Cyclotron damping and Faraday rotation of gravitational waves

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We study the propagation of gravitational waves in a collisionless plasma with an external magnetic field parallel to the direction of propagation. Because of resonant interaction with the plasma particles the gravitational wave experiences cyclotron damping or growth, the latter case being possible if the distribution function for any of the particle species deviates from thermodynamical equilibrium. Furthermore, we examine how the damping and dispersion depends on temperature and on the ratio between the cyclotron and gravitational wave frequency. The presence of the magnetic field leads to different dispersion relations for different polarizations, which in turn imply Faraday rotation of gravitational waves.

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I. INTRODUCTION

The propagation of weak gravitational waves in the presence of matter and electromagnetic fields has been considered by several authors, e.g., Refs. [1–12]. The back reaction on the gravitational waves, resulting in damping and/or dispersion, has been studied by Refs. [1–8], but the gravitational effects of matter (most frequently assumed to be a fluid or neutral gas) and the effects of electromagnetic fields have mainly been treated separately. The interaction of gravitational waves with a plasma—which is the most common state of matter relevant for gravitational wave propagation—has been considered in an astrophysical as well as an cosmological context, see, e.g., Refs. [9–11] and Ref. [12], respectively, and references therein. In general it turns out that matter which is in a plasma state—and thus exhibits electromagnetic properties—has possibilities of more efficient interaction with gravitational radiation, as compared to neutral matter.

In this paper we study the propagation of weak gravitational waves in a collisionless plasma with an external static and homogenous magnetic field, parallel to the direction of propagation. Naturally, the matter and fields produce a background curvature, but nevertheless it is meaningful to treat the background as Minkowski space, provided the wavelength is much shorter than the background curvature (see the Appendix for a detailed discussion). It turns out that there is a new effect on the gravitational waves that appears due to the presence of the external magnetic field—a gravitational analogue of *cyclotron damping* of electromagnetic waves (see Ref. [13]). In the electromagnetic case, the waves may interact resonantly with the gyrating motion of the particles, and the resonance occurs for particles that experience a wave whose Doppler shifted frequency equals the gyrofrequency. In the gravitational analogue, the resonance occurs when the (Doppler shifted) wave frequency is twice the gyrofrequency. A mechanism for gravitational wave damping similar to cyclotron damping has been considered previously, namely, Landau damping [1]. However, cyclotron damping is a potentially more important mechanism, since, for this case, efficient wave-particle interaction may take place without the presence of ultrarelativistic particles. The possibility for cyclotron damping of gravitational waves has been considered in Ref. [8] and recently in Ref. [14]. However, the calculation of the damping coefficient in Ref. [14] does not rest on a self-consistent kinetic theory, and it turns out that our value of the damping based on the Vlasov equation is smaller than theirs by several orders of magnitude. On the other hand, if we take the limit of a nonrelativistic Maxwellian distribution function, our results essentially (see Ref. [15]) reduces to those of Ref. [8].

In the case where the unperturbed distribution function of the particles is not in thermodynamical equilibrium, gravitational wave instabilities rather than damping may occur. We give a condition on the distribution function for instabilities to develop and demonstrate that it may be fulfilled, for instance, by plasmas with a temperature anisotropy. Naturally, the gravitational wave dispersion is also modified by the presence of the magnetic field. The dependence of the damping and dispersion on the temperature and on the cyclotron and gravitational wave frequencies is investigated. Furthermore, we confirm that the natural wave modes are circularly polarized waves also in the relativistic regime and that—as a consequence of the different dispersion relations for these modes (in an electron-ion type of plasma)—gravitational waves experience a phenomena analogous to Faraday rotation of electromagnetic waves in such a medium.

The paper is organized as follows. In Sec. II we present the equations governing our system, using a tetrad description in order to make the interpretation of our results more straightforward. Section III reviews the problem of single particle motion in the presence of a gravitational wave propagating parallel to an external magnetic field. In particular, it is shown that resonant particles experience continuous acceleration with velocities approaching the speed of light in vacuum. In Sec. IV the interaction between the gravitational waves and the plasma is studied self-consistently in the linearized approximation using the Einstein-Maxwell-Vlasov system of equations. The damping and dispersion of the gravitational wave is studied in some detail for an electron-ion plasma as well as for an electron-positron plasma in thermodynamical equilibrium. The case of a plasma which is not in thermodynamical equilibrium is studied in Sec. V. Finally, in Sec. VI, we summarize our results and discuss their implications. There is also an appendix where the problem of separating effects into background curvature effects (''indirect'' matter effect) and ''direct'' matter effects is discussed. The results in this paper, obtained using a tetrad frame formalism, are compared with those of a coordinate frame formalism.

II. BASIC EQUATIONS

We consider the interaction between weak gravitational waves and a collisionless plasma in an external magnetic field. Since we consider nonempty space the background space-time is necessarily curved. However, if the wavelength of the gravitational waves and the interaction region is small relative to the background curvature we may take the background to be flat and static, and the energy-momentum tensor to be the one corresponding to the perturbations of the electromagnetic and material fields (see the Appendix).

For simplicity, the direction of propagation is assumed parallel to the magnetic field which we take to be static and homogeneous. Linearized, the Einstein field equations (EFEs) take the form

$$\Box h_{ab} = -2\kappa\left[\delta T_{ab} - \frac{1}{2}\delta T\,\eta_{ab}\right] \tag{1}$$

provided the gauge condition $h^{ab}_{\ \ ,b}=0$ is fulfilled, which is equivalent to state that only tensorial perturbations are present. That the gauge condition is indeed satisfied will be verified in Sec. III below. Here $\Box \equiv [c^{-2}\partial_t^2 - \partial_z^2]$, h_{ab} is the small deviation from the Minkowski background metric, i.e., $g_{ab}=\eta_{ab}+h_{ab}$, $\kappa \equiv 8\pi G/c^4$, δT_{ab} is the part of the energy-momentum tensor containing small electromagnetic and material field perturbations associated with the gravitational waves and $\delta T = \delta T^a_a$. In the following it is understood that we neglect contributions of second order and higher in h_{ab}. In our notations $a,b,c,\ldots=0,1,2,3$ and $i,j,k,\ldots=1,2,3$ and the metric has the signature $(-+++)$.

In vacuum, a linearized gravitational wave can be transformed into the transverse and traceless (TT) gauge. Then we have the following line-element and corresponding orthonormal frame basis:

$$ds^2 = -c^2dt^2 + [1+h_+(\xi)]dx^2 + [1-h_+(\xi)]dy^2 + 2h_\times(\xi)dx\,dy + dz^2, \tag{2}$$

$$\mathbf{e}_0 \equiv c^{-1}\partial_t, \quad \mathbf{e}_1 \equiv \left(1-\frac{1}{2}h_+\right)\partial_x - \frac{1}{2}h_\times\partial_y,$$

$$\mathbf{e}_2 \equiv \left(1+\frac{1}{2}h_+\right)\partial_y - \frac{1}{2}h_\times\partial_x, \quad \mathbf{e}_3 \equiv \partial_z,$$

where $\xi \equiv z - ct$ and $h_+, h_\times \ll 1$. As it turns out, the gravitational waves take this form also in the particular case (propagation parallel to the magnetic field) we are considering. The difference to the vacuum case will be that $\xi = z - v_{\rm ph}t$, where $v_{\rm ph}$ is the phase velocity of the gravitational wave. From now on we will refer to tetrad components rather than coordinate components.

We follow the approach presented in Ref. [9] for handling gravitational effects on the electromagnetic and material fields. Suppose an observer moves with 4-velocity u^a. This observer will measure the electric and magnetic fields $E_a \equiv F_{ab}u^b$ and $B_a \equiv \frac{1}{2}\epsilon_{abc}F^{bc}$, respectively, where F_{ab} is the electromagnetic field tensor and ϵ_{abc} is the volume element on hypersurfaces orthogonal to u^a. It is convenient to introduce a 3-vector notation $\mathbf{E} \equiv (E^i) = (E^1,E^2,E^3)$ etc., and $\nabla \equiv \mathbf{e}_i$. From now on we will assume that $u^0 = c$ is the only nonzero component of u^a. Generally, the Maxwell equations contain terms coupling the electromagnetic field to the gravitational radiation field. If the gravitational waves propagate parallel to a magnetic field there are no gravitationally induced effects on **E** and **B**. This can be deduced as follows: Given the Ricci rotation coefficients for gravitational waves in the TT gauge, the ''gravitational source terms'' in the Maxwell equations in Ref. [9] vanishes for the given orientation of the magnetic field.

The equation of motion for a particle of mass m and charge q in an electromagnetic and gravitational wave field is

$$\frac{d}{dt}\mathbf{p} = q[\mathbf{E} + (\gamma m)^{-1}\mathbf{p}\times\mathbf{B}] - \mathbf{G}, \tag{3}$$

where $\gamma = \sqrt{1+p_ip^i/(mc)^2}$ and the four-momenta is $p^a = \gamma m dx^a/dt$. The gravitational-force-like term $G^i \equiv \Gamma^i_{ab}p^ap^b/\gamma m$, where Γ^i_{ab} are the Ricci rotation coefficients, becomes

$$G_1 = \frac{1}{2}(v_{\rm ph} - p_z/\gamma m)[\dot h_+ p_1 + \dot h_\times p_2], \tag{4}$$

$$G_2 = \frac{1}{2}(v_{\rm ph} - p_z/\gamma m)[-\dot h_+ p_2 + \dot h_\times p_1], \tag{5}$$

$$G_3 = \frac{1}{2}(\gamma m)^{-1}[\dot h_+(p_1^2 - p_2^2) + 2\dot h_\times p_1 p_2] \tag{6}$$

for weak gravitational waves propagating in the z direction in Minkowski space, where the overdot represents derivative with respect to ξ.

In order to account for resonant wave-particle interactions we apply kinetic plasma theory, representing each particle species by a distribution function f governed by the Vlasov equation. In tetrad form the Vlasov equation reads [16]

$$\mathcal{L}f = 0,$$

where the Liouville operator is

$$\mathcal{L} \equiv \partial_t + (c/p^0)p^ie_i + [F^i_{\rm EM} - \Gamma^i_{ab}p^ap^bc/p^0]\partial_{p^i}$$

and the electromagnetic force responsible for geodesic deviation is $F^i_{\mathrm{EM}} \equiv q(E^i + \epsilon^{ijk} p_j B_k / \gamma m)$. In vector notation the Vlasov equation reads

$$\partial_t f + \frac{\mathbf{p}\cdot\nabla f}{\gamma m} + \left[q\left(\mathbf{E} + \frac{\mathbf{p}\times\mathbf{B}}{\gamma m} \right) - \mathbf{G} \right] \cdot \nabla_{\mathbf{p}} f = 0, \tag{7}$$

where $\nabla_{\mathbf{p}} \equiv (\partial_{p_1}, \partial_{p_2}, \partial_{p_3})$. In the absence of gravitational waves, the Vlasov equation has the following spatially homogeneous (thermodynamical) equilibrium solution, the Synge-Jüttner distribution [17]

$$f_{SJ} = \frac{n_0 \mu}{4\pi (mc)^3 K_2(\mu)} e^{-\mu\gamma}, \tag{8}$$

where n_0 is the spatial particle number density, $\mu \equiv mc^2/k_B T$, k_B is the Boltzmann constant, T the temperature, and $K_2(\mu)$ is a modified Bessel function of second kind. Generally, the unperturbed static solutions to Eq. (7) consistent with a homogeneous and static magnetic field, say in the z direction, but not necessarily in thermodynamical equilibrium are distribution functions $f = f(p_\perp, p_3)$, where $p_\perp = \sqrt{p_1^2 + p_2^2}$.

Since there are no induced electromagnetic fields, the perturbed energy-momentum tensor can be written

$$\delta T_{ab} = \sum_{\mathrm{PS}} \int \frac{p_a p_b}{m\gamma} f_G d^3 p \tag{9}$$

where f_G is the gravitational perturbation of the distribution function and the summation is over particle species (PS). The self-consistent set of equations governing the interaction between the gravitational waves and the plasma are thus the two coupled equations (1) and (7). Obviously, as we have adapted a tetrad formalism, we mean the tetrad equivalence of Eq. (1) [18].

III. CYCLOTRON RESONANCE ACCELERATION

As we will consider effects that are due to wave-particle interaction, it might be in place to first review some results of single particle (test-particle) motion in the presence of gravitational waves and an external magnetic field. Single particle motion in gravitational wave fields has been examined by many authors, see, e.g., Ref. [1], and references therein. Particles moving in a monochromatic gravitational wave field experience periodic changes in its energy and periodic deviation from its mean direction of propagation. If the particle motion is constrained, for instance by a magnetic field, the change in energy and momentum may be cumulative. We refer to this as resonant acceleration.

We focus here on charged particles in a homogenous and static magnetic field parallel to the direction of propagation of gravitational waves. The single particle motion in this situation have been investigated in some detail by Ref. [14], treating it as a Hamiltonian dynamical problem, and also by Ref. [19]. Most noticeable is that not only can resonant acceleration occur, particles can even be ''trapped'' in such a resonant state and experience essentially unlimited linear growth in kinetic energy (linear in coordinate time) and the parallel velocity will approach the velocity of light. The solution we present here does not—in contrast to that of Ref. [14]—result in an *exact* description of the particle motion but does reveal the main effect (resonant acceleration of particles) and has the advantage of being straight forward. It also provides an intuitively clear explanation of the mechanism and the resemblance to electromagnetic cyclotron resonance acceleration [20]. In the electromagnetic (vacuum) case this *phase-lock* situation also exists, i.e., unlimited resonant acceleration of ''trapped'' particles.

Since there are no gravitationally induced electromagnetic fields (linear in h_{ab}), Eq. (3) becomes

$$\frac{d}{dt}\mathbf{p} = (\omega_c/\gamma)\mathbf{p}\times\hat{\mathbf{z}} - \mathbf{G} \equiv \mathbf{F}, \tag{10}$$

where $\omega_c \equiv qB/m$. The gravitational wave is assumed monochromatic and arbitrarily polarized: $h_+ = \hat{h}_+ \exp[i(kz - \omega t)] + \mathrm{c.c.}$ and $h_\times = \hat{h}_\times \exp[i(kz - \omega t)] + \mathrm{c.c.}$, where c.c. stands for complex conjugate. In this section ω and k are assumed real.

We assume $\omega_c/\gamma \gg \omega h_{ab}$ so that the gravitational force in Eq. (10) is small compared to the electromagnetic one. Thus, the particle orbits are close to the gyrating motion in the absence of a gravitational field, and we therefore make the ansatz $p_1 = (1/\sqrt{2})\hat{p}(t)\exp(-i\omega_c t/\gamma) + \mathrm{c.c.}$ and $p_2 = \pm (i/\sqrt{2})\hat{p}(t)\exp(-i\omega_c t/\gamma) + \mathrm{c.c.}$, where the amplitude $\hat{p}(t)$ depends slowly on time [i.e., $|d\hat{p}/dt| \ll |(\omega_c/\gamma)\hat{p}|$] due to the gravitational influence and $\pm = \mathrm{sgn}(q)$. Note that the cyclotron period is not itself a constant, but is depending on the gamma factor that also is varying slowly. Consider now the explicit form of the parallel driving force:

$$\begin{aligned} F_3 = &-\frac{i}{2}\frac{k}{\gamma m}[\hat{h}_+ \hat{p}^2 \pm i\hat{h}_\times \hat{p}^2] e^{i(kz - \omega t - 2\omega_c t/\gamma)} + \mathrm{c.c.}, \\ &-\frac{i}{2}\frac{k}{\gamma m}[\hat{h}_+ \hat{p}^{*2} \mp i\hat{h}_\times \hat{p}^{*2}] e^{i(kz - \omega t + 2\omega_c t/\gamma)} + \mathrm{c.c.}, \end{aligned} \tag{11}$$

where the asterisk denotes complex conjugate. A particle with trajectory $z(t)$ will typically experience an irregular oscillatory force. Unless the particle is resonant (or almost resonant) with the wave, the parallel motion will be random and there will be no net effect—except for the possibility of (small) diffusive acceleration [14]. Particles may, however, be resonant, i.e., have a trajectory such that there will be a nonoscillatory force resulting in a lasting acceleration/deceleration over several time periods of gyration. Almost resonant particles will be acted on by a force varying in time on a time scale (depending on the magnitude of the mismatch) slower than the gravitational wave period. From Eq. (11) we see that there are two possibilities for a particle to be resonant and thus acted on by a constant force:

$$k\frac{d\overline{z}}{dt}-\omega-\frac{2\omega_c}{\gamma}=0,$$

$$k\frac{d\overline{z}}{dt}-\omega+\frac{2\omega_c}{\gamma}=0$$

implying

$$\omega=-\frac{2\omega_c}{\gamma-p_3/mv_{\mathrm{ph}}},\tag{12}$$

$$\omega=\frac{2\omega_c}{\gamma-p_3/mv_{\mathrm{ph}}},\tag{13}$$

respectively, where $v_{\mathrm{ph}}=\omega/k$. By $\overline{z}$ we mean the time averaged trajectory and since we will only consider time-averaged effects, we have put $p_3=m\gamma d\overline{z}/dt$ here and throughout the remainder of this section. Physically, the particles are resonant when they see a wave whose Doppler shifted frequency is twice the gyrofrequency ω_c/γ. The factor 2—not present in the electromagnetic case—is due to the fact that the driving force is quadratic in $\hat{p}$. Note that the two resonance conditions can be satisfied *simultaneously* only by particles that are oppositely charged.

The question is now whether or not the resonant particles remain resonant even though they are accelerated—a change in p_3 and γ may potentially lead to a violation of the resonance condition (12) or (13). Clearly the resonance is preserved only if $\gamma-p_3/mv_{\mathrm{ph}}$ is a constant of motion. The gravitational force, Eqs. (4)–(6), has the property $\mathbf{G}\cdot\mathbf{p}=\gamma m v_{\mathrm{ph}}G_3$ [21]. Thus it holds that

$$\frac{d}{dt}\left(\gamma-\frac{p_3}{mv_{\mathrm{ph}}}\right)=\frac{\mathbf{F}\cdot\mathbf{p}}{\gamma m^2c^2}-\frac{F_3}{mv_{\mathrm{ph}}}=0\tag{14}$$

if and only if $v_{\mathrm{ph}}=\pm c$, i.e., the resonance is preserved in vacuum but generally not in a medium. In many situations for gravitational waves, however, the vacuum relation $v_{\mathrm{ph}}=\pm c$ holds to a very good approximation.

In the remainder of this section we will confine ourselves to the case $v_{\mathrm{ph}}=c$, $\mathrm{sgn}(q)=-1$ and to the resonance condition (13). We define the constant of motion $\alpha\equiv\gamma-p_3/mc$ and observe that $p_3=mc(\gamma-\alpha)$ and $|\hat{p}|=mc\sqrt{2\gamma\alpha-1-\alpha^2}$. By considering the (time averaged) time evolution of γ we find

$$\frac{d\gamma}{dt}=\omega[2\alpha-\gamma^{-1}(1+\alpha^2)][|\hat{h}_+|\sin\varphi+|\hat{h}_\times|\cos\psi],\tag{15}$$

where $\varphi\equiv\arg(\hat{h}_+\hat{p}^{*2}/\gamma m)$ and $\psi\equiv\arg(\hat{h}_\times\hat{p}^{*2}/\gamma m)$. Apparently the kinetic energy is a monotonously increasing function for particles with suitable initial phase of the gyrating motion, i.e., for

$$h\equiv|\hat{h}_+|\cos\varphi+|\hat{h}_\times|\sin\psi>0$$

independently of the initial magnitude of $\hat{p}$ and p_3. From now on we limit ourselves to such particles. For large times when $\gamma\gg1$, Eq. (15) implies $\gamma\propto2\alpha h\omega t$. Using the expressions for $\hat{p}$ and p_3 in terms of γ and α we note that coordinate momenta $\hat{p}/\gamma\to0$ as $t\to\infty$ whereas p_3/γ scales as

$$p_3/\gamma\to mc-\frac{mc}{2h\omega t}.\tag{16}$$

This implies the possibility of unlimited acceleration in which $p_3/\gamma m$ approaches c on a time scale $t\sim(h\omega)^{-1}$. Note, however, that the assumption $\omega_c/\gamma\gg\omega h_{ab}$ formally restricts the predicting power of these results. However, as demonstrated by Ref. [14] the linear growth of γ (due to parallel acceleration) remain even beyond $\omega_c/\gamma\gg\omega h_{ab}$.

The sections to follow will concentrate on linearized gravitational wave propagation, in which case the particles will be assumed to deviate only slightly from the unperturbed orbits. It should be noted, however, that cyclotron acceleration may have interesting applications in the vicinity of a binary pulsars close to merging. Practically speaking the effective distance of acceleration close to the source will be limited by effects due to a three-dimensional (3D) geometry. On the other hand, particles close to pulsars are likely to have a relativistic background temperature, in which case the resonant ones may be accelerated to ultrahigh energies.

IV. CYCLOTRON DAMPING

As seen in the preceding section, charged particles in a homogenous static magnetic field can be accelerated and decelerated by gravitational waves. Thus it should not be surprising that the gravitational wave will be damped—or be unstable and experience growth—as it propagates through a collisionless plasma. The damping (or growth) rate will depend on how the particles are distributed in momentum space. In order to incorporate the damping effect due to this resonant wave-particle interaction mechanism, we use a kinetic description of the plasma, i.e., each plasma component is represented by a distribution function $f(x^a,p^i)$. The gravitational waves $h_+=\hat{h}_+\exp[i(kz-\omega t)]$ and $h_\times=\hat{h}_\times\exp[i(kz-\omega t)]$ are superimposed on the Minkowski background metric (see the Appendix), and are assumed to be associated with a small perturbation $f_G=\hat{f}_G\exp[i(kz-\omega t)]$, of the distribution function, i.e., $f=f_0+f_G$, where f_0 is a stationary solution to the Vlasov equation (7) in the absence of gravitational waves. This means that the background distribution function f_0 is a function of $p_\perp$ and p_3, where $p_\perp=\sqrt{p_1^2+p_2^2}$.

A. Linearized Vlasov equation

We begin by calculating the perturbed distribution function that results from a gravitational wave propagating parallel to an external magnetic field. Linearizing the Vlasov Eq. (7) in h_+, $h_\times$, and f_G gives

$$i\left[k\frac{p_3}{\gamma m}-\omega\right]f_G+\frac{q}{\gamma m}\mathbf{p}\times\mathbf{B}\cdot\boldsymbol{\nabla}_{\mathbf{p}}f_G=\mathbf{G}\cdot\boldsymbol{\nabla}_{\mathbf{p}}f_0. \tag{17}$$

It is convenient to change to cylindrical coordinates in momentum space, i.e., we define $p_1\equiv p_\perp\cos\phi$, $p_2\equiv p_\perp\sin\phi$ and $p_3\equiv p_\parallel$, so that $(q/\gamma m)\mathbf{p}\times\mathbf{B}\cdot\boldsymbol{\nabla}_{\mathbf{p}}=-\gamma^{-1}\omega_c\partial_\phi$ and Eq. (17) thus becomes

$$\left[i\left(\frac{kp_\parallel}{\gamma m}-\omega\right)-\frac{\omega_c}{\gamma}\partial_\phi\right]f_G=i\omega[h_+\cos 2\phi+h_\times\sin 2\phi]\mathcal{F}_0, \tag{18}$$

where

$$\mathcal{F}_0\equiv\frac{p_\perp}{2v_{ph}}\left[\left(v_{ph}-\frac{p_\parallel}{\gamma m}\right)\partial_{p_\perp}f_0+\frac{p_\perp}{\gamma m}\partial_{p_\parallel}f_0\right]. \tag{19}$$

Equation (18) has the solution

$$f_G=\frac{1}{2}\frac{\gamma\omega\mathcal{F}_0e^{i2\phi}}{kp_\parallel/m-\gamma\omega-2\omega_c}(h_+-ih_\times)+\frac{1}{2}\frac{\gamma\omega\mathcal{F}_0e^{-i2\phi}}{kp_\parallel/m-\gamma\omega+2\omega_c}(h_++ih_\times). \tag{20}$$

The occurrence of singularities in Eq. (20) indicate that there is a resonant interaction. In the case $\omega_c=0$ there are no singularities, because without magnetic field $\omega\geqslant ck$ in our approximation, and therefore Landau damping of gravitational waves do not occur. For any finite value of ω_c, however, the expression for f_G has singularities and we therefore expect cyclotron damping to occur.

B. Dispersion relation

We now solve the tetrad equivalence [18] of EFE (1). Using Eqs. (9) and (20) it is straightforward to confirm that the TT gauge is a consistent choice in our case. The two diagonal elements both read

$$\begin{aligned}[\omega^2-c^2k^2]h_+&=2c^2\kappa\sum_{\text{PS}}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\frac{p_\perp^3}{\gamma m}\cos^2\phi f_G dp_\perp d\phi dp_\parallel\\&=\frac{1}{2}(h_+-ih_\times)I_-+\frac{1}{2}(h_++ih_\times)I_+\end{aligned} \tag{21}$$

and the two off diagonal elements give

$$\begin{aligned}[\omega^2-c^2k^2]h_\times&=2c^2\kappa\sum_{\text{PS}}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\frac{p_\perp^3}{\gamma m}\cos\phi\sin\phi f_G dp_\perp d\phi dp_\parallel\\&=\frac{i}{2}(h_+-ih_\times)I_--\frac{i}{2}(h_++ih_\times)I_+,\end{aligned} \tag{22}$$

where we have made use of Eq. (20) and

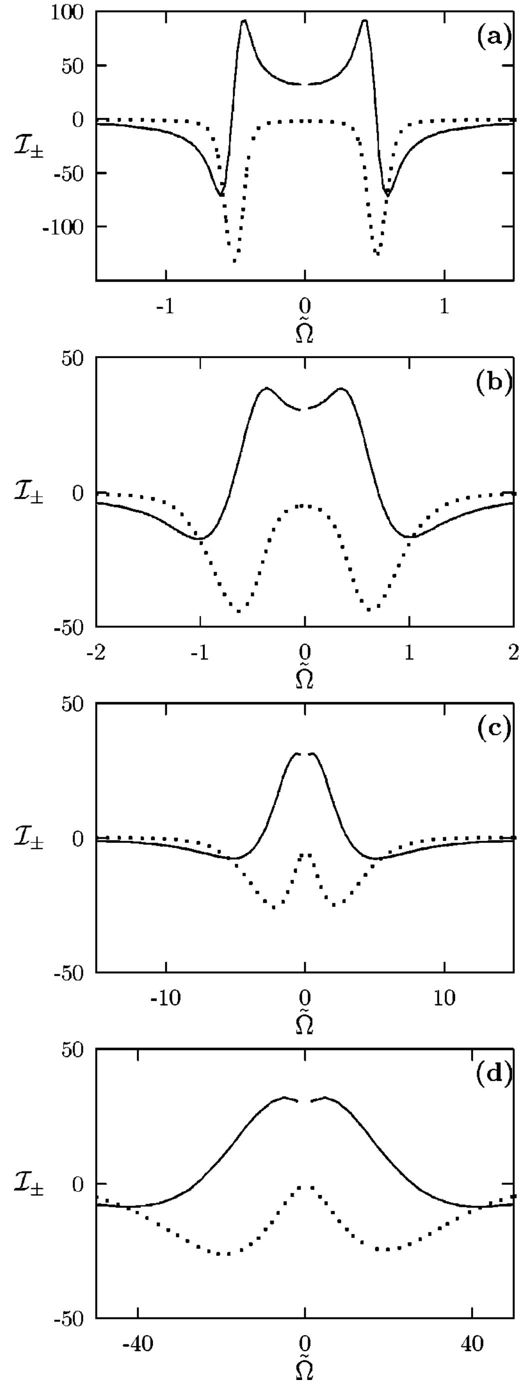

FIG. 1. The real (solid line) and imaginary (dotted line) part of $\mathcal{I}_\pm\equiv I_\pm\mu\tau_{ep}^2$ for an electron-positron plasma at four different temperatures: (a) $\mu=100$, (b) $\mu=10$, (c) $\mu=1$, and (d) $\mu=0.1$.

$$I_\pm=\sum_{\text{PS}}C\int_{-\infty}^{\infty}\int_0^{\infty}p_\perp^3\frac{\mathcal{F}_0}{kp_\parallel/m-\gamma\omega\mp 2\omega_c}dp_\perp dp_\parallel \tag{23}$$

together with $C\equiv\pi\omega\kappa c^2/m$. The two equations (21) and (22) combine to

$$[\omega^2-c^2k^2-A_+-iB_+](h_++ih_\times)=0, \tag{24}$$

$$[\omega^2-c^2k^2-A_--iB_-](h_+-ih_\times)=0, \tag{25}$$

where

$$A_{\pm}\equiv \mathrm{Re}\, I_{\pm} \quad \text{and} \quad B_{\pm}\equiv \mathrm{Im}\, I_{\pm}\,.$$

The natural gravitational wave modes for EFE with the given source are thus the circularly polarized modes $h_{+}+ih_{\times}$ and $h_{+}-ih_{\times}$. Equations (24) and (25) contain the information about the dispersion and the damping or growth (due to $B_{\pm}\neq 0$). Due to the smallness of the gravitational coupling constant, $A_{\pm}$ and $B_{\pm}$ can typically be considered small in the sense that the dispersion relations Eqs. (24) and (25) read

$$\omega \approx ck + A_{\pm}/2\omega + iB_{\pm}/2\omega. \tag{26}$$

Throughout the remainder of this paper we will have this approximation in mind and we will occasionally make use of $\omega\approx ck$ to simplify the expressions for $I_{\pm}$.

The fact that $A_{+}\neq A_{-}$, unless the plasma components are of equal particle masses and as long $\omega_c\neq 0$, implies that the two gravitational wave modes have slightly different phase velocities. Thus, an incident linearly polarized wave—being a superposition of the two circularly polarized states—will experience a polarization shift, i.e., the direction of linear polarization will be rotated as it propagates through the medium. In the case of electromagnetic waves this is known as Faraday rotation. In principle this could be an important result as the polarization of a gravitational wave carries valuable information about the emitting source [22], e.g., the inclination of the spin axis of a quadrupole moment source.

The pole contribution in the integral in Eq. (23) is dealt with in the standard fashion, i.e., by letting the contour of integration pass below the pole. It should be noted that this approach disregards what happens with the distribution function close to the singularity, and it also misses some other aspects of the damping process, see, e.g., Ref. [23], but it is the simplest way to find the main effect due to the pole, and for our purposes it suffices.

C. Equilibrium plasma

For a plasma in a state of thermodynamical equilibrium the unperturbed relativistic expression for the distribution function is the Synge-Jüttner distribution $f_0=f_{\mathrm{SJ}}$, defined in Eq. (8), and for this choice $\mathcal{F}_0$ reduces to

$$\mathcal{F}_0=-\frac{\mu p_{\perp}^2}{2\gamma(mc)^2}f_{\mathrm{SJ}}\,. \tag{27}$$

As no instabilities can develop the gravitational waves will exhibit damping. In this section we examine how the damping and dispersion depends on the ratio ω_c/ω and on the temperature of the plasma. First we examine the nonrelativistic regime for which analytic results can be obtained. In the relativistic regime we then present results for an electron-positron plasma and an electron-ion plasma obtained by numerical integration of Eq. (23). For this purpose it is practical to introduce normalized momenta $\mathfrak{p}_{\perp}\equiv p_{\perp}/mc$, $\mathfrak{p}_{\parallel}\equiv p_{\parallel}/mc$, and a dimensionless frequency ratio $\Omega=\omega_c/\omega$.

1. Nonrelativistic temperature

In the regime of nonrelativistic particle velocities Eq. (23) together with Eq. (27) becomes

$$I_{\pm}=-\sum_{\mathrm{PS}}\frac{32\pi}{\tau^2}\left(\frac{\mu}{2\pi}\right)^{3/2}\mu^{-2}\int_{-\infty}^{\infty}\frac{e^{-\mu\mathfrak{p}_{\parallel}^2/2}}{\mathfrak{p}_{\parallel}-1\mp 2\Omega}\,d\mathfrak{p}_{\parallel} \tag{28}$$

after performing the $p_{\perp}$ integration, where we have introduced $\tau\equiv 1/\sqrt{\pi m n_0 G}$, which is the characteristic time for gravitational contraction of a gas with density n_0 and particle mass m. In the case of a cosmological plasma τ coincides, apart from a factor $\sqrt{3/8}$, with the Hubble time. From Eq. (28) we obtain to lowest order in the temperature

$$A_{\pm}=\sum_{\mathrm{PS}}\frac{16}{\tau^2\mu}\frac{1}{1\pm 2\Omega}, \tag{29}$$

$$B_{\pm}=-\sum_{\mathrm{PS}}\frac{8\sqrt{2\pi}}{\tau^2\sqrt{\mu}}e^{-\mu(1\mp 2\Omega)^2/2}. \tag{30}$$

Note that for Eq. (29) to apply we must assume $\mu^{-1}(1\pm 2\Omega)^{-1}\ll 1$, i.e., for the wave frequency close enough to twice the gyrofrequency, higher order thermal effects is always important for the wave dispersion, whereas Eq. (30)—which follows from the residue theorem—holds for all values of Ω. Still, it is clear that the dispersion can be enhanced by the magnetic field due to the existence of the resonance. The exponential decrease of the damping with $1\mp 2\Omega$ implies that significant damping only occurs in a limited region in frequency space close to $1\mp 2\Omega=0$ for a low temperature plasma. In the limit of zero temperature the size of this region tends to zero. The magnitude of both $A_{\pm}$ and $B_{\pm}$ are monotonically increasing with temperature and vanishes at zero temperature. The results in this subsection essentially agree with Ref. [8], see Ref. [15]. In the case of no magnetic field, $\Omega=0$, the dispersion relations (24) and (25) both reduce to $\omega^2-c^2k^2-A=0$, where $A\equiv\Sigma_{\mathrm{PS}}16/\tau^2\mu$. This expression is in agreement [24] with Refs. [1–5].

2. Electron-positron plasma

Denote $\tau_{ep}\equiv 1/\sqrt{\pi m_e n_0 G}$ (e and p stands for electron and positron, respectively). In this case only the sign of the charge differs for the two particle species and we may write Eq. (23), applying Eq. (27), as

$$I_{\pm}(\tilde{\Omega})=-\tau_{ep}^{-2}\sum_{\mathrm{PS}}\frac{\mu^2}{K_2(\mu)}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\mathfrak{p}_{\perp}^5}{\sqrt{1+\mathfrak{p}_{\perp}^2+\mathfrak{p}_{\parallel}^2}}\,\frac{e^{-\mu\sqrt{1+\mathfrak{p}_{\perp}^2+\mathfrak{p}_{\parallel}^2}}}{\mathfrak{p}_{\parallel}-\sqrt{1+\mathfrak{p}_{\perp}^2+\mathfrak{p}_{\parallel}^2}\mp 2\,\mathrm{sgn}(q)\tilde{\Omega}}\,d\mathfrak{p}_{\perp}\,d\mathfrak{p}_{\parallel}\,, \tag{31}$$

where $\tilde{\Omega}=\tilde{\omega}_c/\omega$ and $\tilde{\omega}_c=|q|B/m$. Note that $I_\pm(\tilde{\Omega})=I_\mp(\tilde{\Omega})=I_\pm(-\tilde{\Omega})$. The normalized function $\mathcal{I}_\pm(\tilde{\Omega})\equiv I_\pm\mu\tau_{ep}^2$ is numerically calculated for four different temperatures, namely, $\mu=100$ (''nonrelativistic'') 10, 1, and 0.1 (''ultrarelativistic''). The results are displayed in Fig. 1 (note that $\mathcal{I}_\pm$ is normalized against the temperature). In the nonrelativistic case [see Fig. 1(a)] we have a finite region, $-1/2\lesssim\tilde{\Omega}\lesssim 1/2$, where $\mathrm{Re}\,I_\pm$ is positive. $\mathrm{Re}\,I_\pm$ changes sign at $\tilde{\Omega}\approx\pm 1/2$ and approaches zero as $1/(\mp\tilde{\Omega})$ in the limit $\tilde{\Omega}\to\pm\infty$. $\mathrm{Im}\,I_\pm$ is negative definite (and thus there is indeed *damping* of the gravitational wave for any finite $\tilde{\Omega}$) and has a Gaussian shape around the resonances $\tilde{\Omega}\approx\pm 1/2$.

This is also the characteristic behavior for $\mathrm{Re}\,I_\pm$ and $\mathrm{Im}\,I_\pm$ at higher temperatures. The relativistic effects on the dispersion and the damping are the following: (i) The resonance peaks of $\mathrm{Im}\,I_\pm$ (which occur at $\tilde{\Omega}=\pm 1/2$ at zero temperature) are shifted to higher values of $|\tilde{\Omega}|$ for higher temperatures. Also $\mathrm{Re}\,I_\pm$ experiences a similar shift. In the ultrarelativistic case [see Fig. 1(d)], $\mathrm{Re}\,I_\pm$ changes sign at $\tilde{\Omega}\approx\pm 25$ and $\mathrm{Im}\,I_\pm$ is centered about $\tilde{\Omega}\approx\pm 20$. (ii) The magnitude of both $\mathrm{Re}\,I_\pm$ and $\mathrm{Im}\,I_\pm$ *increases* roughly linearly with temperature [note that the curves are normalized against $(\mu\tau_{ep}^2)^{-1}$ in Fig. 1], which can be compared with the temperature dependence in the nonrelativistic regime that is given by Eqs. (29) and (30). (iii) The region of damping *broadens*. In the cold limit $\mathrm{Im}\,I_\pm$ takes the form of two separated Gaussian functions, the width tending to zero with diminishing temperature. These regions are widened and the gaussian shape is deformed with increasing temperature. In the ultrarelativistic case $\mathrm{Im}\,I_\pm$ decays exponentially as $\tilde{\Omega}\to\pm\infty$ but approaches zero more abruptly as $\tilde{\Omega}\to 0$.

3. Electron-ion plasma

Denote $\tau=\tau_{ei}\equiv 1/\sqrt{\pi m_i n_0 G}$ (e and i stand for electron and ion, respectively). For an electron-ion plasma there is an asymmetry between the particle species due to the small mass ratio $\varepsilon\equiv m_e/m_i$, giving different order of magnitudes for the two cyclotron frequencies ω_{ce} and ω_{ci}. Thus the resonances will occur for very different gravitational wave frequencies. Given Eq. (27), we have the following form of Eq. (23):

$$I_\pm=-\tau_{ei}^{-2}\frac{\varepsilon\mu_e^2}{K_2(\mu_e)}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\mathfrak{p}_\perp^5}{\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}}\,\frac{e^{-\mu_e\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}}}{\mathfrak{p}_\parallel-\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}\pm 2\tilde{\Omega}_e}\,d\mathfrak{p}_\perp d\mathfrak{p}_\parallel$$

$$-\tau_{ei}^{-2}\frac{\mu_i^2}{K_2(\mu_i)}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\mathfrak{p}_\perp^5}{\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}}\,\frac{e^{-\mu_i\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}}}{\mathfrak{p}_\parallel-\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}\mp 2\tilde{\Omega}_i}\,d\mathfrak{p}_\perp d\mathfrak{p}_\parallel, \qquad (32)$$

where

$$\mu_{e/i}=\frac{m_{e/i}c^2}{k_B T_{e/i}} \quad \text{and} \quad \tilde{\Omega}_{e/i}\equiv\frac{\tilde{\omega}_{ce/i}}{\omega}=\frac{|q|B}{m_{e/i}\omega}.$$

Provided that $T_e=T_i$, μ_i and μ_e differs typically by three orders in magnitude. This implies that the ions can be considered nonrelativistic even in the regime of ultrarelativistic electrons ($\mu_e\sim 0.1$). It is convenient to consider the two frequency domains $\tilde{\Omega}_e\sim 1$ and $\tilde{\Omega}_i\sim 1$ separately. Noting that $\tilde{\Omega}_i=\varepsilon\tilde{\Omega}_e$ and $\mu_i=\varepsilon^{-1}\mu_e$, in the region $\tilde{\Omega}_e\sim 1$ we have

$$I_\pm(\tilde{\Omega}_e)\approx-\tau_{ep}^{-2}\frac{\mu_e^2}{K_2(\mu_e)}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\mathfrak{p}_\perp^5}{\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}}\,\frac{e^{-\mu_e\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}}}{\mathfrak{p}_\parallel-\sqrt{1+\mathfrak{p}_\perp^2+\mathfrak{p}_\parallel^2}\pm 2\tilde{\Omega}_e}\,d\mathfrak{p}_\perp d\mathfrak{p}_\parallel+\frac{16}{\tau_{ep}^2\mu_e}\,\frac{1}{1\pm 2\varepsilon\tilde{\Omega}_e}. \qquad (33)$$

The normalized function $\mathcal{I}_+(\tilde{\Omega}_e)\equiv I_+\mu\tau_{ep}^2$ is displayed in Fig. 2 for $\mu_e=100$ (''nonrelativistic'') 10, 1, and 0.1 (''ultrarelativistic''). By symmetry, $I_-(\tilde{\Omega}_e)$ is the mirror image of $I_+(\tilde{\Omega}_e)$, i.e., $I_-(\tilde{\Omega}_e)=I_+(-\tilde{\Omega}_e)$ and therefore this curve is not presented. Except for the lack of symmetry about $\tilde{\Omega}_e=0$ the result is similar to that of the electron-positron plasma. In the given temperature and frequency domains the ion contribution to the normalized function $I_\pm(\tilde{\Omega}_e)$ is just the approximately constant value 16.

The effect of resonant ions becomes important in the frequency domain $\tilde{\Omega}_i\sim 1$. In Fig. 3 we show $\mathcal{I}_+(\tilde{\Omega}_i)\equiv I_+\mu\tau_{ep}^2$ for $\mu_e=0.1$. The corresponding figures are qualitatively similar in the entire temperature domain $0.1-100$, however, and therefore only one of them is shown. The small region near $\tilde{\Omega}_i=0$ ($\tilde{\Omega}_i\sim\varepsilon$ to be specific) that contains the electron contribution has been left out.

4. The group velocity

The fact that $A_\pm$ is at some points negative implies, together with Eq. (26), that for some wavelengths and frequencies the group velocity of the waves exceeds the speed of light. Superluminal group velocities for gravitational waves

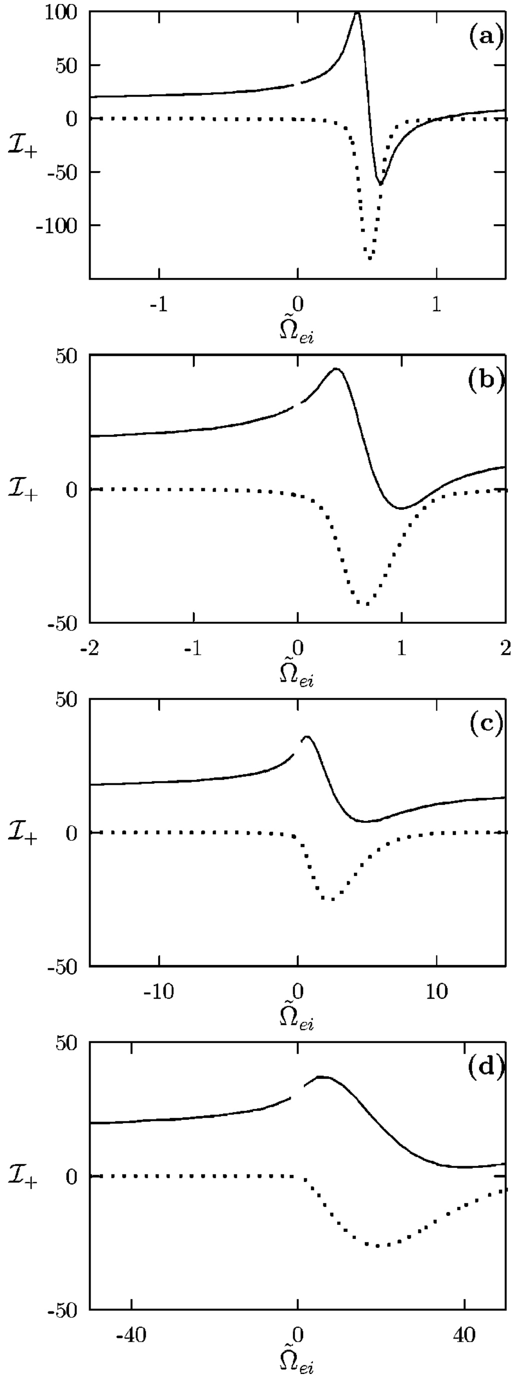

FIG. 2. The real (solid line) and imaginary (dotted line) part of $\mathcal{I}_+ \equiv I_+ \mu \tau_{ep}^2$ for an electron-ion plasma at four different temperatures: (a) $\mu = 100$, (b) $\mu = 10$, (c) $\mu = 1$, and (d) $\mu = 0.1$.

have been found before, see Ref. [25], and references therein. In most cases, but not in all, it has been an effect of the background curvature. It should be noted that several results from the literature are in contradiction with each other. Naturally, in our case the superluminal group velocity is a direct effect of the medium. For the case of an electron-positron plasma the group velocity corresponding to Eq. (26) is

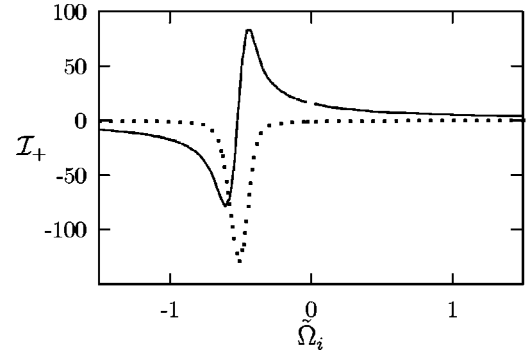

FIG. 3. The real (solid line) and imaginary (dotted line) part of $\mathcal{I}_+ \equiv I_+ \mu \tau_{ep}^2$ for an electron-ion plasma for $\mu_e = 0.1$. The small region near $\tilde{\Omega}_i = 0$ containing the electron contribution has been left out.

$$v_g \equiv \frac{d\omega}{dk} = c\left[1 - \frac{1}{2\omega^2}\left(A_\pm + \frac{\tilde{\omega}_c}{\omega} A'_\pm\right)\right], \tag{34}$$

where the prime denotes derivative with respect to $\tilde{\omega}_c/\omega$. For $\omega_c = 0$, i.e., in the case of no magnetic field, the medium is just a collisionless gas of charged particles and the group velocity is smaller than the velocity of light, in agreement with the results of previous authors, e.g., Ref. [1]. From Fig. 1 it is clear that there are regions where $v_g > c$ is realized, for instance about the point $\tilde{\omega}_c/\omega \approx 0.6$ (where $A_\pm$ has a local minima) in Fig. 1(a). Similarly, also an electron-ion plasma allows superluminal group velocities. Group velocities that exceeds the speed of light is not necessarily at odds with causality—an issue explored, for instance, in Ref. [25]. The group velocity can simply not be interpreted as the (gravitational wave) signal velocity in this situation.

V. NONEQUILIBRIUM PLASMA

In the case of thermodynamical nonequilibrium, the system has free energy that may feed a gravitational wave instability. This occurs whenever the imaginary part of $I_\pm$, defined by Eq. (23), is somewhere positive. Applying the residue theorem, Eq. (23) gives

$$B_\pm = \pi \sum_{\mathrm{PS}} C \int_0^\infty p_\perp^3 \left[\frac{\mathcal{F}_0}{\Delta_r k/m}\right]_{p_\parallel = p_r} dp_\perp\,, \tag{35}$$

where $p_r \equiv mc(\gamma_\mathrm{r} \pm 2\omega_c/\omega)$ is the resonant parallel momenta,

$$\gamma_r \equiv \frac{2\omega_c}{\omega^2 - c^2k^2}\left\{\omega - ck\left[1 - \frac{(\omega^2 - c^2k^2)}{4\omega_c^2}\left(1 + \frac{p_\perp^2}{m^2c^2}\right)\right]^{1/2}\right\}$$

is the gamma factor evaluated at the resonant momenta and

$$\Delta_r \equiv 1 - \frac{\omega}{ck}\frac{p_r}{\gamma_r mc}.$$

Instabilities occur for distribution functions and frequencies such that the condition $B_{\pm}>0$ is satisfied. For simplicity we focus on nonrelativistic temperatures from now on, in which case Eq. (35) reduces to

$$B_{\pm}=\sum_{\mathrm{PS}}\frac{4\pi^2}{n_0(mc)^2\tau^2}\int_0^\infty p_\perp^3[4p_c f_0+p_\perp^2\partial_{p_\parallel}f_0]_{p_\parallel=p_r}dp_\perp, \tag{36}$$

where $p_c\equiv\pm 2mc\omega_c/\omega$. In order to show that there indeed exist instabilities we consider the following example of a temperature-anisotropic ''drifting Maxwellian'' distribution function:

$$f_{\mathrm{drift}}=\frac{n_0}{\pi^{3/2}p_{\mathrm{th}\parallel}p_{\mathrm{th}\perp}^2}e^{-[(p_\parallel-p_{\mathrm{d}})^2/p_{\mathrm{th}\parallel}^2+p_\perp^2/p_{\mathrm{th}\perp}^2]}, \tag{37}$$

where $p_{\mathrm{th}\parallel}\equiv\sqrt{2mk_BT_\parallel}$, $p_{\mathrm{th}}\perp\equiv\sqrt{2mk_BT_\perp}$, and p_d is the drift momenta, for which $B_{\pm}$ becomes

$$B_{\pm}=\sum_{\mathrm{PS}}\frac{8\sqrt{\pi}}{\tau^2}\frac{p_{\mathrm{th}\perp}^2}{mcp_{\mathrm{th}\parallel}}\left[\frac{p_{\mathrm{d}}}{mc}-1-\alpha\Theta\right]e^{-m^2c^2\Theta^2/p_{\mathrm{th}\parallel}^2}, \tag{38}$$

where $\alpha\equiv T_\perp/T_\parallel-1$ and $\Theta\equiv 1\pm 2\omega_c/\omega-p_d/mc$. It is easily seen that $B_{\pm}>0$ for certain values of α and ω_c/ω. Note that if $\alpha=0$ then $B_{\pm}<0$ for all values of ω_c/ω and hence there can be no temperature-isotropic beam instability [26]. On the other hand, putting $p_d=0$, it is clear that a temperature-anisotropic distribution function without a drift can be the source of an instability.

The gravitational waves produced by a homogeneous plasma, due to the above cyclotron resonance instability will have a frequency of the order of the cyclotron frequency, at least in the nonrelativistic temperature limit considered here. In principle this opens up the possibility of emission of high frequency gravitational waves through large magnetic fields (of the order of 10^{-6} T or larger), i.e., frequencies well above the frequency range expected from ''conventional'' gravitational wave sources, such as compact binaries, neutron-star normal modes and gravitationally collapsing objects, reaching up to 10 kHz. Still the radiation considered here is generated by a collective process where the amount of matter interacting *coherently* can be as large as in other astrophysical examples.

It is not so easy to find astrophysical applications of the cyclotron resonance instability, however: First there must be a magnetized plasma cloud, with a nonequilibrium distribution function that fulfills the condition $B_{\pm}>0$. Such a cloud could in principle be generated, for example, if there is a magnetic field geometry that allows for a loss-cone distribution to evolve. However, there is an obvious risk that there will be purely electromagnetic instabilities, which typically have much higher growth rates than the gravitational one, that will dominate the picture. Secondly, for significant gravitational generation, the plasma cloud must be very much denser than the average density of the universe, otherwise the growth of the amplitude will take place slowly even compared to a cosmological time scale. In a dense plasma cloud, on the other hand, gravitation must be balanced by pressure gradients in order not to self-contract before significant radiation generation due to the cyclotron resonance instability occurs. This suggests that maybe plasma inhomogeneities should be included in our treatment, although it seems likely that the effects of inhomogeneity may be neglected as long as the gradient scale lengths is much longer than the wavelength of the gravitational radiation.

VI. SUMMARY AND DISCUSSION

We have considered linearized gravitational waves in the short wavelength approximation, propagating in a plasma parallel to an external magnetic field. In vacuum, there is the possibility of cyclotron acceleration of charged particles up to velocities arbitrarily close to the speed of light. Taking the collective effects of particle distributions on the gravitational waves into account, it follows that the (gravitational) normal modes for the system are circularly polarized gravitational waves, and we derive the corresponding dispersion relations, which coincide with that of Ref. [8] in the limit of a low-temperature Maxwellian plasma. In the case of an equilibrium plasma the waves are shown to be damped due to resonant interaction with the plasma particles and the dispersion is modified and enhanced as compared to the case of no magnetic field. In the case of thermodynamical nonequilibrium, there is the possibility of gravitational wave instabilities. To show in a concrete way that this can be realized, we demonstrate that there are temperature anisotropic distribution functions that are unstable. Furthermore, we have examined how the damping and dispersion in an electron-positron type of plasma and an electron-ion plasma, respectively, depends on the ratio ω_c/ω and on T in the regime of relativistic temperature. The strongest effects occurs when ω and ω_c are comparable and the effect increases with temperature and density.

The question is whether cyclotron damping can be observed, if we assume that gravitational wave astronomy [22] develops successfully. The calculations made in Ref. [14] end up with an estimate of 10% damping during a propagation distance of the order of 30 kpc, which suggests that there is at least some chance of gravitational cyclotron damping to be observed. Combining Eq. (26) with the definition $\mathcal{I}_{\pm}\equiv I_{\pm}\mu\tau^2$ (recall that $\mathcal{I}_{\pm}$ is the normalized value of $I_{\pm}$ displayed in Figs. 1, 2, and 3) we note that our damping rate is

$$\Gamma_{\pm}=\frac{\mathrm{Im}\,\mathcal{I}_{\pm}}{2\omega\mu\tau^2}. \tag{39}$$

Similarly, the time scale for gravitational wave dispersion is $\mu\omega\tau^2/\mathrm{Re}\mathcal{I}_{\pm}$. For typical values for (equilibrium) plasma in interstellar space, it is clear that the damping rate predicted by Eq. (39) is several orders of magnitude smaller than the estimation made in Ref. [14]. Presumably the discrepancy is due to the fact that in Ref. [14], only the effect of *acceleration* of particles (corresponding to energy being transported from the gravitational wave) is considered. Generally when the damping due to wave-particle interaction is small, it should be noted that as $\omega t_{\rm prop}$ (where $t_{\rm prop}$ is the time of wave-particle interaction) grows, the energy loss of the decelerated particles becomes very close to the energy gained by the accelerated particles, and thus the omission of the contribution from decelerated particles leads to large errors in the damping coefficient. If the singularities are treated properly, the effect of decelerating as well as accelerating particles on the wave is included automatically in a kinetic description. In general for parallel propagation we find that cyclotron damping as well as dispersion of gravitational waves through interstellar space is negligible, in the sense that there is essentially no hope of detecting it with gravitational wave detectors with realistic sensitivity.

In general gravitational waves propagate in an angle to the magnetic field. It is well-known (see, e.g., Ref. [1]) that this leads to generation of electromagnetic fields by the gravitational wave, and a dispersion relation governing propagation perpendicular to a magnetic field in a plasma has been derived by Ref. [8]. The results show that the gravitational wave is most affected by the matter when the frequency matches one of the natural frequencies of the system, such as the cyclotron frequency or the plasma frequency. It should be noted that in an inhomogeneous medium with slowly varying background parameters we will typically reach a point where the generated electromagnetic fields fulfills the dispersion relation of some natural plasma mode. In that case linear mode conversion, which is a resonant process involving *all particles*, may take place. However, such a problem remains an issue for future research.

APPENDIX

At a first sight our results seem to disagree with those of Ref. [8]. However, we will show below that our results essentially are in agreement [15], although the comparison is nontrivial. In general the effect of a curved background space-time cannot be neglected. However, for linearized gravitational waves with short wavelength compared to the background curvature, the back reaction on the gravitational wave can be separated into two effects, those respective contribution can be added to the flat vacuum dispersion relation $\omega^2 - k^2c^2 = 0$, see Ref. [1], p. 427. The first effect (that is considered by us) is a direct consequence of matter and fields (where the background curvature can be neglected), and the second effect is an indirect consequence due to the background curvature produced by the matter and fields. How to find the curved background contribution to the gravitational wave dispersion relation for a Robertson-Walker or a Schwarzschild background metric, see Refs. [1] and [7] respectively. Here a warning is strongly motivated, however. In general the separation of contributions to the dispersion relation into the ''direct'' and ''indirect'' effect is *not completely unique*, but to some extent depends on the formalism used. Thus great care must be taken when adding a ''direct'' and an ''indirect'' contribution to the dispersion relation derived by different authors, to see that the formalisms that have produced the different expressions are compatible.

To illustrate the above matters we compare our calculations using a tetrad frame formalism with those of a coordinate frame formalism. In the absence of gravitational wave perturbations the Einstein field equation reads

$$R^{(0)}_{\mu\nu} = \kappa\left[T^{(0)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}T^{(0)}\right]. \tag{A1}$$

In this section we use Greek indices $\mu,\nu,\ldots = 0,1,2,3$ (or t,x,y,z) for coordinate components and reserve $a,b,\ldots = 0,1,2,3$ for tetrad components. Adding a small perturbation, so that $R_{\mu\nu} = R^{(0)}_{\mu\nu} + \delta R_{\mu\nu}$, $T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu}$, and $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, gives the linearized equation

$$\delta R_{\mu\nu} = \kappa\,\delta\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]. \tag{A2}$$

Focusing on the direct effect of matter and assuming the short wavelength regime we use $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, and after suitable gauge transformations we obtain

$$\Box h_{\mu\nu} = -2\kappa\left[\delta T_{\mu\nu} - \frac{1}{2}\delta T\,\eta_{\mu\nu}\right]. \tag{A3}$$

It should be noted that in principle the left-hand side of Eq. (A3) should contain ''cross terms'' proportional to the product of $h_{\mu\nu}$ and components of the background part of the Riemann tensor (see Ref. [1]), corresponding to the ''indirect effects of matter'' but these terms are omitted, not because they are small, but because in the short wavelength limit their contribution can be calculated separately and added afterwards. For the same reason also a term proportional to $T^{(0)}h_{\mu\nu} = -R^{(0)}h_{\mu\nu}$ have been omitted from the right-hand side. Equation (A3) also applies to the tetrad description if it is understood that h_{ab} denotes what δR_{ab} reduces to in this approximation, i.e., $h_{11} = -h_{22} = h_+$ and $h_{12} = h_{21} = h_\times$ in the TT gauge. The short wavelength approximation and TT gauge is now assumed throughout the remainder of this section if nothing else is said.

The total energy-momentum tensor due to electromagnetic fields and matter is

$$T_{\mu\nu} = \mu_0^{-1}\left(F^{\sigma}_{\mu}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\sigma\tau}F_{\sigma\tau}\right) + \sum_{\rm PS}\int p_\mu p_\nu f\frac{\sqrt{|g|}}{m\gamma}d^3p,$$

where μ_0 is the magnetic permeability, $\gamma \equiv p_t/mc = [1 + \mathbf{p}^2/(mc)^2 - h^{\mu\nu}p_\mu p_\nu/(mc)^2]^{1/2}$ and p_t is the zero component of the four-momenta satisfying $p^\mu p_\mu = -m^2c^2$. Note that the definition $\gamma \equiv p_t/mc$ differs from the tetrad formalism. The background magnetic field is taken to be $F_{12} = F^{12} = B$ (being *identical* to that in the tetrad description). Hence the linearized energy-momentum tensor reads

$$\delta T_{\mu\nu}=\mu_0^{-1}\left(-h^{\sigma\tau}F_{\sigma\mu}F_{\tau\nu}-\frac{1}{2}h_{\mu\nu}B^2\right)+\sum_{\rm PS}\int p_\mu p_\nu\frac{f_0\delta\gamma}{m\gamma_0}d^3p+\sum_{\rm PS}\int p_\mu p_\nu\frac{\delta f}{m\gamma_0}d^3p, \tag{A4}$$

where $\delta\gamma=-\frac{1}{2}h^{\mu\nu}p_\mu p_\nu/(\gamma_0 mc)^2$ and $\gamma_0=[1+\mathbf{p}^2/(mc)^2]^{1/2}$. Note that only the last term on the right-hand side appears in the tetrad equation corresponding to Eq. (A4), since we have $g_{ab}=\eta_{ab}$ in that case. In some sense the perturbation of the electromagnetic part of the energy momentum tensor is somewhat artificial, since the electromagnetic field itself is not perturbed, although it is clear from a technical point of view that all terms in Eq. (A4) must be included.

In the coordinate description the Vlasov equation reads

$$\partial_t f+\frac{\mathbf{p}}{m\gamma}\cdot\nabla f+[\mathbf{F}-\mathbf{G}]\cdot\nabla_{\mathbf{p}}f=0,$$

where $F^i=qp^\mu F^i_{\ \mu}/\gamma m$, $G^i=\Gamma^i_{\mu\nu}p^\mu p^\nu/\gamma m$, $\Gamma^i_{\mu\nu}$ are the Christoffel symbols corresponding to $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ and $\nabla=(\partial_x,\partial_y,\partial_z)$. The linearized Vlasov equation for the gravitationally perturbed part of the distribution function, f_G, thus becomes

$$\left[\partial_t+\frac{\mathbf{p}}{m\gamma_0}\cdot\nabla+\mathbf{F}_0\cdot\nabla_{\mathbf{p}}\right]f_G+[\delta\mathbf{F}-\mathbf{G}]\cdot\nabla_{\mathbf{p}}f_0=0 \tag{A5}$$

where

$$\delta\mathbf{F}\cdot\nabla_{\mathbf{p}}f_0=2\omega_c\gamma_0^{-1}[h_+\sin 2\phi-h_\times\cos 2\phi]\mathcal{F}_0 \tag{A6}$$

$$\mathbf{G}\cdot\nabla_{\mathbf{p}}f_0=i(2\omega-kp_\parallel(\gamma_0 m)^{-1})[h_+\cos 2\phi-h_\times\sin 2\phi]\mathcal{F}_0 \tag{A7}$$

and $\mathcal{F}_0$ was defined in Eq. (19). The term $\delta\mathbf{F}$ in Eq. (A5) has its origin from the lowering of an index on $F^i_{\ \mu}$ and therefore does not occur in the tetrad description [15]. Also the term $\mathbf{G}\cdot\nabla_{\mathbf{p}}f_0$ differs from the tetrad description because $G_1=\frac{1}{2}G_x$, $G_2=\frac{1}{2}G_y$, and $G_3=G_z$.

Solving Eq. (A5) and applying the solution to Eq. (A3) gives the dispersion relations

$$[\omega^2-c^2k^2-A_+-iB_+](h_++ih_\times)=0, \tag{A8}$$

$$[\omega^2-c^2k^2-A_--iB_-](h_+-ih_\times)=0, \tag{A9}$$

where $A_\pm=\mathrm{Re}\,I_\pm+A_0$, $B_\pm=\mathrm{Im}\,I_\pm$, and

$$I_\pm=\frac{\pi\kappa c^2}{m}\sum_{\rm PS}\int_{-\infty}^{\infty}\int_0^{\infty}p_\perp^3\frac{2\omega-kp_\parallel/(\gamma_0 m)\pm 2\omega_c/\gamma_0}{kp_\parallel/m-\gamma_0\omega\mp 2\omega_c}\times\mathcal{F}_0\,dp_\perp dp_\parallel, \tag{A10}$$

$$A_0=\kappa c^2\mu_0^{-1}B^2+\frac{\pi\kappa}{2}\sum_{\rm PS}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{p_\perp^5}{(\gamma_0 m)^3}f_0\,dp_\parallel dp_\perp. \tag{A11}$$

A couple of things should be noted. First, the coordinate frame description gives a term contained in A_0 proportional to B^2 that remains even in the absence of particles. This term has no correspondence in the tetrad formalism, and the reason is that there are no perturbations of the electromagnetic part of the energy momentum tensor when we study the projections on the tetrad basis vectors, in contrast to what happens when we look at the coordinate basis components [see Eqs. (9) and (A4)]. However, the division into the perturbed and unperturbed *tensors* are the same independent of formalism, and thus the ‘‘direct term’’ proportional to B^2 in the coordinate frame formalism is compensated by a term associated with the background curvature produced by the unperturbed magnetic field in the tetrad formalism. On the other hand, the division into a direct effect of matter and an indirect (background curvature) effect is not completely artificial: The direct effect is determined *entirely* by the local matter content, whereas the background curvature effect is determined by the matter content both locally and globally. Furthermore, the background curvature effect will necessarily give contributions to Eqs. (A8) and (A9) that are real and *independent* of the gravitational wave frequency [see the discussion after Eq. (A3)]. Noting that

$$\frac{2\omega-kp_\parallel/(\gamma_0 m)\pm 2\omega_c/\gamma_0}{kp_\parallel/m-\gamma_0\omega\mp 2\omega_c}=\frac{\omega}{kp_\parallel/m-\gamma_0\omega\mp 2\omega_c}-\gamma_0 \tag{A12}$$

it is clear that the correction to the vacuum dispersion relations in the coordinate and tetrad frame formalism deviates only by a frequency independent real constant, and in particular the cyclotron damping agrees perfectly within the short wavelength approximation scheme. Adding the contribution to $I_\pm$ from the second term in Eq. (A12) to A_0 then all differences between the tetrad and coordinate frame formalisms are collected in this term:

$$\kappa c^2\mu_0^{-1}B^2+\pi\kappa\sum_{\rm PS}\int_{-\infty}^{\infty}\int_0^{\infty}\left[\frac{p_\perp^5}{2(\gamma_0 m)^3}f_0-\gamma_0 p_\perp^3\mathcal{F}_0 c^2/m\right]dp_\parallel dp_\perp. \tag{A13}$$

Relaxing the assumption of Minkowskian background, i.e., adding ‘‘cross terms’’ (see Ref. [1]) to Eq. (A3) (which of course requires a different tetrad basis in that formalism), we could in principle confirm the agreement of the total dispersion relations including both direct and indirect effects. This would be a tedious task, however, since we must then solve for the background configuration including the anisotropic magnetic field contribution. Since we have shown that the separation into background curvature and ‘‘direct’’ matter effects is not unique, one can question the relevance of the figures showing the *real* (dispersive) part of $I_\pm$ since only the direct matter effect is contained (recall that the damping contribution is unique, however, and due to the direct effect only). But it turns out that there are three reasons that make also the real plots of $I_\pm$ relevant: First, the main contribution to the real value of $I_\pm$ occurs in the frequency regime when cyclotron resonance effects plays a role, which always is a direct effect of matter *independent of formalism*. Secondly, for many cases the dominant contribution to the background curvature comes from the rest mass (T_{00}) part of the energy momentum tensor, and if that is the case, the difference displayed in Eq. (A13) is negligible as compared to other background curvature contributions [27]. For example, we could add the background curvature contributions for a Robertson-Walker or a Schwarzschild background metric, using the results of Refs. [1] and [7], respectively, since the difference in formalisms used in most cases would give only a negligible contribution in comparison with that included. Thirdly, since the background curvature gives only a real constant contribution to the dispersion relations, our plots always reveal the frequency dependence of the dispersion.

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