

Free energy of hot gauge theories

Rajesh R. Parwani

Departmento di Fisica, Universita' di Lecce, and Istituto Nazionale di Fisica Nucleare, Sezione di Lecce, Via Arnesano, 73100 Lecce, Italy

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The total perturbative contribution to the free energy of hot $SU(3)$ gauge theory is argued to lie significantly higher than the full result obtained by lattice simulations. This then suggests the existence of large nonperturbative corrections even at temperatures a few times above the critical temperature. Some speculations are then made on the nature and origin of the nonperturbative corrections. The analysis is then carried out for quantum chromodynamics, $SU(N_c)$ gauge theories, and quantum electrodynamics, leading to a conjecture and one more speculation.

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I. INTRODUCTION

The most convincing evidence for a phase transition in thermal Yang-Mills theories is provided by direct lattice simulation of the partition function:

$$Z = \text{Tr} e^{-\beta H}, \quad (1)$$

where $\beta = 1/T$ is the inverse temperature. Over the years the lattice data for $SU(3)$ theory, the purely gluonic sector of quantum chromodynamics (QCD), has become increasingly accurate, with various systematic errors brought under control [1]. Figure 1 shows the normalized free-energy density, $F = -T \ln Z/V$, of $SU(3)$ gauge theory taken from the first of Ref. [1]. Plots such as this have supported a picture of a low-temperature phase of glueballs melting above some critical temperature to produce a deconfined phase of weakly interacting gluons: As the gluons are liberated the number of degrees of freedom increases causing the free-energy density to rise, while asymptotic freedom guarantees the gluons are weakly interacting at sufficiently high temperature.

Though the numerical data for QCD is less accurate, due to technical difficulties in simulating fermions, the accumulated data continue to support a phase transition. It is generally believed that this is a transition from a low-temperature hadronic phase to a high-temperature phase of quarks and gluons. This ‘‘quark-gluon plasma’’ is the new phase of matter which experiments at Brookhaven and CERN hope to detect in the near future.

For the most part of this paper the focus will be on pure $SU(3)$ theory, since the accurate lattice data allow a direct comparison with theory. Referring to Fig. 1, there is one feature which is ignored by some, commented on by many and which has bothered a few. While there is little doubt that at infinite temperature a description in terms of gluons is tenable, this is less clear at temperatures a few times the critical temperature, $T_c \sim 270$ MeV. For example, at $3T_c$, the curve lies 20% below that of an ideal gas of gluons.

What is the origin of this large deviation? Is it due to (i) perturbative corrections to the ideal gas value, (ii) non-perturbative effects in the plasma, (iii) an equally important combination of (i) and (ii) or, (iv) is this an irrelevant ques-

tion arising from an improper insistence of describing the high-temperature phase in terms of weakly coupled quasi-particles?

Several viewpoints have been expressed in the literature. Some believe that the deviation is mainly a non-perturbative correction to a gas of weakly coupled gluons and parametrize it in terms of a phenomenological ‘‘bag constant.’’ Others have attempted a phenomenological description of the high-temperature phase in terms of generalized quasi-particles. For a discussion and detailed references to these phenomenological approaches see, for example, [2]. On the other hand, a few have suggested that the best consistent description of the high-temperature phase might be in terms of novel structures [3,4].

In order to help discern among the various possibilities, this paper will focus on estimating the total perturbative contribution to the free energy. It is important to first agree on some terminology so as to avoid confusion due to an overuse of some phrases in the literature. The partition function depends on the Yang-Mills coupling g , and has a natural representation as an Euclidean path integral [5]

$$Z(g) = \int D\phi e^{-\int_0^\beta d\tau \int d^3x \mathcal{L}(\phi(x, \tau))} \quad (2)$$

where ϕ collectively denotes the gauge and ghost fields, and \mathcal{L} the gauge-fixed Lagrangian density of $SU(3)$ gauge theory. An expansion of this path integral, and hence the free-energy density, around $g=0$ leads to the usual Feynman

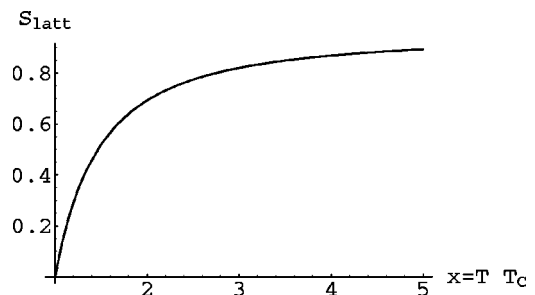


FIG. 1. Mean lattice results for the free-energy density of hot $SU(3)$ gauge theory from Ref. [1]. Here S_{latt} refers to the free energy divided by the free energy of an ideal gas of gluons.

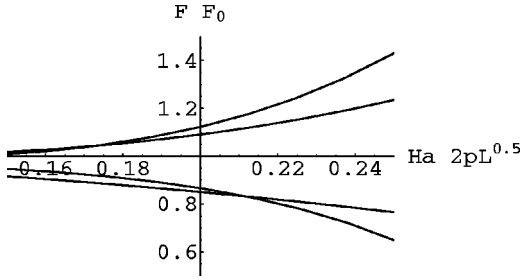


FIG. 2. The divergent perturbative free-energy density of $SU(3)$ gauge theory given in Eq. (3). Starting from the lowest curve at $(\alpha/2\pi)^{0.5}=0.24$, one has $N=2,5,3,4$.

perturbation theory and contributions of the form g^n , with infrared effects occasionally generating logarithms multiplying the power terms, $g^n(\ln(g))^m$. These terms will be called perturbative. What is invisible in a diagrammatic expansion about $g=0$ are terms such as e^{-1/g^2} , associated for example with instantons [5]. Such terms, which are exponentially suppressed as $g \rightarrow 0$, will be called non-perturbative.

Note that at non-zero temperature, odd powers of the coupling, such as g^3 , appear [5]. These are perfectly natural and represent collective effects in the plasma. Though they sample interesting long-distance physics, mathematically they fall into our definition of perturbative corrections. Similarly, Linde [6] had shown that at order g^6 the free energy receives contributions from an infinite number of topologically distinct Feynman diagrams. Though the calculation of that contribution is difficult, it is possible in principle [7], and anyway does not qualify as a non-perturbative contribution according to the definition above.

Following the heroic work of Arnold and Zhai, a completely analytical calculation of the free energy of thermal gauge theory to order g^5 has been obtained [8,7]. For $SU(3)$ gauge theory the result can be summarized as follows:

$$\begin{aligned} \frac{F}{F_0} = & 1 - \frac{15}{4} \left(\frac{\alpha}{\pi} \right) + 30 \left(\frac{\alpha}{\pi} \right)^{3/2} + \left(67.5 \ln \left(\frac{\alpha}{\pi} \right) + 237.2 \right. \\ & \left. - 20.63 \ln \left(\frac{\bar{\mu}}{2\pi T} \right) \right) \left(\frac{\alpha}{\pi} \right)^2 - \left(799.2 - 247.5 \ln \left(\frac{\bar{\mu}}{2\pi T} \right) \right) \\ & \times \left(\frac{\alpha}{\pi} \right)^{5/2}, \end{aligned} \quad (3)$$

where $F_0 = -8\pi^2 T^4/45$ is the contribution of non-interacting gluons, $\alpha = g^2/4\pi$, and $\bar{\mu}$ is the renormalization scale in the modified minimal subtraction (MS) scheme. Unfortunately Eq. (3) is an oscillatory, non-convergent, series even for α as small as 0.2, which is close to the value of physical interest. A plot of Eq. (3) at different orders, at the scale $\bar{\mu} = 2\pi T$ is shown in Fig. 2. The poor convergence of Eq. (3) does not allow a direct comparison of the perturbative results with lattice data. Furthermore, the result (3) is actually strongly dependent on the arbitrary value of $\bar{\mu}$. Inspired by the relative success of Padé resummation in other areas of physics, Hatsuda [9] and Kastening [10] studied the

Padé improvement of the divergent series (3). Their conclusion was that the convergence could be improved, and the dependence on the scale $\bar{\mu}$ reduced. However they did not attempt a direct comparison of their improved results with the lattice data, though Hatsuda did conclude that for the case of four fermion flavors, the deviation of the fifth-order Padé improved perturbative results from the ideal gas value was less than 10%.

Not all seem to agree that a resummation of perturbative results as in [9,10] sheds sufficient light on the lattice data. For example, Andersen *et al.* [11] and Blaizot *et al.* [12] have abandoned the expansion of the free energy in terms of any formal parameter, but use instead gauge invariance as the main guiding principle to sum select classes of diagrams. Their low order results seem to be close to the lattice data, but unfortunately because of the complexity of the calculations and the absence of an expansion parameter, it is not at all obvious what the magnitude of “higher order” corrections is. A completely different approach has been taken by Kajantie *et al.* In [13] an attempt has been made to numerically estimate the net contribution of long-distance effects, summarized in a dimensionally reduced effective theory, to the free-energy density. As will be discussed later, the calculation of [13] probably contains some of the non-perturbative effects defined above but might miss out on some others.

This author believes that the declared demise of information content in perturbative results such as Eq. (3) is premature. In Ref. [14] a resummation scheme was introduced to obtain an estimate of the total perturbative contribution to the free-energy density of $SU(3)$ theory. The methodology of Ref. [14] has been further developed and applied to other problems in [15,16]. In Secs. II–V an explicit and improved discussion of the results in [14] is given, leading to the conclusion that the *total* perturbative contribution to the free-energy density lies significantly above the full lattice data. In Sec. VI I discuss the consequent magnitude of non-perturbative contributions, and speculate on their possible origin. Section VII contains an analysis of the perturbative free energy of generalized QCD, with N_f fermions, and a brief comparison with the available lattice data which is less precise. $SU(N_c)$ gauge theory is discussed in Sec. VIII, and an apparently universal relation noted. Section IX considers quantum electrodynamics (QED) and some speculations about its high-temperature phase. A summary and the conclusion is in Sec. IX.

II. THE RESUMMATION SCHEME

The truncated perturbative expansion of the normalized free-energy density can be written as

$$\hat{S}_N(\lambda) = 1 + \sum_{n=1}^N f_n \lambda^n, \quad (4)$$

where $\lambda = (\alpha/\pi)^{1/2}$ is the coupling constant, and where following [9,10], possible logarithms of the coupling constant are absorbed into the coefficients f_n . The poor convergence of Eq. (4) is obviously due to the large coefficients at high

order. Indeed such divergence of perturbative expansions is generic in quantum field theory and one expects the coefficients f_n to grow as $n!$ for large n [17]. This leads to the introduction of the Borel transform

$$B_N(z) = 1 + \sum_{n=1}^N \frac{f_n z^n}{n!} \quad (5)$$

which has better convergence properties than Eq. (4). The series (4) may then be recovered using the Borel integral

$$\hat{S}_N(\lambda) = \frac{1}{\lambda} \int_0^\infty dz e^{-z/\lambda} B_N(z). \quad (6)$$

The logic of Borel resummation is to define the total sum $S(\lambda)$ of the perturbation expansion as the $N \rightarrow \infty$ limit of Eq. (6). This of course requires knowledge of $B(z) \equiv B_\infty(z)$ and the existence of the Borel integral. Lacking knowledge of the exact $B(z)$, one therefore attempts to reconstruct an approximation to S by replacing the partial series $B_N(z)$ in Eq. (6) by a possible analytical continuation thereof. A simple and popular method to achieve this is to use Padé approximants, leading therefore to a Borel-Padé resummation of the series (4). This method will be briefly discussed in Sec. V. Here instead I will proceed as suggested by Loeffel [18] and change variables in Eq. (6) through a conformal map. For a positive parameter p , define

$$w(z) = \frac{\sqrt{1+pz}-1}{\sqrt{1+pz}+1} \quad (7)$$

which maps the complex z -plane (Borel plane) to a unit circle in the w -plane. The inverse of Eq. (7) is given by

$$z = \frac{4w}{p} \frac{1}{(1-w)^2}. \quad (8)$$

The idea [19] is to rewrite Eq. (6) in terms of the variable w . Therefore, using Eq. (8), z^n is expanded to order N in w and substituted into Eqs. (5),(6). The result is

$$S_N(\lambda) = 1 + \frac{1}{\lambda} \sum_{n=1}^N \frac{f_n}{n!} \left(\frac{4}{p}\right)^n \sum_{k=0}^{N-n} \frac{(2n+k-1)!}{k! (2n-1)!} \times \int_0^\infty e^{-z/\lambda} w(z)^{(k+n)} dz, \quad (9)$$

where $w(z)$ is given by Eq. (7). Equation (9) represents a highly nontrivial resummation of the original series (4) [15]. In the pioneering application of the Borel-conformal-map technique in condensed matter physics by Le Guillou and Zinn-Justin [19], the parameter p was a fixed constant which determined the precise location of the instanton singularity at $z = -1/p$. In some more recent QCD applications [20], the fixed constant p determines the ultraviolet renormalon singularity closest to the origin in the Borel plane [21].

The novelty introduced in Ref. [14] and further developed in [15] was to consider p as a variational parameter determined according to the condition

$$\frac{\partial S_N(\lambda_o, p)}{\partial p} = 0, \quad (10)$$

where λ_o is a convenient reference value. For the problem at hand, because f_2 is negative, the solutions $p(N)$ to Eq. (10) will be positions of minima [15]. Denote the value of Eq. (9) at $p = p(N)$ by $S_N(\lambda)$. Notice that although $p(N)$ is determined at the reference value λ_o , $S_N(\lambda)$ is defined for all λ . The reason why this is sufficient has been explained in [15] and will be discussed further in the next section.

It must be stressed that, in general (see [15]), the variational parameter $p(N)$ is not related to possible singularities of $B(z)$. Rather, it is determined according to the extremum condition (10). Thus the presentation here is a slight departure from that in [14] and represents the developments in Ref. [15].

Sufficient information is now available to construct the resummed approximants S_N from $N=3$ up to $N=5$ in the next section. The approximant S_1 of course does not exist since $f_1=0$, while S_2 cannot be constructed because no solution exists to equations such as Eq. (10) at the first non-trivial order. For $N>2$, the solutions $p(N)$ will be positions of global (local) minima if the sign of f_N is positive (negative) [15].

III. SU(3): RESUMMATION UP TO FIFTH ORDER

In order to make contact with lattice data which show a temperature dependent curve, one must use in Eq. (4) a temperature dependent coupling. Let us begin by using the one-loop running coupling defined by [11,12]

$$\lambda(c, x) = \frac{2}{\sqrt{11L(c, x)}}, \quad (11)$$

where $L(c, x) = \ln((2.28\pi cx)^2)$, $c = \bar{\mu}/2\pi T$ and $x = T/T_c$, with $T_c \sim 270$ MeV the critical temperature which separates the low and high temperature phases [1,22].

Fixing first the reference values $c_0=1$, $x_0=1$, which fixes the reference value of λ_o , the results of Eq. (10) are: $p(3)=3.2$, $p(4)=7.9$, $p(5)=13.7$. The curves for $S_N(\lambda)$ are shown in Fig. 3(a) at the renormalization scale $c=1$. Notice the behavior $S_5 > S_4 > S_3$ and how these all lie significantly above the lattice curve in Fig. 1. The results do depend on the renormalization scale, denoted here by the dimensionless parameter c . It has been suggested [11,12] that a suitable choice for such a parameter is $0.5 < c < 2$, corresponding to $\pi T < \bar{\mu} < 4\pi T$. Certainly this is the natural energy range for the high-temperature phase. Figure 3(b) shows the mild dependence of S_5 on the renormalization scale.

The results above were obtained by solving Eq. (10) at the point $c_0=x_0=1$. Now consider changing the reference values to $c_0=1$, $x_0=3$, that is, a more central value for the temperature. The solutions are: $p(3)=3.2$, $p(4)=7.8$, $p(5)=13.4$. These values are hardly different from those above. This is firstly due to the fact that Eq. (9) is a much slower varying function of the coupling than the original divergent series. Furthermore, for the present problem,

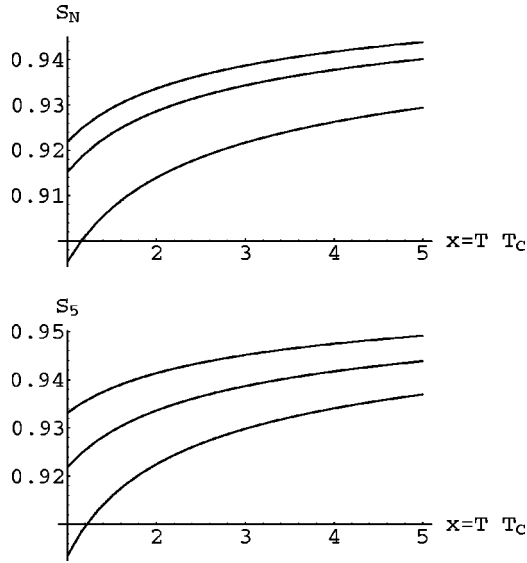


FIG. 3. (a) The resummed perturbative free-energy density of hot $SU(3)$ gauge theory for $N=3, 4$ and 5 , using a one-loop running coupling, the reference values $c_0=1, x_0=1$, and the renormalization scale $c=1$. The curves move upwards as N increases. (b) The fifth order resummed perturbative free-energy density of (a) at three different renormalization scales, $c=0.5, 1$ and 2 . The free-energy density increases with increasing c .

the coupling itself varies slower than logarithmically with c and x (the c and x dependence of the coefficients f_4 and f_5 is also only logarithmic). The curves for the re-optimized S_N are essentially identical to those shown in Figs. 3(a,b), the difference being only at the fifth decimal point. For example, the value of S_5 in Fig. 3(a) at $x=3$ is 0.938684 , while that for S_5 optimized at $x_0=3$ [and hence evaluated at $p(5)=13.4$], is 0.938672 at $x=3$. This confirms the assertion in [14] that the results are quite insensitive to the exact reference values chosen to solve Eq. (10).

We now proceed to test the sensitivity of the results to the approximation used for the running coupling (11). The approximate two-loop running coupling is given by [11,12]

$$\lambda(c,x) = \frac{2}{\sqrt{11L(c,x)}} \left(1 - \frac{51}{121} \frac{\ln(L(c,x))}{L(c,x)} \right) \quad (12)$$

with the symbols having the same meaning as before. In Fig. 4, the one-loop running coupling (11) and the approximate two-loop running coupling (12) are plotted at $c=1$ to show their difference. At $x=3$, the value for the approximate two-loop coupling is about 20% lower than the one-loop result. Nevertheless, because of the above-mentioned property of the resummed series, we shall see that the final results do not shift dramatically. Using Eq. (12), the solution of Eq. (10) at the reference point $c_0=1, x_0=3$ are: $p(3)=3.2, p(4)=7.6, p(5)=13.1$. The corresponding curves shown in Fig. 5(a) have moved up slightly compared to those in Fig. 3(a). The ‘‘two-loop’’ value of $S_5(c=1, x=3) = 0.9473$ should be compared to the ‘‘one-loop’’ value 0.9387 obtained above. The mild renormalization scale dependence of the new S_5 is shown in Fig. 5(b).

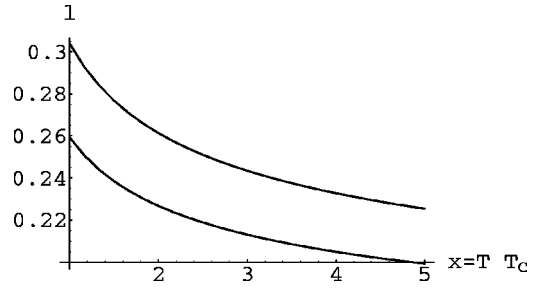


FIG. 4. The one-loop (upper curve) and approximate two-loop running couplings for $SU(3)$ gauge theory at the renormalization scale $c=1$.

In summary, it has been demonstrated that the resummed approximants S_3, S_4, S_5 all lie significantly above the lattice data and satisfy the monotonicity condition $S_5 > S_4 > S_3$. The result is insensitive to the reference value used to solve Eq. (10), for the range of interest $0.5 < c < 2, 1 < x < 5$. The result is also insensitive to the approximation used for the running coupling constant and in fact better approximations for the coupling seem to move the values of S_N further away from the lattice data. Finally it should be noted that the values S_N also appear to converge as N increases.

The only way to force the values of S_N down closer to the lattice data is to choose very low values for the renormalization scale, $c \sim 0.05$. Of course this is not only unnatural but increases the effective value of the coupling content beyond what one would believe is physically reasonable for a perturbative treatment. That is by making an artificially low choice for the renormalization scale, one cannot escape the conclusion stated in the abstract of large non-perturbative corrections.

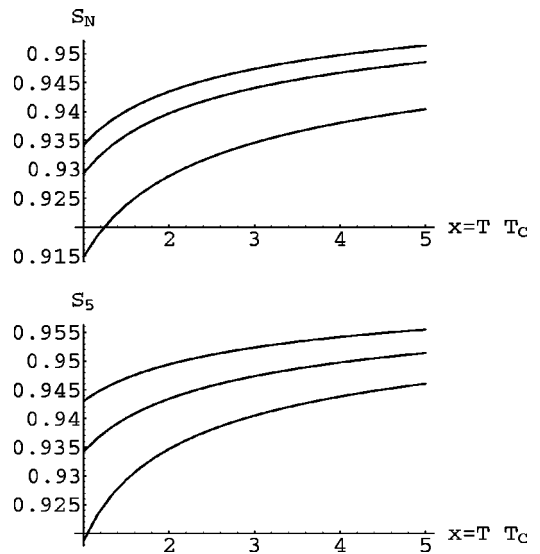


FIG. 5. (a) The resummed perturbative free-energy density of hot $SU(3)$ gauge theory for $N=3, 4$ and 5 , using a two-loop running coupling, the reference values $c_0=1, x_0=3$, and the renormalization scale $c=1$. The curves move upwards as N increases from 3 to 5. (b) The fifth order resummed perturbative free-energy density of (a) at three different renormalization scales, $c=0.5, 1$ and 2 . The free-energy density increases with increasing c .

IV. HIGHER ORDER CORRECTIONS

Due to technical complications the sixth order contribution, λ^6 , to the free-energy density has not been calculated although an algorithm for it exists [7]. There is a misconception that because that contribution is due to an infinite number of topologically distinct diagrams, its value must be very large. A counter-example is provided by the magnetic screening mass [5], which suffers from the same disease but whose approximate calculations in the literature show it to be of ordinary magnitude [23].

Having said that, let us see what is the worst that can happen. It has been suggested [24] that Padé approximants can be used to estimate the next term of a truncated perturbation series. That is, after approximating the truncated series by the ratio of two polynomials, the Padé approximant is re-expanded as a power series to estimate the next term in the series. Well, why not also use Borel-Padé approximants for the same purpose? Using the fifth order result (3) together with the two-loop running coupling (12), and choosing the central values $c=1, x=3$, all fifth order Padé and Borel-Padé approximants were constructed and then re-expanded to give an estimate of the coefficient f_6 . The largest value obtained was 30 000 and the smallest $-30\,000$. Note that the fifth order coefficient at $c=1$ is -800 , so the estimated magnitude of f_6 is about 37 times larger. Since the coupling λ is about 0.2 at $x=3, c=1$, the total value of the sixth order contribution to Eq. (3) is therefore estimated to be almost 8 times in magnitude compared to the fifth order contribution. These are big numbers and should be expected to cause some damage.

Using Eq. (9) with $f_6=30\,000$, the two-loop coupling (12), and solving Eq. (10) at the reference point $c_0=1, x_0=3$ gives $p(6)=19.75$ and $S_6(c=1, x=3)=0.9490$. Repeating for $f_6=-30,000$ gives $p(6)=19.5$ and $S_6(c=1, x=3)=0.9489$. Notice the negligible change in the value of S_6 even when wildly differing values have been used for f_6 . Those values should be compared with the fifth order approximant of Fig. 3(a), which gives $S_5(c=1, x=3)=0.9473$. The large estimated sixth order corrections to the divergent perturbation expansion (4) cause a change of only 0.002 to the values of the resummed series, and more importantly the shift is upwards, $S_6>S_5$, preserving the lower order monotonicity, regardless of the sign of f_6 .

Kajantie *et al.* [13] have suggested that the sixth order contribution, $f_6\lambda^6$ be of order 10. For a coupling $\lambda\sim 0.2$, this translates into the astronomical value ± 156250 for f_6 . Solving Eq. (10) at $c_0=1, x_0=3$ gives $p(6)=19.8$ and $S_6(c=1, x=3)=0.9491$ for the positive f_6 , and $p(6)=19.4$ and $S_6(c=1, x=3)=0.9489$ for the negative f_6 . Despite the anomalously large value of the sixth order contribution proposed in [13], the conclusion here is still $S_6>S_5$, and an increment of only 0.002.

To highlight the above result in a more dramatic way, suppose the sixth order coefficient vanishes, $f_6=0$. Then because of the non-trivial way the resummation is done in Eq. (9), the solution to Eq. (10) for $N=6$ will still be different from that of $N=5$. At $c_0=1, x_0=3$ I find $p(6)=19.5$ and then $S_6(c=1, x=3)=0.9489$ at four decimal places, which is

almost identical to the values obtained above for various large values of f_6 . This sounds incredible but is actually not once one remembers that large corrections to the divergent series (4) do not translate into large corrections to the resummed series (9). In fact those large values are suppressed in various ways. First, in the re-organization of the series in Eq. (9), less weight is given to higher order corrections. Secondly, the variational procedure chooses values of $p(N)$ which in this example increase with N , and so suppress further the value of S_N .

More understanding of the above results can be obtained through a large N analysis carried out for the general Eqs. (9),(10) in [15]. It was shown in [15] that if $p(N)$ increases for the first few values of N , then that trend will continue. Let $c(N)\equiv p(N+1)/p(N)$. In the large N and large $p(N)$ limit one can show that [15]

$$\frac{1}{c(N+1)} = 1 - \frac{1}{c(N)} + \frac{1}{c(N)^2}. \quad (13)$$

One consequence of this is that $c(N+1)<c(N)$ and $c(N)\rightarrow 1^+$ as $N\rightarrow\infty$. This is indeed observed for the present problem already at low N . Numerically, Eq. (13) too is not a bad approximation at small N . In fact using the values found for $p(N)$ in the last section, one has $p(4)/p(3)=7.6/3.2=2.375$. With this as input for $c(3)$, Eq. (13) gives the estimate $c(4)\sim 1.32234$, to be compared with the actual value $p(5)/p(4)=13.1/7.6=1.7$. Next using $c(4)=1.7$ as the exact input, Eq. (13) gives $c(5)\sim 1.32201$ and thus an estimate of $p(6)\sim 1.3\times 13.1=17.03$. On the other hand, using $p(6)\sim 19.5$, as determined by various estimates above, gives $c(5)=19.5/13.1=1.49$ and then through Eq. (13) the estimate $c(6)\sim 1.28$ and hence $p(7)\sim 1.28\times 19.5=25$.

Equation (13) was derived in [15] for the case $f_1\neq 0$. For the present case where $f_1=0$ one will actually obtain the slightly more accurate equation

$$\frac{1}{c(N+1)^2} = 1 - \frac{1}{c(N)} + \frac{1}{c(N)^3}, \quad (14)$$

but in the large N limit where $c\rightarrow 1^+$ this is clearly equivalent to Eq. (13).

Note that the recursion relations (13),(14) make no explicit reference to the values of the f_n which in the derivation in [15] were assumed to be generic, that is, diverging at most factorially with n . Indeed the fact that various different assumptions about the value of f_6 earlier in this section led to essentially the same value for $p(6)\sim 19.5$ supports the f_n independence of Eq. (13) already at $N\sim 6$.

From the general analysis in [15], one also deduces that for large N and large $p(N)$, the monotonicity $\Delta S_N\equiv S_{N+1}-S_N>0$ is guaranteed by the fact $f_2<0$, and that $\Delta S_N\sim 1/N^3$ as $N\rightarrow\infty$. Since the explicit $N\leq 5$ calculations and the estimated $N=6$ result already support $c(N)>1$ and large values of $p(N)$ at low N , this suggests that the continued monotonicity and rapid convergence of the S_N is assured by the large N analysis.

V. LOWER BOUND AND OTHER ESTIMATES

From the explicit low N calculations, and the large N analysis, one concludes that for $N > 2$,

$$S_N < S_{N+1} \quad (15)$$

for all N , and furthermore, the difference $S_{N+1} - S_N$ decreases as N increases, showing a rapid convergence of the approximants. However, in general, it is not quite correct to say that the approximants converge to the total sum of the series [15]. For each N , let $p^*(N)$ be the value of p that is optimal, that is, it is the value which when used in Eq. (9) gives the best estimate of S , the total sum of the series. Define, $S_N^* = S_N(\lambda, p^*(N))$. Then for those $p(N)$ which are positions of global minima one has by definition

$$S_N \leq S_N^*. \quad (16)$$

It is S_N^* which presumably converges to S as $N \rightarrow \infty$. (This implicitly assumes that the sub-sequence of global minima is infinite: That is, given any positive integer N_0 , there is some $n > N_0$ for which f_n is positive.)

Hence if one accepts the two assumptions above, then combining Eq. (15) with Eq. (16),

$$S_N \leq S \quad (17)$$

for all $N > 2$, and one may conclude that the S_N are lower bounds to the sum of the full perturbation series.

In particular that conclusion implies that the $N=5$ curve in Fig. 5(a) is a lower bound on the *total perturbative* contribution to the free-energy density of hot $SU(3)$ theory. The statement has three qualifications. Firstly, it involves the technical assumptions mentioned above. Secondly, as discussed before, better approximations to the running coupling can move the bounds, but it was seen that a 20% improvement in the coupling shifted the bound *upwards* by 1%. Secondly, the bounds shift by $\pm 1\%$ when the renormalization scale is varied by a factor of two from its central value $\bar{\mu} = 2\pi T$. Thus it might be more appropriate to call the bounds as ‘‘plausible soft lower bounds’’ with an uncertainty depicted in Fig. 5(b).

Given that the lower bound obtained above involves some unproved technical assumptions, it is useful to compare the above results with those obtained using different resummation schemes for the divergent series (4). I briefly state here the main results obtained using a Borel-Padé resummation of Eq. (4), with the two-loop approximation for the coupling (12) and the central value $c=1$. The approximants will be denoted as $[P, Q]$, referring of course to the particular Padé approximant used for the partial Borel series (5) constructed from Eq. (4). The only approximants which did not develop poles and which gave a resummed value below one in the temperature range $2 < x < 5$ were $[1,2]$, $[2,1]$ and $[2,2]$. These are displayed in Fig. 6. The $[3,2]$ and $[4,1]$ approximants did not develop poles but gave a value above one. If the approximants which developed a pole are defined through a principal value prescription, then the lowest value was given by $[2,3]$: $0.91 \rightarrow 0.94$ as x increased from $2 \rightarrow 5$.

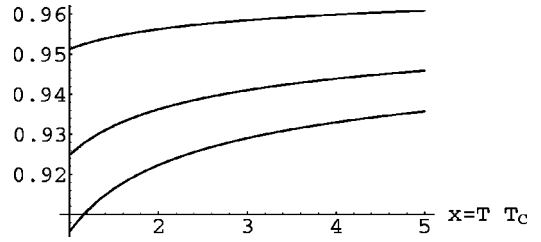


FIG. 6. The $[1,2]$, $[2,1]$ and $[2,2]$ Borel-Padé approximants to the perturbative free-energy density, with a two-loop running coupling, and the renormalization scale $c=1$. Starting with the lowest curve at $x=5$, one has $[2,1]$, $[1,2]$, $[2,2]$.

The $[1,3]$ and $[1,4]$ approximants gave values above 0.98 in the range of interest while $[3,1]$ gave a value above one.

Thus in the Borel-Padé method, the minimum estimate for the fifth order resummed series is given by the principal value regulated $[2,3]$. The highest values were all above one. If one keeps only the fifth order estimates below one (thus giving a very conservative lower value), then the average of the $[2,3]$ and $[1,4]$ is greater than 0.94 for the entire range $2 < x < 5$. At $x=3$ the estimates are 0.95 ± 0.04 . Of course including also the values above one would push this average higher. Clearly the Borel-Padé estimates are comparable to the bounds obtained using the resummation technique of Sec. II and should reassure some readers about the novel resummation used here.

For completeness, I mention an alternative way of thinking about divergent series such as Eq. (4). For QCD at zero temperature, a paradox is that one-loop results give remarkable agreement with experimental data even when the energy scale is relatively low. As the running coupling is then large it is not obvious why higher-loop perturbative corrections are suppressed. It has been suggested [25,26] that the explanation might lie in the probable asymptotic nature of the QCD perturbation series. Recall that in an asymptotic series the best estimate of the full sum, *at a given value of the coupling*, is obtained when only an optimal number of terms is kept and the rest discarded (even if they are large). Thus if one knew the general behavior, at least at large order, of the series (4) and assumed that it was asymptotic, then one could have obtained a reasonable estimate of the full sum by simply adding the optimal first few terms. What has been done in the previous sections, and this is what various resummation schemes try to do, is to instead sum up the whole series to get an even better estimate of the total perturbation series (and this has the greater advantage of giving a good result for a large range of couplings). Also note that thinking of Eq. (4) as an asymptotic series does not say anything about explicit non-perturbative corrections [25,26].

VI. NON-PERTURBATIVE CORRECTIONS

The total perturbative contribution to the free-energy density of $SU(3)$ gauge theory has been argued to be close to, or above, the $N=5$ curve in Fig. 5(a). A residual uncertainty that could lower the curve of Fig. 5(a) is the exact value of the renormalization scale. For a natural range of parameters, the lower curve in Fig. 5(b) is the result. On the other hand

the full result as given by lattice simulations is shown in Fig. 1. Lattice errors have been stated to be under 5% [1]. Taken together, the conclusion appears inescapable: *Even at temperatures a few times above the transition temperature, there are large negative non-perturbative contributions to the free-energy density.* For example, at $T=3T_c \sim 700$ MeV, the lattice results for the normalized free-energy density are 0.8 ± 0.04 while the lower bound on the perturbative contribution is 0.947 ± 0.007 , implying a minimum non-perturbative correction of 10% (and as high as 20%).

Thus an answer has been given to the questions raised in the introduction. The deviation of the lattice data from the ideal gas value is apparently caused mainly by non-perturbative corrections, with perturbative corrections accounting for a much smaller amount. At $T \sim 3T_c$ the relative contributions are $\sim 15\%$ and $\sim 5\%$.

I speculate now on possible sources of the non-perturbative corrections. First there are the familiar instantons, already present in the classical action, and which contribute terms of the order e^{-1/λ^2} . Secondly there are the magnetic monopoles. There is by now overwhelming evidence that confinement at zero temperature is caused by the 't Hooft–Mandelstam mechanism of condensing monopoles (the dual superconducting vacuum). Thus it is possible that the monopole condensate has not completely melted above the critical temperature. Note that since the classical theory does not support finite energy monopoles, these must be of quantum origin, and so their contribution might be larger than those of the instantons.

In fact contributions which are exponentially small but much larger than those of the instantons are suggested by the Borel resummation itself. It is known that Yang-Mills theories are not Borel summable [20,21]. That is, the function $B(z)$ has singularities for positive z , making the Borel integral ill-defined. One can nevertheless define the sum of the perturbation series using the Borel integral if a prescription is used to handle the singularities. It is generally believed that the prescription dependent ambiguity disappears when explicit non-perturbative contributions are taken into account for the physical quantity in question. Indeed the nature of singularity itself suggests the form for the non-perturbative contribution. If there is a pole at $z=q$, then the non-perturbative contribution will be of the form $\sim e^{-q/\lambda}$, which is larger than the instanton contribution for small λ . An explicit mathematical model which illustrates the interplay between Borel non-summability and non-perturbative contributions has been given in [15].

Notice that the non-perturbative corrections suggested by the Borel method at non-zero temperature are very different from those at zero temperature. In the latter case the expansion parameter is g^2 and so the contribution is $\sim e^{-q/g^2}$, which translates into a power suppressed contribution $\sim 1/(Q)^b$ when g^2 is replaced by the running coupling $\sim 1/\ln(Q/\Lambda)$. In cases where the physical quantity can also be analyzed using the operator-product expansion (OPE), these power suppressed contributions to perturbative results correspond in the OPE picture to vacuum condensates [20,21].

At nonzero temperature, since the natural expansion pa-

rameter is $\lambda = \sqrt{g^2/4\pi^2} \sim 1/\sqrt{\ln(T/\Lambda)}$, one does not get a simple power suppression from $e^{-1/\lambda}$. Nevertheless, the analogy with zero-temperature results suggests that such contributions might be due to some condensates. Thus the conventional condensates discussed for example in [13] are plausibly part of the non-perturbative contributions. The form however suggests even more novel condensates. These might be, for example, those of DeTar [3] or Pisarski [4].

It is worth noting an explicit instance of a theory displaying exponentially small non-perturbative effects which are larger than those due to standard solitons. In fundamental string theory where the coupling is g , there are the usual solitons but there are also novel ‘‘D-instantons’’ which give a larger contribution $e^{-1/g}$ [27]. I also mention in passing the recurrent and intriguing relationship between gauge theories and strings [28] which leads one to wonder whether that is a possible route to understanding the non-perturbative structure of hot gauge theories.

Using the $N=5$ curve of Fig. 5(a) as a reasonable estimate of the full perturbative result, and assuming a nonperturbative component of the form

$$S_{np} = \frac{A}{\lambda} e^{-q/\lambda}, \quad (18)$$

as suggested by the Borel method, one can determine the constants A and q by comparing the lattice data of Fig. 1 with the perturbative result. In [14] it was shown that

$$S_{latt} = S_{pert} - \frac{1}{\lambda(x)} e^{8.7 - 2.62/\lambda(x)}, \quad (19)$$

where S_{latt} represents the lattice data for the free energy, and S_{pert} the resummed perturbative result, both normalized with respect to the ideal gas value, and $\lambda(x)$ is given by Eq. (12) at $c=1$. Equation (19) is a phenomenological equation of state for the free energy which generalizes the usual discussions in the literature where the second term on the right-hand side of Eq. (19) is called a ‘‘bag constant.’’ In this case the ‘‘constant’’ is really temperature dependent and represents a non-perturbative contribution to the free energy that vanishes at infinite temperature. It is important to note that the non-perturbative contribution is negative, since the perturbative result is above the full lattice data, and thus consistent with the usual interpretation in the literature.

So far the discussion has implicitly assumed an additive picture of perturbative and non-perturbative contributions, with both components clearly distinguished. It might be that in reality the best description of the high-temperature phase is in terms of completely novel structures [3,4]. In that case a forced expansion of those alternatives about $\lambda=0$ must give something like Eq. (19) and the subsequent distinction between perturbative and non-perturbative contributions. The mathematical toy model of [15] illustrates this.

Within the framework of this paper, one can distinguish three versions of the popular concept of ‘‘quasi-particle.’’ First there are the ‘‘perturbative quasi-particles’’ which are deformations of the gluon formed by a particular reorganization of the perturbative Feynman diagram expansion. The

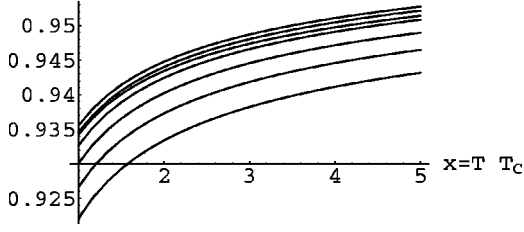


FIG. 7. The fifth order resummed perturbative free-energy density of $SU(3)$ gauge theory coupled to N_f fermions as discussed in Sec. VII. Starting from below at $x=5$, the curves label $N_f = 6, 5, 4, 3, 0, 2, 1$.

results above suggest that if all the contributions of such quasi-particles to the free energy are added up, the net result will lie above the lattice data, and only truly non-perturbative contributions, as defined in Sec. I, may give the final agreement. Secondly there are the “nonperturbative quasi-particles,” which are excitations about the nontrivial thermal vacuum that includes condensates, and so forth. Currently there is not sufficient control over the theory to construct these objects. Finally there are the “phenomenological quasi-particles” which simply aim to give numerical agreement with the lattice data within a simple *ansatz*. The ultimate justification for these phenomenological constructs must surely come from the “nonperturbative quasiparticles.”

VII. QCD

In this section lower bounds (within assumptions similar to those made previously) are obtained for the perturbative free-energy density of hot $SU(3)$ coupled to N_f flavors of fundamental fermions. As the essential features are very similar to the pure gauge case, the discussion here will be brief. The fifth order perturbative results in the \overline{MS} scheme can be read off from the landmark papers [8]. The approximate two-loop coupling that is used here is given by

$$\lambda(c, x) = \frac{1}{\sqrt{4\pi^2\beta_0 L(c, x)}} \left(1 - \frac{\beta_1}{2\beta_0^2} \frac{\ln(L(c, x))}{L(c, x)} \right) \quad (20)$$

with

$$\beta_0 = \frac{11N_c - 2N_f}{48\pi^2}, \quad (21)$$

$$\beta_1 = \frac{1}{3(4\pi)^4} (34N_c^2 - 13N_c N_f + 3N_f/N_c). \quad (22)$$

Following [12] I also assume a relative N_f independence of $L(c, x)$, and thus use for it the same expression as used in Eq. (11). The extremization condition (10) is solved at the reference point $c_0 = 1$, $x_0 = 3$ for $N_c = 3$ and $1 \leq N_f \leq 6$. The results obtained are all extremely similar: In each case the convergence of the S_N is monotonic and rapid as in the pure glue case in Fig. 5(a). For this reason only the $N=5$ curves are displayed in Fig. 7 for the various number of flavors. For comparison the pure glue result ($N_f=0$) is also included.

(Each curve has been normalized with respect to the ideal gas value for that number of flavors.)

The $N=5$ curves in Fig. 7 can be taken as plausible lower bounds, or estimates, to the total perturbative free energy density of QCD with N_f fundamental fermions. Lattice results for QCD contain large systematic errors compared to those for $SU(3)$ and so a precise comparison is not possible. After making some assumptions about the size of the systematic errors, the authors in Ref. [29] determine that for $N_f = 2$ the free-energy density lies about 15–20% below the ideal gas limit. This is similar to the case of pure $SU(3)$. Comparing this lattice estimate with the estimate on the perturbative result in Fig. 7 one is again led to suggest that there are large non-perturbative corrections to the naive picture of a weakly interacting quark-gluon plasma.

Of course, given the physical relevance of QCD, it would be preferable to have more precise numbers from the lattice, and especially for other values of N_f . However it seems that a non-perturbative component of 10–15% at temperatures a few times T_c is likely to be generic.

When the draft of this paper was complete, I came across [22] which gives for $N_f = 2$ a $T_c/\Lambda_{\overline{MS}} \sim 0.5$, a factor of two lower than that for the pure gauge theory. This has the consequence that c in $L(c, x)$ should be replaced by about $c/2$. However as the reader can surmise by now, this has hardly any impact on the results above, for this is equivalent to shifting the renormalization scale c by a factor of two, which as we have seen causes only a 1% shift of the curves. In any case this serves to remind that the lattice results for N_f are in a state of flux.

VIII. $SU(N)_c$

Define

$$\lambda(N_c) = \left(\frac{N_c}{3} \right)^{1/2} \left(\frac{\alpha}{\pi} \right)^{1/2}. \quad (23)$$

Then the free-energy density of pure $SU(N_c)$ theory up to fifth order [8,7] is given by the expression (3), with $(\alpha/\pi)^{1/2}$ replaced everywhere (including inside the logs) by $\lambda(N_c)$. Thus there is no explicit N_c dependence of the free-energy density when written in terms of $\lambda(N_c)$.

To examine the N_c dependence of the new coupling (23), consider the approximate two-loop running coupling given by

$$\left(\frac{\alpha(T)}{\pi} \right)^{1/2} = \frac{1}{\sqrt{4\pi^2\beta_0 L(T, \bar{\Lambda})}} \left(1 - \frac{\beta_1}{2\beta_0^2} \frac{\ln(L(T, \bar{\Lambda}))}{L(T, \bar{\Lambda})} \right) \quad (24)$$

with

$$L(T, \bar{\Lambda}) \equiv 2 \ln \left(\frac{2c' \pi T}{\bar{\Lambda}} \right), \quad (25)$$

$$\beta_0 = \frac{11N_c}{48\pi^2}, \quad (26)$$

$$\beta_1 = \frac{1}{3(4\pi)^4} (34N_c^2), \quad (27)$$

and with $\bar{\Lambda} = \bar{\Lambda}(N_c)$ the $SU(N_c)$ gauge theory scale parameter in the $\overline{\text{MS}}$ scheme. In L , the constant c' is $\bar{\mu}/2\pi T$. Comparing the various equations, one comes to the remarkable conclusion that the new running coupling $\lambda(N_c, T)$ will be independent of N_c if the $\overline{\text{MS}}$ scale parameter $\Lambda(N_c)$ is itself independent of N_c when expressed in terms of some physical length scale. By comparing some data for $N_c = 2, 3, 4$, Teper [30] has concluded that this is indeed the case.

Therefore, accepting the result of [30], one deduces that the $N=5$ curve in Fig. 5(a) is a plausible lower bound, or estimate, to the total perturbative free energy density of hot $SU(N_c)$ theory when the x axis is interpreted as $T/\bar{\Lambda}$ instead of T/T_c . This is then a universal relation (at least for low N_c), and one suspects that the corresponding full lattice results might also obey a universal curve, thus leading to the guess that the non-perturbative component of an $SU(N_c)$ plasma is about 10–15% for temperatures a few times above the critical temperature.

IX. QED

Though the fine structure constant α of QED is small at everyday energies, it is interesting to consider super-high temperatures where it will be large. The free-energy density of massless QED at temperature (T), has been computed up to order $\alpha^{5/2}$ in Refs. [31,8,7,32]. Denoting as usual $\bar{\lambda} = (\alpha/\pi)^{1/2}$, the normalized free-energy density at the $\overline{\text{MS}}$ renormalization scale $\bar{\mu} = 2\pi T$, is given by [31,8]

$$\begin{aligned} F/F_0 = & 1 - 1.13636\bar{\lambda}^2 + 2.09946\bar{\lambda}^3 + 0.488875\bar{\lambda}^4 \\ & - 6.34112\bar{\lambda}^5, \end{aligned} \quad (28)$$

where $F_0 = 11\pi^2 T^4/180$ is the free-energy density of a non-interacting plasma of electrons, positrons and photons. Figure 8(a) shows the plot of Eq. (28) at different orders. The series diverges at large coupling (super-high temperatures), exhibiting a behavior similar to that of Yang-Mills theory at low temperatures. The convergence at large coupling can be improved by using the resummation technique (9),(10). Using the coefficients from Eq. (28), the solutions of Eq. (10) at the reference value $\lambda_0 = 0.5$ are (minima): $p(3) = 0.7$, $p(4) = 1.75$, $p(5) = 3$.

The resummed series, with its much improved convergence, is shown in Fig. 8(b). The $N=5$ curve can be taken as a lower bound to the full perturbative result. If one assumes that the potential non-perturbative contributions lower the perturbative result, as happens in QCD, or are very small in magnitude, then one may conclude from Fig. 8(b) that super-hot QED undergoes a phase transition. This speculated high-temperature phase of QED might then be analogous to the low-temperature phase of QCD with various bound states. Or, it might resemble the alternative picture of low-energy QCD: that of flux tubes [33]. It is unfortunate that no lattice

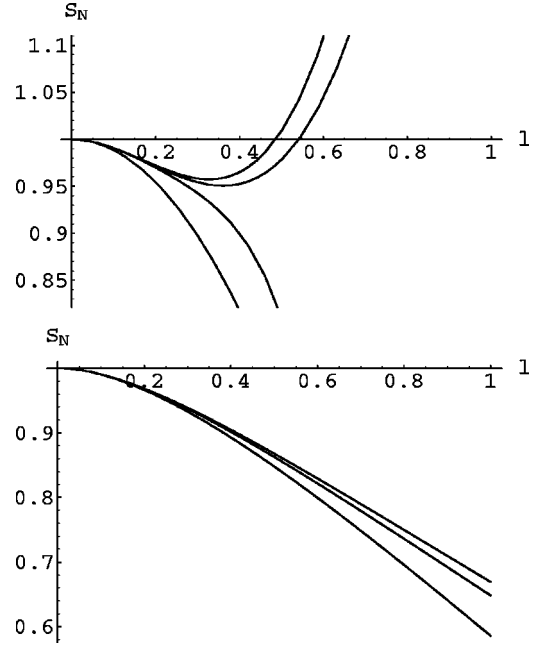


FIG. 8. (a) The divergent perturbative free-energy density of QED, given in Sec. IX. Starting from the lowest curve and moving upwards, one has $N=2, 5, 3, 4$. (b) The resummed perturbative free-energy density of QED. The curves move upwards as N increases from 3 to 5.

or other non-perturbative information is currently available about the high-temperature phase of QED.

X. CONCLUSION

The phrase ‘‘nonperturbative’’ is used often and loosely with regard to field theories at non-zero temperature. This has caused a great deal of semantic confusion and misunderstanding. For the purpose of uncovering the cause of the deviation of the result in Fig. 1 from the ideal gas value, it has been proposed to term ‘‘perturbative’’ all power like (modulo logarithms) contributions to the free-energy density. Such perturbative contributions follow from the usual Feynman diagram expansion of the partition function around zero coupling.

For $SU(3)$ gauge theory a plausible lower bound was obtained on the totality of such perturbative contributions to the free-energy density. The derivation of that lower bound using the variational conformal map involved some technical assumptions, and so one may instead wish to consider it only as an estimate of the total perturbative contribution. The estimate is comparable to that obtained using Padé or Borel-Padé resummation methods and lies significantly higher than the full lattice result, thus suggesting that large and truly non-perturbative corrections exist. As discussed in Sec. VI, these non-perturbative corrections might include the usual instantons, magnetic monopoles, the usual condensates, and perhaps also more novel condensates and extended structures as suggested by the Borel form $e^{-1/\lambda}$ of the non-perturbative contributions. As to which of these possibilities dominates is an interesting question left for future work.

The equation of state for hot $SU(3)$ can be summarized

by the phenomenological relation [14]

$$S_{latt} = S_{pert} - \frac{1}{\lambda(x)} e^{8.7 - 2.62/\lambda(x)}, \quad (29)$$

where S_{latt} represents the lattice data for the free energy, and S_{pert} the resummed perturbative result, both normalized with respect to the ideal gas value, and where λ is the temperature dependent coupling (12) at $c=1$. There is a slight ambiguity in the estimate of the magnitude of non-perturbative corrections coming from the residual renormalization scale ambiguity of the resummed perturbative results. For the natural range $\pi T < \bar{\mu} < 4\pi T$, the ambiguity is less than one percent. Such an ambiguity between the perturbative and non-perturbative components is understandable, as only the full physical quantity can be demanded to be scale independent, and not separately its perturbative and non-perturbative components, though the latter simplifying assumption is often made.

By choosing an anomalously low value for the scale $\bar{\mu}$ one can fit the purely perturbative results to the lattice data but at the price of a very large effective coupling constant, and thus an unsuppressed contribution from common place objects like instantons. Therefore this route does not offer an escape from the conclusion of large non-perturbative corrections.

The results of this paper do not imply that the description of the free-energy density must be as in Eq. (29). Rather, completely novel descriptions in terms of various extended structures are possible [3,4], but in a forced expansion of those alternative descriptions about $g=0$, one must recover something like Eq. (19). On the other hand, the results here *do* imply that an accounting of the lattice free-energy density in terms of a subset of Feynman diagrams of the perturbative thermal vacuum might miss some essential physics.

Being restricted to an analysis of the free-energy density, the results here of course do not imply that every observable must have large non-perturbative contributions, but it is likely that this is true for most of the bulk thermodynamic quantities.

An analysis was also carried out for generalized QCD with N_f quarks. Currently the lattice data is only approximate and so the conclusions are less definitive. However accepting the estimates in [29], the conclusion is the same as before: There appear to be large non-perturbative corrections to the free-energy density of hot $SU(3)$ gauge theory coupled to N_f quarks. Thus in particular, ‘‘quark-gluon plasma’’ seems to be an incomplete description even at temperatures several times above the transition temperature.

A simple relation was noted for the perturbative free-energy density of $SU(N_c)$ gauge theory in Sec. VIII. That result, combined with the results for generalized QCD in Sec. VII, and the available lattice data, leads one to a universality *conjecture*: For the high-temperature ($T \geq 2T_c$) phase of $SU(N_c)$ gauge theory coupled to N_f fundamental quarks, and for all moderate values of N_c and N_f , at most 5% of the deviation of the free-energy density from the ideal gas value is due to perturbative effects while non-perturbative effects contribute a larger 10–15%.

It has been speculated in Sec. IX that QED might have an interesting high-temperature phase.

Finally, the methods of this paper might be of some use for the study of supersymmetric theories at non-zero temperature, a topic of interest in recent developments [28].

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