expect the metric to factorize into a center of mass piece, parametrizing the overall position of the kinks, together with the internal degree of freedom arising from shifts of the tri-holomorphic Killing vector field $\mu_I k^I$. The moduli space is thus

$$\mathcal{M}^n=\mathbb{R}\times\frac{\mathbb{R}\times\tilde{\mathcal{M}}^n}{G} \tag{42}$$

where G is a discrete normal subgroup of the isometries. Supersymmetry requirements ensure that the metric (14) is Kähler. Moreover, the symmetries of the original massive sigma-model descend to the low-energy dynamics, ensuring that the metric on $\mathcal{M}^n$ is toric Kähler i.e. admits n holomorphic $U(1)$ isometries.[2] We denote these by l^I, $I=1,\ldots,n$.

There exists a generalization of the static domain walls that we have been considering so far to dyonic domain walls, or Q-kinks [5]. These objects, which are 1/2-BPS [7], carry both topological charge as well as Noether charge associated with the isometry $\mu_I k^I$. They solve the time dependent BPS equations (11). Within the low-energy description of motion on the kink moduli space, they are described by excitations along the $\mathbb{R}$ factor of the numerator in Eq. (42).

We would now like to demonstrate the existence of 1/4-BPS O-kinks and explain how they arise in the low-energy dynamics. The analysis is identical to that of 1/4-BPS monopoles, so we will be brief. These objects are related to the intersecting domain wall solutions discussed in [7]. As in that reference, the important point is that the potential (19) is not the most general potential allowed by supersymmetry. Rather, a HK sigma-model with 8 supercharges in D space-time dimensions admits the sum of $(6-D)$ potentials, each the length squared of a mutually commuting tri-holomorphic Killing vector [13,7]. In order to build 1/4-BPS objects, we require two such potentials and must therefore be in a space-time dimension $D\leqslant 4$, with a target space of dimension $\geqslant 8$. For the Calabi metrics, we take the potential to be of the form

$$V=\tfrac{1}{2}\mu_I\mu_J(U^{-1})^{IJ}+\tfrac{1}{2}\lambda_I\lambda_J(U^{-1})^{IJ}. \tag{43}$$

The Bogomol'nyi equations for the 1/4-BPS Q-kinks are derived thus

$$\begin{aligned}\mathcal{E}=\int dx\{&U_{IJ}(\mathbf{X}^{I\prime}-\mu_K(U^{-1})^{IK}\mathbf{n})\cdot(\mathbf{X}^{J\prime}-\mu_L(U^{-1})^{JL}\mathbf{n})\\&+U_{IJ}\dot{\mathbf{X}}^I\cdot\dot{\mathbf{X}}^J+(U^{-1})^{IJ}(\psi_I'+\boldsymbol{\omega}_{IK}\cdot\mathbf{X}^{K\prime})(\psi_J'+\boldsymbol{\omega}_{JL}\cdot\mathbf{X}^{L\prime})\\&+(U^{-1})^{IJ}(\dot{\psi}_I+\boldsymbol{\omega}_{IK}\cdot\dot{\mathbf{X}}^K-\lambda_I)(\dot{\psi}_J+\boldsymbol{\omega}_{JL}\cdot\dot{\mathbf{X}}^L-\lambda_J)\}\\&+\mu_I[\mathbf{X}^I\cdot\mathbf{n}]_{-\infty}^{+\infty}+\int dx\{\lambda_I(U^{-1})^{IJ}(\dot{\psi}_J+\boldsymbol{\omega}_{JK}\cdot\dot{\mathbf{X}}^K)\}.\end{aligned} \tag{44}$$

The Bogomol'nyi equations are now given by Eqs. (23) and (24), together with

$$\dot{\mathbf{X}}^I=0 \tag{45}$$

$$\dot{\psi}_I=\lambda_I \tag{46}$$

in which case the mass of the Q-kink interpolating between vacua I and J is given by

$$E_{IJ}=m|\nu_I-\nu_J|+\lambda_K Q^K \tag{47}$$

where ν_I are defined in Eq. (27) and

$$Q^K=\int dx(U^{-1})^{KL}\dot{\psi}_L \tag{48}$$

is recognized as the Noether charge associated with the Killing vector field ∂_{ψ_K}.

As is usual for 1/4-BPS states, it is possible to rewrite the Noether charge in terms of a potential on the kink moduli space $\mathcal{M}^n$ associated with the Killing vectors l^I [17]:

$$\lambda_I Q^I=G_{ab}(\lambda_I l^{Ia})(\lambda_J l^{Jb}). \tag{49}$$

[2] Note that the potential (19) breaks the $SU(n+1)$ isometry of the target space to $U(1)^n$ and thus the domain wall moduli space inherits only these Abelian isometries.

Finally, note that there is a single condition relating the topological and Noether charges which ensures that the dyonic state is truly bound rather than, as appears from the energy (47), marginally bound. This relation is

$$\lambda_I[\mathbf{X}^I\cdot\mathbf{n}]_{-\infty}^{+\infty}=\int dx\lambda_I\mu_J(U^{-1})^{IJ}=\mu_I Q^I. \tag{50}$$

The dynamics of 1/4-BPS monopoles has been discussed in [18,19,13,20,21], and for instantons in [22]. In both cases, the low-energy dynamics is described by a massive sigma-model on the soliton moduli space. The same is true here. The relevant potential on $\mathcal{M}^n$ is given by $V=\lambda_I Q^I$ which, by Eq. (49), we know can be expressed as the length of a holomorphic Killing vector, ensuring that the 4 supercharges of the low-energy dynamics are preserved.[3] The 1/4-BPS Q-kinks are then recovered as 1/2-BPS solutions of the low-energy dynamics.

There is another class of 1/4-BPS solitons which may exist in these models, namely orthogonally intersecting domain walls. These were discussed in [7]. In the context of the low-energy dynamics, they occur if the potential V on $\mathcal{M}^n$ has more than one isolated minima. In this case, if the original domain wall had spatial world-volume dimension $\geqslant 2$, we could consider a ''kink-within-a-kink,'' in which we build domain wall within the low-energy effective theory. We do not know at present if the Calabi metrics admit such intersections, but the above observation reduces this question to understanding the $\mathbf{T}^n$ action on $\mathcal{M}^n$.

V. DISCUSSION

We have shown that massive Kähler sigma models with compact target spaces $\mathcal{N}$ and massive hyper-Kähler sigma models with compact target spaces $T^\star(\mathcal{N})$ admit a moduli space of domain wall solutions. We examined these solutions in detail for $\mathcal{N}=\mathbb{CP}^n$ and showed that the collective coordinates of the solution have the interpretation of the positions, together with internal degrees of freedom, of n parallel fundamental domain walls. The domain wall moduli space is identified with the space of Morse flows on $\mathcal{N}$, where the morse function is related to the sigma-model potential on $\mathcal{N}$. There is a natural Kähler metric on this moduli space.

We close with a few applications. First, there is a remarkable similarity between kinks in the $\mathbb{CP}^n$ model and monopoles in $SU(n+1)$ Yang-Mills-Higgs theory. This fact has been noted before [5,23,24] and is underscored in the present work. Specifically, the moduli space of a (1,1, . . . ,1) monopole (n 1's) has a toric HK $4n$-metric [25–28]. Here we have shown that the moduli space of the highest kink in $\mathbb{CP}^n$ has a toric Kähler moduli space of dimension $2n$. Moreover, the construction of 1/4-BPS dyon solutions in the two theories is entirely analogous. In [23,24] this correspondence between kinks and monopoles was made quantitative; it was shown that the BPS mass spectrum of the two dimensional $\mathcal{N}=(2,2)$ $\mathbb{CP}^n$ massive sigma model and the four dimensional $\mathcal{N}=2$ $SU(n+1)$ Yang-Mills theory coincide. This correspondence exists at both classical and quantum levels. Subsequently, it has been understood that the four-dimensional theory also contains ''1/4-BPS-like'' states [20,21].[4] The discussion of Sec. IV suggests that analogous states also exist within the two-dimensional $\mathbb{CP}^n$ sigma model. It would be interesting to verify this by semi-classical methods. Note that if they do exist, the calculation of the central charges performed in [23,24] guarantees that their mass coincides with that of the monopoles.

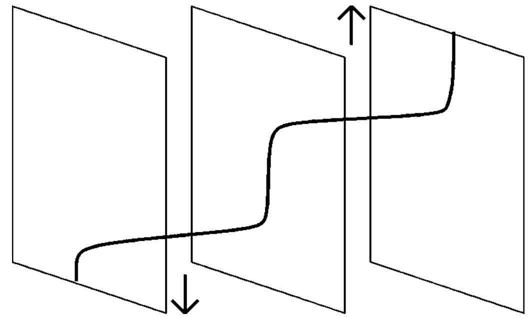

FIG. 3. Twice as kinky: the $D1-D5$ system in background NS B-field. The single D-string has a modulus in which the two kinks move apart as shown by the arrows.

Finally, we mention an application of our results to the $D1-D5$ system of type IIB string theory. Consider a single $D1$-brane in the presence of n parallel but separated $D5$-branes. Turning on a background NS B-field results in an attractive force between the D-string and the $D5$-branes. The D-string has n vacuum states in which it lies within a single $D5$-brane, where it appears as an instanton in noncommutative $U(n)$ gauge theory, broken to the Cartan subalgebra [29]. For small separations between the $D5$-branes, the low-energy dynamics of the D-string is described by the massive sigma-model on $T^\star(\mathbb{CP}^n)$ considered in Sec. III. The kink solutions (31) have the interpretation of the D-string interpolating from one $D5$-brane to another [30,31]. The results of Sec. III make it clear that this kinky string has a moduli in which the two kinks move apart. This is shown in Fig. 3. It would be interesting if one could understand this motion in terms of a gauge theory along the lines of [32].

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[3]Note that the original HK sigma model with two potentials exists in $D\leqslant 4$ space-time dimensions. The kink solutions then have world-volume of dimension $d\leqslant 3$ as is required to construct a massive supersymmetric sigma-model description of the dynamics with four supercharges with a potential given by the length of a holomorphic Killing vector.

[4]These states have non-parallel electric and magnetic charge vectors which would make them 1/4-BPS in the $\mathcal{N}=4$ theory. However, in the $\mathcal{N}=2$ theory, where no 1/4-BPS particle states exist, they preserve 1/2 supersymmetry.

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