Holographic renormalization group flow and the low-energy, strong coupling, large *N* limit

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From the AdS/CFT correspondence, we learn that the classical evolution of supergravity in the bulk can be reduced to a renormalization group (RG-) flow equation for the dual low-energy, strongly coupled and large *N* gauge theory on the boundary. This result has been used to obtain interesting relations between the various terms in the gravitational part of the boundary effective action, in particular the terms that affect the cosmological constant. It is found that once the cosmological constant is canceled in the UV theory, the RG-flow symmetry of the boundary effective action automatically implies the existence of zero cosmological constant solutions that extend all the way into the IR. Given the standard (and well founded) contradiction between the RG-flow idea and the observational evidence of a small cosmological constant, this is considered to be important progress, albeit incomplete, towards the final solution. Motivated by this success, it would be interesting to see whether this RG stability extends outside the scope of strong 't Hooft coupling and the large *N* regime that are implicitly assumed in the de Boer–Verlinde–Verlinde Hamilton-Jacobi formulation of the holographic RG-flow equations of the boundary theory. In this paper, we address this question, where we start first by identifying the modifications that are required in the Hamilton-Jacobi formulation of the bulk supergravity theory when the strong 't Hooft coupling and the large *N* limits are relaxed. Next, taking into account the leading order corrections in these parameters, we derive new bulk-boundary relations, from which one can read all the local terms in the boundary effective action. Finally, we use the resulting new constraints to examine whether the RG stability of the cosmological extends to the new coupling regime. It would be also interesting to use these constraints to study the Randall-Sundrum scenario in this case.

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I. INTRODUCTION

According to the holographic principle [1,2], a macroscopic region of space and everything inside it can be represented by a boundary theory living on the boundary of that region.[1] Furthermore, the boundary theory should not contain more than one degree of freedom per Planck area. This holographic principle has in the past few years found a remarkable realization in superstring theory due to new insights gained from the investigation of various superstring dualities. At the heart of this string theory incarnation of the holographic principle is growing evidence for an intimate connection between quantum phenomena in gauge theory and classical aspects of gravity. Early examples illustrating such a relation are D-branes [7,8], black hole entropy counting [9], and matrix theory [10]. However, the clearest statement about the duality between gauge theory and gravity is made within the framework of the recently discovered AdS_D $(D-1)$-dimensional conformal field theory (CFT_{D-1}) correspondence [11–13]. According to this correspondence the strong 't Hooft coupling, i.e., $g_{\mathrm{YM}}^2 N \gg 1$, and the large N limit, i.e., $N \gg 1$, of certain $(D-1)$-dimensional gauge theories have a dual description in terms of a supergravity theory defined on one higher-dimensional bulk space. An important feature of this duality is the existence of an intriguing relationship between infrared (IR) effects in the bulk theory and ultraviolet (UV) ones on the boundary. In a succeeding work [14], this relation was shown to be crucial in yielding the bound of one degree of freedom per Planck area as required by the holographic principle.

An immediate follow-up of the IR-UV relation above, which is important to our work in this paper, is the interpretation of the extra ''radial'' Dth coordinate r, in the bulk space, as a renormalization group (RG) parameter of the $(D-1)$-dimensional quantum field theory living at its boundary. Indeed, the radial evolution of the D-dimensional bulk fields was shown to share many features with an RG flow [15–19]. This fact was made elegantly more transparent in the work of [15] by casting the Einstein equations in the D-dimensional bulk into the form of a Hamiltonian flow across constant-r timelike foliations. Specifically, it has been shown that the Hamilton-Jacobi (HJ) equation for the D-dimensional Einstein gravity in the bulk, with the latter taken to be sliced along timelike foliations, can be written in the form of first-order RG-flow equations of the classical supergravity action. Furthermore, in the asymptotic limit where the UV boundary extends all the way to infinity, these RG-flow equations reduce to the standard Callan-Symanzik equation including the the conformal anomaly terms [20], in full accordance with the RG-flow ideas in quantum field theory. This result lends support to the identification of the bulk classical supergravity action with the boundary quantum effective action of the gauge theory as suggested in Refs. [11,13].

In the standard AdS_D-CFT_{D-1} correspondence (as described above), where the bulk spacetime is taken to be non-

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[1]Recently, however, the entropy bound on spacelike and lightlike surfaces has been generalized to the case of flat Robertson-Walker geometries in Ref. [3] and to more general geometries in Refs. [4,5]. See also Ref. [6] for work related to the role of focusing mechanism in holography.

compact, the dual boundary theory is at infinite bulk radius. As such, it must have an infinite energy UV cutoff by virtue of the IR-UV relation. Therefore, the *D*-dimensional bulk graviton modes that extend all the way to the UV boundary are *not normalizable*, and hence gravity decouples totally form the boundary, leaving out there pure Yang-Mills theory. However, as first pointed out in Ref. [21] by Randall and Sundrum (RS), this situation changes as soon as one considers the transverse bulk radius to be of *finite* range. This in effect translates into having a dual boundary theory at finite bulk radius, and hence with a finite UV cutoff due to the IR-UV relation. In this case, there will exist *normalizable* fluctuations of the *D*-dimensional metric that propagate and couple as graviton modes of the $(D-1)$-dimensional boundary theory. This generalization of the $\mathrm{AdS}_D/\mathrm{CFT}_{D-1}$ correspondence leads also to a remarkable interplay between Einstein equations of the coupled gravity-matter theory on the boundary and the RG-flow equations [15–17,21]. In addition, it provides interesting relations between the various terms in the boundary quantum effective action, in particular the boundary Newton constant, the cosmological constant and the scalar potential [15–17,21]. As a result, a cosmological constant is naturally prevented from being generated dynamically along the RG-flow once it has been canceled at higher energies inside the bulk, as pointed out in Ref. [15]. These results join and corroborate earlier findings on the role of large extra dimensions in the resolution of the cosmological constant puzzle [22].

In principle, the above results should continue to hold for any $(D-1)$-dimensional gauge theory provided that it can be represented as a relevant or marginal perturbation (in the sense of Ref. [23]) of a large *N* superconformal field theory or any deformation of it, for which the AdS_D-CFT_{D-1} correspondence has been established. It is important to point out, though, that two main assumptions went into the derivation of the RG-flow equation of the boundary gauge theory from the HJ equation of the classical supergravity action in the bulk, as presented first in Ref. [15]. These two assumptions are, inherently, part of the conditions that are involved in the derivation AdS_D-CFT_{D-1} correspondence. The first assumption concerns the requirement that the gauge theory must have a large $N\gg 1$, (and thus a large gauge group) so that one can neglect the string loop effects represented by the $1/N^2$ corrections. Secondly, the gauge theory is required to have a large 't Hooft coupling, $g_{\mathrm{YM}}^2 N\gg 1$, which amounts to taking the energy scale in the theory to be low enough so that one can ignore quantum gravity effects controlled by the "stringy" α'/R^2 corrections. α' denotes as usual the square of the string length, and *R* represents some characteristic radius of the bulk geometry.[2] Therefore, one expect to have significant modifications of the HJ equation and hence the RG-flow equations outside this low-energy strongly coupled, large *N* regime. It is the purpose of this paper to identify the changes that are brought in the derivation of the RG-flow equation from the equation when the limits $N\gg 1$ and $\alpha'/R^2\ll 1$ are relaxed. In other words, we are interested in the calculation of the leading-order corrections, in the parameters $1/N$ and α'/R^2, to the HJ equation of the bulk supergravity, and in the study of their consequences.

We start in Sec. II by reviewing briefly the AdS_D-CFT_{D-1} correspondence to set notation and especially to emphasis the emergence of the large *N* and large 't Hooft coupling. In Sec. III, we introduce the leading α' corrections in the bulk supergravity action [24–26]. These corrections have their origin in the vanishing of the beta function of the string theory nonlinear sigma model. They are represented by higher-derivative local effective interactions involving the higher-curvature gravitational terms. Next, we give a Hamiltonian formulation of the the bulk higher-curvature supergravity action so obtained. As expected, we find that the HJ equations are changed since the canonical conjugate momentum to the metric inherits in this case new terms coming from the α' corrections. Even though it is tedious to calculate the changes that are brought by the α' corrections to the HJ equations, their form and how they appear as higher-derivative nonrenormalizable effective interactions can be derived systematically in string theory using effective field theory language [23,27]. In Sec. IV, we deal with the question of how to incorporate the $1/N$ corrections in the HJ equations. There is a striking similarity between our problem here and the one we face when we make the transition from the *classical* HJ equations to the *quantum* Schrödinger equation. In that context, using the WKB or semiclassical theory, the leading quantum corrections linear in $\hbar$ are found to be proportional to the second order variation of the action *S*. In a similar manner, the $1/N$ corrections which would change the RG-flow equations are taken to be represented by second order variations of the supergravity bulk action. The interpretation of the HJ constraints of the bulk theory as giving us the RG-flow equations of the boundary theory taken at the radius where the HJ constraints are satisfied, rests also upon their strong resemblance with Polchinski's exact *RG* equation [28]. Therefore, in Sec. IV, we also use this connection to motivate the addition of the second order variations of the action as representing the $1/N$ corrections. After adding the α' and $1/N$ corrections, we look in Sec. V for their implication on the relations between quantities in the boundary action previously derived in Refs. [15,17], in particular those involving the Newton constant, the scalar potential and the cosmological constant. Furthermore, it would be interesting to see whether the solution to the cosmological constant problem as proposed in Refs. [15,17] is affected in this case. Finally, in Sec. VII, we discuss our results and offer suggestions for future directions. The Hamiltonian formulation of general relativity in the presence of higher-curvature terms is presented in the Appendix.

II. AdS_D-CFT_{D-1} CORRESPONDENCE AND HOLOGRAPHY

We start by reviewing quickly some basic elements of the AdS_D-CFT_{D-1} correspondence. Our main concern here will

[2]To better understand these limits, we refer the reader to Sec. II, where we show that for type IIB superstrings on $\mathrm{AdS}_5\times S^5$, the string coupling is $g_{\mathrm{st}}\sim g_{\mathrm{YM}}^2\sim 1/N^2$, and the radius is $R^2\sim\alpha'\sqrt{g_{\mathrm{YM}}^2 N}$.

be to motivate the large $N \gg 1$, and the large 't Hooft coupling $g_{\rm YM}^2 N \gg 1$ limit, involved in the correspondence. Furthermore, to simplify our presentation, we focus only on the $D=5$ case since many of the features found in this case continue to hold for general D. The most studied example in this category is the proposed duality between four-dimensional Yang-Mills theory with $\mathcal{N}=4$ supersymmetries and type-IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. At the heart of this duality is the existence of the relation between the two different descriptions of a stack of N parallel extremal D3-branes. One in terms of the low-energy four-dimensional $U(N)$, $\mathcal{N}=4$ supersymmetric gauge theory on its world-volume, and the other in terms of the classical supergravity background of the type-II closed superstring theory. An essential step in the derivation of the AdS_5-CFT_4 correspondence is the understanding of the range of validity of each of the description above. For the classical supergravity description, we need the form of the background string metric, the dilaton and the RR-gauge field for the stack of N parallel extremal D3-branes. This is given by the following form:

$$ds^2=(1+R^4/r^4)^{-1/2}dx_{//}^2+(1+R^4/r^4)^{1/2}(dr^2+r^2 d\Omega_5^2), \tag{1}$$

$$e^{\phi}=g_{\rm st}, \tag{2}$$

$$C_{0123}=(1+R^4/r^4)^{-1}-1, \tag{3}$$

where $dx_{//}^2$ denotes the flat four-dimensional metric for the coordinates parallel to the D3-branes, and the radius R is $R^2=\alpha' \sqrt{g_{st}N}$. For the low-energy supersymmetric Yang-Mills description on the D3-branes worldvolume, we need the relation $g_{\rm YM}^2=g_{\rm st}$ between the couplings.[3]

Another piece of knowledge which played an important role in the formulation of the standard AdS_5-CFT_4 correspondence is the realization that the low-energy limit of the gauge theory on the D3-branes world volume, corresponding to $\alpha' \to 0$, may be taken directly in the supergravity description. On the supergravity side, the limit amounts simply to taking the near horizon geometry corresponding to the $r \to 0$ limit. Thus, finally, in the limit $\alpha' \to 0$ and $r \to 0$, with r/α' fixed, one finds that the metric in Eq. (1) reduces to the form

$$ds^2=\frac{r^2}{R^2}dx_{//}^2+\frac{R^2}{r^2}dr^2+R^2 d\Omega_5^2, \tag{4}$$

which describes the product-space geometry $\mathrm{AdS}_5 \times \mathrm{S}^5$, where both factors have radius $R^2=\alpha' \sqrt{g_{\rm YM}^2 N}$. Furthermore, we know that the classical supergravity description can be trusted only if the length scale of the D3-brane solution, given by the metric (4), is much larger than the string scale $\sqrt{\alpha'}$, which allows for the ''stringy'' quantum gravity effects to be neglected. This condition translates into $R^2 \gg \alpha'$, which yields the large 't Hooft coupling limit for the gauge theory on the D3-branes world volume, i.e., $g_{\rm YM}^2 N \gg 1$. In order to suppress the string loop corrections, we also need to take $g_{\rm st} \to 0$, and hence $g_{\rm YM}^2 \to 0$, which amounts to taking the large N limit, $N \gg 1$. To summarize, the supergravity solution is expected to give exact information about the $\mathcal{N}=4$ supersymmetric Yang-Mills theory on the D3-branes world volume, in the limit of large $N \gg 1$ and large 't Hooft coupling $g_{\rm YM}^2 N \gg 1$. More on the two limits above after introducing another key feature of the AdS_5-CFT_4 correspondence below, that is, the idea of RG flow and holography.

From the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry in Eq. (4), we can see that the coordinate r transverse to the D3-branes can be regarded as a renormalization group scale. Indeed, two excitations in the gauge theory on the D3-branes world volume, which are related by a scale transformations

$$x_{//} \to e^{\tau} x_{//}, \tag{5}$$

translate on the AdS factor of the geometry into two excitations concentrated around different locations in the transverse r direction, and which are related by the following transformation [12,29]:

$$r \to e^{-\tau} r. \tag{6}$$

The AdS_5-CFT_4 correspondence provides us thus with a holographic map between physics in the gauge theory on the world-volume, which can be thought of as living on the AdS_5 boundary, and physics in one higher dimension in AdS_5 bulk space. This holographic map is at the center of the IR-UV relation according to which (IR) effects in the bulk theory are related to (UV) ones on the boundary. This relation turned out to be very crucial in yielding the holographic bound of one degree of freedom per Planck area as required by the holographic principle [15].

In the original AdS_5-CFT_4 correspondence, the AdS_5 boundary is taken to be at $r=+\infty$, and as a result the range of the r values extends all the way to infinity. Therefore, while the theory in the AdS_5 bulk space contains gravity, the dual CFT_4 theory on the boundary does not. This happens because the bulk gravitational modes that propagate all the way to infinity are not normalizable, and therefore do not fluctuate. In this paper, however, we are interested in the much more general situation where gravity does not decouple at the boundary. For this to happen, we follow the Randall-Sundrum proposal in Ref. [21], and choose the AdS_5 transverse r coordinate to run over a finite range, $r \leq r_0$, instead over an infinite range. An immediate consequence of this is that, there exists now a normalizable gravitational collective mode at the boundary, which in this case is living at finite the radius $r=r_0$. Furthermore, in view of the IR-UV

[3]For a general Dp-brane, the relation between the couplings is $g_{\rm YM}^2=g_{\rm st}(\alpha')^{p-3}$, and the dimensionless effective coupling, at energy scale E, is $g_{\rm eff}^2(E)=g_{\rm YM}^2 N E^{p-3}$. Perturbation theory applies in UV for $p<3$, and in IR for $p>3$, and the two cases may be related by S duality [12]. The special case $p=3$, presented in Sec. II, corresponds to $\mathcal{N}=4$ supersymmetric Yang-Mills theory in $D=4$, which is known to be a finite, conformally invariant quantum field theory.

relation, truncating the bulk theory to r values larger (or smaller) than some finite $r=r_0$ amounts to introducing a finite UV (or IR) cutoff in the theory at the boundary [15]. Therefore, allowing for the bulk transverse r direction to be interpreted as an RG scale. Indeed, by casting the bulk Einstein equations into the form of Hamiltonian flow across timelike boundaries, the r evolution of the bulk fields were shown in Ref. [15] to share many features with an RG flow on the boundary.

Combining this holographic perspective of AdS_5-CFT_4 correspondence with the RG scale interpretation of the bulk transverse r coordinate, one aims to derive the low-energy quantum effective action $\mathcal{S}_b$ on the boundary from the knowledge of the bulk supergravity theory. As explained above, we shall take the boundary to be at finite radius r_0 so that gravity does not decouple from the boundary theory. To this end, we start by defining some classical action for the supergravity theory in the bulk, which we denote by $S_T[\phi^I,g]$. In addition the bulk metric $\mathcal{G}_{AB}$, $S_T[\phi^I,g]$ also depends on some scalar fields ϕ^I that represent the various couplings of the boundary theory. In fact, it is the evolution of these scalar fields as a function of the bulk transverse r coordinate that eventually lead to the RG-flow equations on the boundary theory.[4] For later reference, we choose the bulk metric to be of the form

$$
\begin{aligned}
ds^2 &= \mathcal{G}_{AB}dx^A dx^B \\
&= (N^2+N_\mu N^\mu)dr^2 + 2N_\mu dx^\mu dr \\
&\quad + g_{\mu\nu}(x,r)dx^\mu dx^\nu,
\end{aligned} \tag{7}
$$

where A and B are taken to denote the bulk coordinates (r,x^μ) and μ and ν denote the boundary coordinates. We assume the boundary metric $g_{\mu\nu}(x,r)$ to be of Euclidean signature, and we allow the scalars $\phi^I(x,r)$ to depend on all bulk coordinates (r,x^μ). N and N^μ are the lapse and shift functions, respectively. A convenient choice of coordinates are the Gaussian normal coordinates, where $N^\mu=0$ and $N=-1$. Using such coordinates, the metric in Eq. (7) takes on the simple form

$$
ds^2 = dr^2 + g_{\mu\nu}(x,r)dx^\mu dx^\nu. \tag{8}
$$

(More details on our notation and convention are presented in the Appendix.)

Finally, one of the main ingredients in the AdS_5-CFT_4 correspondence is the identification of the classical supergravity action $S_T[\phi^I,g]$ evaluated on a classical solution, with specified boundary values $g_{\mu\nu}(x,r_0)$ and $\phi^I(x,r_0)$, with the generating functional of gauge invariant correlators of gauge invariant observables $\mathcal{O}_I$ in the boundary theory living at $r=r_0$, that is, we have

[4]Because of the stress energy-momentum tensor of the scalar fields ϕ^I, the background geometry in the bulk will deviate from that of a pure AdS_5 form.

$$
\langle \mathcal{O}_{I_1}(x_1)\cdots\mathcal{O}_{I_n}(x_n)\rangle
= \frac{1}{\sqrt{g(x_1)}}\frac{\delta}{\delta\phi^{I_1}(x_1)}\cdots\frac{1}{\sqrt{g(x_n)}}\frac{\delta}{\delta\phi^{I_n}(x_n)}S_T[\phi^I,g]\bigg|_{r=r_0}. \tag{9}
$$

By requiring that the scalar fields ϕ^I and the metric $g_{\mu\nu}$ stay regular inside the bulk, there is in principle one unique supergravity classical solution for a given boundary value for ϕ^I and $g_{\mu\nu}$. If we put the scalar fields ϕ^I to zero after doing the variation, we do obtain the gauge invariant correlators of the unperturbed $\mathcal{N}=4$ supersymmetric Yang-Mills boundary theory. If the fields ϕ^I are put to finite values, however, the resulting boundary theory will correspond to a finitely perturbed $\mathcal{N}=4$ supersymmetric Yang-Mills theory.

Although the discussion, in this section, was so far limited to the AdS_5-CFT_4 correspondence, one could easily generalize it to include the higher-dimensional AdS_D spaces. We would be then talking about an AdS_D-CFT_{D-1} correspondence. In similarity with the AdS_5-CFT_4 correspondence, the large N limit, $N\gg 1$, and the large 't Hooft coupling limit, $R^2/\alpha' = \sqrt{g_{\rm YM}^2 N}\gg 1$ will also be involved in this case. In particular, the interpretation of the radial AdS_D coordinate with an RG scale will also allow in this case for an identification of the radial evolution of the bulk fields with a RG flow. Thus, by working within the general framework of AdS_D-CFT_{D-1} correspondence, our purpose next will be to go beyond the large N, and large R^2/α' limit, and consider the leading corrections in $1/N$ and α'/R^2 to the RG-flow equations derived from the bulk HJ constraint.

III. HJ EQUATIONS AND THE HIGHER-CURVATURE TERMS

In this section, we consider the derivation of the HJ constraint of the D-dimensional bulk supergravity theory in the presence of the α' corrections coming from a quantum theory of gravity such as string theory. In string theory, the lowest-order α'-corrections to the low-energy effective action involve the higher-curvature terms, which are controlled by the expansion parameter α'/R^2, where R is the characteristic radius of the bulk space. Therefore, by virtue of the relation $R^2/\alpha' = \sqrt{g_{\rm YM}^2 N}$, the addition of the higher-curvature terms will necessarily affect the large 't Hooft coupling limit $g_{\rm YM}^2 N\gg 1$ involved in the AdS_D/CFT_{D-1} correspondence as well the the RG-flow equations derived from it. As in the approach of Refs. [15,17], the changes that are brought by the higher-curvature corrections are most conveniently analyzed using the HJ theory of the D-dimensional bulk supergravity theory. By casting the Einstein's equations in the bulk into the form of a Hamiltonian evolution across timelike boundaries, one is led to the familiar HJ constraint of the canonical formalism of gravity. It has been shown in Refs. [15,17] that this constraint play a key role in the bulk-boundary correspondence, as they allow for a systematic derivation of the $(D-1)$-dimensional quantum effective action of the boundary theory from the knowledge of the bulk theory. Furthermore, combining these results with the IR-UV relation, a holographic RG-flow picture of the dual boundary theory naturally emerges, as changes in the bulk coordinate $r=r_0$ translates now into shifts in the energy scale $\mu=r/\alpha'$ of the dual boundary theory.

For the purpose of deriving the changes to the HJ constraint due to the higher-curvature corrections, we choose for the D-dimensional bulk spacetime the following supergravity action:[5]

$$S_T=S_1+S_2+S_3, \tag{10}$$

where S_1, S_2, and S_3 are given by

$$S_1=\int_D \sqrt{\mathcal{G}}dr d^{D-1}x[\,^{\{D\}}R+2\Lambda+a_0\,^{\{D\}}R^2+b_0\,^{\{D\}}R_{\mu\nu}\,^{\{D\}}R^{\mu\nu}+c_0\,^{\{D\}}R_{\mu\nu\rho\sigma}\,^{\{D\}}R^{\mu\nu\rho\sigma}], \tag{11}$$

$$S_2=2\int_{D-1}\sqrt{g}d^{D-1}x[K+\mathcal{K}(K,\nabla K;a,b,c)], \tag{12}$$

$$S_3=\int_D\sqrt{\mathcal{G}}dr d^{D-1}x\left[V(\phi^I)-\frac{1}{2}\nabla_\mu\phi^I\nabla^\mu\phi^I\right]. \tag{13}$$

In addition to curvature action S_1, the D-dimensional bulk supergravity theory contains the matter action S_3 for the bulk scalar fields ϕ^I, which through the bulk-boundary correspondence represent the various coupling of the dual boundary theory. The sum over repeated index I of the bulk scalar fields is assumed throughout all the paper. The curvature scalar $^{\{D\}}R$, and the tensors $^{\{D\}}R_{\mu\nu}$ and $^{\{D\}}R_{\mu\nu\rho\sigma}$ appearing in the action S_1 are calculated using the D-dimensional bulk metric $\mathcal{G}_{AB}$, with the coefficients a_0, b_0, and c_0 parametrizing for the time being only the α'/R^2 corrections.[6] Using the Gaussian normal coordinates, $\mathcal{G}_{AB}$ takes the form

$$ds^2=\mathcal{G}_{AB}dx^A dx^B=dr^2+g_{\mu\nu}(x,r)dx^\mu dx^\nu. \tag{14}$$

Following Sec. I, our notation will be to take the upper case Latin letters such as A and B to denote the D-dimensional bulk coordinates (x^μ,r), where the lower case Greek indices such as μ and ν are taken to denote the coordinates $(x^\mu;\mu=0,1,\dots,D-2)$ of the $(D-1)$-dimensional boundary theory. As usual, K is taken to denote the extrinsic curvature of the boundary surface, whose form is given by $K_{\mu\nu}=\frac{1}{2}\mathcal{L}_r g_{\mu\nu}=g_\mu{}^\rho\nabla_\rho n_\nu$. Whereas $\mathcal{K}(K,\nabla K;a,b,c)$ is taken to represent the additional surface terms corresponding to the higher-curvature terms in S_1. The derivation of $\mathcal{K}(K,\nabla K;a,b,c)$ was carried out in Ref. [24,25], but its exact form will not be necessary for our work here. It suffices to recall that the normal-derivative terms from S_1 are canceled by the variation of the surface terms in S_2.

Using the choice of the metric in Eq. (14), the D-dimensional bulk equations of motion can be cast into a form of a Hamiltonian flow across the $(D-1)$-dimensional timelike boundaries, with the bulk transverse r coordinate playing the role of time. Based on the bulk action S_T in Eq. (10), the Hamiltonian that generates this radial flow is explicitly derived in the Appendix, where we found

$$\begin{aligned}-\frac{1}{\sqrt{g}}\mathcal{H}_T=&(\mathcal{R}+2\Lambda+a_0\mathcal{R}^2+b_0\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\\&+c_0\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma})+\left(V(\phi)-\frac{1}{2}\nabla_\mu\phi^I\nabla^\mu\phi^I\right)\\&+\left(\pi_{\mu\nu}\pi^{\mu\nu}-\frac{\pi^2}{D-2}\right)+\frac{1}{2}\pi_I\pi_I-2a_0\mathcal{R}\\&\times\left(\pi_{\mu\nu}\pi^{\mu\nu}-\frac{\pi^2}{D-2}\right)-2b_0\mathcal{R}_{\mu\nu}\\&\times\left(\pi^\mu{}_\rho\pi^{\rho\nu}-\frac{\pi\pi^{\mu\nu}}{D-2}\right)+4c_0\mathcal{R}_{\mu\nu\rho\sigma}\pi^{\mu\rho}\pi^{\nu\sigma}\\&+\frac{4c_0}{(D-2)^2}\mathcal{R}\pi^2-\frac{8c_0}{D-2}\mathcal{R}_{\mu\nu}\pi\pi^{\mu\nu},\end{aligned} \tag{15}$$

with $\pi_{\mu\nu}$ and π_I are the canonical momentum variables conjugate to $g^{\mu\nu}$ and π^I, respectively. It is a standard fact, well known in classical mechanics, that (given the bulk action S_T) the value of the canonical momentum π_I conjugate to ϕ^I, at a given slice $r=$const, is equal to the functional derivative of the bulk action $\mathcal{S}_b$ induced on that slice, with respect to ϕ^I. Furthermore, since the conjugate momentum π_I is related to the radial flow of ϕ^I, we have

$$\frac{1}{\sqrt{g}}\frac{\delta\mathcal{S}_b}{\delta\phi^I}=\pi^I=\mathcal{L}_r\phi^I=-\dot{\phi}^I. \tag{16}$$

Similarly, for the canonical momentum variable $\pi^{\mu\nu}$ conjugate to the the metric $g_{\mu\nu}$, we have

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta\mathcal{S}_b}{\delta g^{\mu\nu}}=&\pi_{\mu\nu}\\=&(K_{\mu\nu}-Kg_{\mu\nu})+2a_0\mathcal{R}(K_{\mu\nu}-Kg_{\mu\nu})\\&-b_0(K\mathcal{R}_{\mu\nu}+\mathcal{R}^{\rho\sigma}K_{\rho\sigma}g_{\mu\nu})\\&+b_0(\mathcal{R}^\rho_\mu K_{\rho\nu}+\mathcal{R}^\rho_\nu K_{\rho\mu})-4c_0\mathcal{R}_{\mu\rho\nu\sigma}K^{\rho\sigma}\\&+\mathcal{O}(K^3),\end{aligned} \tag{17}$$

or by taking the trace

$$\begin{aligned}\pi=g^{\mu\nu}\pi_{\mu\nu}=\pi^\mu{}_\mu=&-(D-2)K-(2a_0D-4a_0+b_0)\mathcal{R}K\\&-(b_0D-3b_0+4c_0)\mathcal{R}_{\mu\nu}K^{\mu\nu}+\mathcal{O}(K^3).\end{aligned} \tag{18}$$

[5] Since we choose to work within the Einstein frame, we can use the D-dimensional Planck unit such that $\kappa_D=1$.

[6] Later, in Sec. IV, the coefficients of the higher-curvature terms will include, in addition to a_0, b_0, and c_0, the contributions a_1, b_1, and c_1 from the $1/N$ corrections.

(The details about the calculation of the conjugate momenta π_I and $\pi^{\mu\nu}$ can be found in the Appendix.)

Hamiltonian flow across the boundary is a constrained system, since it is still endowed with redundancies. The choice of the foliation is arbitrary, and even after fixing one, the system is still endowed with redundancies. To remove completely these redundancies, two set of constraint equations on the initial data at the boundary are necessary. The first constraint translates simply into a statement regarding Poincaré invariance on the boundary slices. It ensures that the boundary effective action is invariant under $(D-1)$-dimensional coordinate transformations. The second constraint, which is most important for our purposes, is the Hamilton constraint. It requires to set $\mathcal{H}_T=0$, which ensures invariance of the constant-r slices under the local shifts. Using Eq. (15) for the Hamiltonian $\mathcal{H}_T$, we obtain

$$\left(\frac{\pi^2}{D-2}-\pi_{\mu\nu}\pi^{\mu\nu}\right)-\frac{1}{2}\pi_I\pi_I+2a_0\mathcal{R}\left(\pi_{\mu\nu}\pi^{\mu\nu}-\frac{\pi^2}{D-2}\right)$$
$$+2b_0\mathcal{R}_{\mu\nu}\left(\pi^{\mu}{}_{\rho}\pi^{\rho\nu}-\frac{\pi\pi^{\mu\nu}}{D-2}\right)-4c_0\mathcal{R}_{\mu\nu\rho\sigma}\pi^{\mu\rho}\pi^{\nu\sigma}$$
$$-\frac{4c_0}{(D-2)^2}\mathcal{R}\pi^2+\frac{8c_0}{D-2}\mathcal{R}_{\mu\nu}\pi\pi^{\mu\nu}$$
$$=(\mathcal{R}+2\Lambda+a_0\mathcal{R}^2+b_0\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+c_0\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma})$$
$$+\left(V(\phi)-\frac{1}{2}\nabla_\mu\phi^I\nabla^\mu\phi^I\right). \tag{19}$$

To obtain the HJ constraint at $r=r_0$, we simply have to replace the canonical momenta in Eq. (19) by the functional derivatives of the bulk action $\mathcal{S}_b$ induced on $r=r_0$, with respect to the conjugate variables. In terms of the action $\mathcal{S}_b$, the HJ constraint reads

$$2a_0\frac{\mathcal{R}}{\sqrt{g}}\left[\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\frac{\delta\mathcal{S}_b}{\delta g^{\mu\nu}}-\frac{1}{D-2}\left(g_{\mu\nu}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\right)^2\right]$$
$$+2b_0\frac{\mathcal{R}_{\mu\nu}}{\sqrt{g}}\left[\frac{\delta\mathcal{S}_b}{\delta g^{\rho}_{\mu}}\frac{\delta\mathcal{S}_b}{\delta g_{\rho\nu}}-\frac{g_{\rho\sigma}}{D-2}\frac{\delta\mathcal{S}_b}{\delta g_{\rho\sigma}}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\right]$$
$$-4c_0\frac{\mathcal{R}_{\mu\nu\rho\sigma}}{\sqrt{g}}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\rho}}\frac{\delta\mathcal{S}_b}{\delta g_{\nu\sigma}}-\frac{4c_0}{(D-2)^2}\frac{\mathcal{R}}{\sqrt{g}}\left(g_{\mu\nu}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\right)^2$$
$$+\frac{8c_0}{D-2}\frac{\mathcal{R}_{\mu\nu}}{\sqrt{g}}g_{\rho\sigma}\frac{\delta\mathcal{S}_b}{\delta g_{\rho\sigma}}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}+\frac{1}{\sqrt{g}}\left[\frac{1}{D-2}\left(g_{\mu\nu}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\right)^2\right.$$
$$\left.-\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\frac{\delta\mathcal{S}_b}{\delta g^{\mu\nu}}-\frac{1}{2}\frac{\delta\mathcal{S}_b}{\delta\phi^I}\frac{\delta\mathcal{S}_b}{\delta\phi^I}\right]$$
$$=\sqrt{g}\left[V(\phi)-\frac{1}{2}\nabla_\mu\phi^I\nabla^\mu\phi^I+\mathcal{R}+2\Lambda+a_0\mathcal{R}^2\right.$$
$$\left.+b_0\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+c_0\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\right]. \tag{20}$$

As advertised earlier, the α' corrections from the bulk, in the form of higher-curvature terms, modifies the HJ constraint. This HJ constraint will play a central role in the remainder. Indeed, the bulk-boundary correspondence proposes to replace the bulk action $\mathcal{S}_b$ in Eq. (20), induced on the timelike foliations, with that of an effective $(D-1)$-dimensional boundary theory. It is easy to see then that the HJ constraint in Eq. (20) allows us to determine the coefficients of all the local terms in that boundary action, which will in effect include contributions from the higher-curvature corrections. In relation with the higher-curvature corrections, we should also notice that they do induce in the boundary Lagrangian, quartic powers of the extrinsic curvature K, schematically denotes as K^4, in addition to the quadratic terms K^2. As a consequence, the Hamiltonian $\mathcal{H}_T$ in Eq. (15) must also include terms that are quartic in the conjugate momentum $\pi^{\mu\nu}$, such as as π^4. The reason we chose not to include the π^4 terms in $\mathcal{H}_T$, and focus only on the corrections coming from the quadratic terms in K, is that it is in principle possible to generate the quartic terms such as K^4, in a Wilsonian manner,[7] as effective interactions. This is done by integrating out some very heavy auxiliary field χ, with mass much higher than the cutoff scale in the boundary theory, and which enters the boundary Lagrangian in the form $M_\chi^2\chi^2+\xi\chi K^2$.

Finally, using the definition of the extrinsic curvature given by Eq. (A14) of the Appendix, the radial flow of $g_{\mu\nu}$ follows straightforwardly from the expression of the canonical momentum $\pi^{\mu\nu}$ in Eq. (17), and it is found to be

$$K_{\mu\nu}=\frac{1}{2}\mathcal{L}_r g_{\mu\nu}$$
$$=-\frac{1}{2}\dot{g}_{\mu\nu}$$
$$=\left(\pi_{\mu\nu}-\frac{\pi}{D-2}g_{\mu\nu}\right)$$
$$+\left(2a_0+\frac{4c_0}{D-2}\right)\frac{\mathcal{R}\pi}{D-2}g_{\mu\nu}+(b_0-4c_0)\frac{\mathcal{R}_{\rho\sigma}\pi^{\rho\sigma}}{D-2}g_{\mu\nu}$$
$$-2a_0\mathcal{R}\pi_{\mu\nu}+(b_0-4c_0)\frac{\mathcal{R}_{\mu\nu}\pi}{D-2}-b_0$$
$$\times(\mathcal{R}^{\rho}_{\mu}\pi_{\rho\nu}+\mathcal{R}^{\rho}_{\nu}\pi_{\rho\mu})+4c_0\mathcal{R}_{\mu\rho\nu\sigma}\pi^{\rho\sigma}, \tag{21}$$

$$K=K^{\mu}{}_{\mu}$$
$$=-\frac{\pi}{D-2}+\left(2a_0+b_0+\frac{4c_0}{D-2}\right)\frac{\mathcal{R}\pi}{D-2}$$
$$+(3b_0-Db_0-4c_0)\frac{\mathcal{R}_{\mu\nu}\pi^{\mu\nu}}{D-2}. \tag{22}$$

[7] This way of viewing the higher-curvature corrections was suggested to us by Herman Verlinde.

Therefore, given the functional form of the boundary action $\mathcal{S}_b$ at slice $r=r_0$, and using the first-order equations (16) and (21), one can unambiguously compute the radial evolution of the couplings ϕ^I and the metric $g_{\mu\nu}$ in terms of their values on that slice.

IV. 1/*N* CORRECTIONS AS A WKB APPROXIMATION

We have seen in Sec. I that the AdS_D-CFT_{D-1} correspondence involves both the large 't Hooft coupling $g^2_{\mathrm{YM}}N \gg 1$, and the large $N \gg 1$ limit. Because of the relation $R^2/\alpha' = \sqrt{g^2_{\mathrm{YM}}N}$, relaxing the limit $g^2_{\mathrm{YM}}N \gg 1$ on the 't Hooft coupling reduces simply to the problem of incorporating the α'/R^2 corrections, as we have seen in Sec. II. This was carried out systematically, by considering the effects of the bulk higher-curvature terms on the boundary theory. When we turn to the large $N \gg 1$ limit, the derivation of the $1/N$ corrections to the HJ constraint does not, unfortunately, enjoy the same degree of simplicity. What we seem to be missing here is a systematic method, analogous to the α'/R^2 corrections case, where the $1/N$ corrections could be, for example, derived from first principles such as the open-closed string duality relation proposed in Ref. [30]. In the absence of such systematic methods, our derivation of the $1/N$ corrections relies simply on our experience and intuition based on similar problems in other physical examples. One such (well known) example is the problem we face when we make the transition from the *classical* HJ equation to the *quantum* Schrödinger equation. To see this, we recall from quantum mechanics that the wave amplitude to be associated with the mechanical motion of a particle of mass m have the form

$$\psi = \psi_0 e^{(i/\hbar)S}, \tag{23}$$

where ψ satisfies the Schrödinger wave equation

$$\frac{\hbar^2}{2m}\nabla^2\psi - V\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial t}. \tag{24}$$

In terms of the action S the Schrödinger equation can be written as

$$\left[\frac{1}{2m}(\boldsymbol{\nabla} S)^2 + V\right] + \frac{\partial S}{\partial t} = \frac{i\hbar}{2m}\boldsymbol{\nabla}^2 S. \tag{25}$$

The last equation may be called the quantum-mechanical HJ equation; it reduces to the classical HJ equation in the limit as $\hbar$, and therefore the Compton wavelength of the particle, goes to zero. Indeed, one is to note that $\nabla^2 S$ arises in association with $(\nabla S)^2$ in the evaluation of $\nabla^2\psi$ in the quantum-mechanical wave equation. Therefore, Eq. (25) would be the classical HJ equation if $\hbar\nabla^2 S \ll (\nabla S)^2$, or, equivalently, if $\lambda/2\pi \ll p/(\nabla\cdot\mathbf{p})$.

It is clear from the discussion above that the key element in the transition from the classical to the quantum HJ equation is the relation between the wave function ψ and the action given by Eq. (23), and the Schrödinger wave equation (24) describing the propagation of ψ. It is exactly the analog of these relations that we would need in the AdS_D-CFT_{D-1} correspondence, to be able to derive the $1/N$ corrections to the HJ constraint in a systematic way, and which we do not have.[8] Despite this difficulty, one can still use the above analogy, in particular the quantum-mechanical HJ equation in Eq. (25), to discuss the $1/N$ corrections. The analogy becomes even more clear if we think of $1/N$ as $\sqrt{\hbar}$. With this in mind, it is natural to write down the following equation for the HJ constraint:

$$\begin{aligned}
&2a\frac{\mathcal{R}}{\sqrt{g}}\left[\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\frac{\delta\mathcal{S}_b}{\delta g^{\mu\nu}} - \frac{1}{D-2}\left(g_{\mu\nu}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\right)^2\right] + 2b\frac{\mathcal{R}_{\mu\nu}}{\sqrt{g}}\left[\frac{\delta\mathcal{S}_b}{\delta g^{\rho}_{\mu}}\frac{\delta\mathcal{S}_b}{\delta g_{\rho\nu}} - \frac{g_{\rho\sigma}}{D-2}\frac{\delta\mathcal{S}_b}{\delta g_{\rho\sigma}}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\right] - 4c\frac{\mathcal{R}_{\mu\nu\rho\sigma}}{\sqrt{g}}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\rho}}\frac{\delta\mathcal{S}_b}{\delta g_{\nu\sigma}}\\
&\quad - \frac{4c}{(D-2)^2}\frac{\mathcal{R}}{\sqrt{g}}\left(g_{\mu\nu}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\right)^2 + \frac{8c}{D-2}\frac{\mathcal{R}_{\mu\nu}}{\sqrt{g}}g_{\rho\sigma}\frac{\delta\mathcal{S}_b}{\delta g_{\rho\sigma}}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}} + \frac{1}{\sqrt{g}}\left[e_1\frac{\delta^2 S_b}{\delta g_{\mu\nu}\delta g^{\mu\nu}}\right.\\
&\quad \left. + e_2 g_{\rho\sigma}\frac{\delta}{\delta g_{\rho\sigma}}g^{\mu\nu}\frac{\delta\mathcal{S}_b}{\delta g^{\mu\nu}} + e_3\frac{\delta^2\mathcal{S}_b}{\delta\phi^I\delta\phi^I}\right] + \frac{1}{\sqrt{g}}\left[\frac{1}{D-2}\left(g_{\mu\nu}\frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\right)^2 - \frac{\delta\mathcal{S}_b}{\delta g_{\mu\nu}}\frac{\delta\mathcal{S}_b}{\delta g^{\mu\nu}} - \frac{1}{2}\frac{\delta\mathcal{S}_b}{\delta\phi^I}\frac{\delta\mathcal{S}_b}{\delta\phi^I}\right]\\
&= \sqrt{g}\left[V(\phi) - \frac{1}{2}\nabla_\mu\phi^I\nabla^\mu\phi^I + \mathcal{R} + 2\Lambda + a\mathcal{R}^2 + b\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + c\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\right],
\end{aligned} \tag{26}$$

where e_1, e_2, and e_3 are the coefficients parametrizing the $1/N$ corrections to the HJ constraint in the same way that a_0, b_0, and c_0 parametrize the α' corrections in Eq. (20). In fact, in writing down the HJ constraint (26), we have replaced a_0, b_0 and c_0 by the new coefficients a, b, and c allowing the latter to include extra $1/N$ contributions besides the α' corrections. Therefore, we can write

$$a = a_0 + a_1, \quad b = b_0 + b_1, \quad c = c_0 + c_1, \tag{27}$$

where a_1, b_1, and c_1 are taken to parametrize the $1/N$ corrections.

[8] In Ref. [15], it was suggested that the HJ constraint can be considered as the classical limit of the quantum Wheeler-DeWitt equation, which when written as $e^{i/\hbar S}$, contains an additional term proportional to a second order variation of the action S.

In this paper, we take the point of view that Eq. (27) represents the correct HJ constraint taking into account the leading order corrections in α' and $1/N$. Using Eq. (27), we shall determine in the next section the various bulk-boundary relations that follow from it.

V. THE LOCAL BOUNDARY ACTION TERMS REVISITED

One of the remarkable features of the AdS$_D$-CFT$_{D-1}$ correspondence is that the bulk-boundary correspondence is captured by the HJ constraint in Eq. (26). The latter has, in particular, the advantage of containing both the α' and $1/N$ corrections (in the leading order). Extending, therefore, previous work on the RG-flow beyond the low-energy, strong coupling, large N limit. It is also important to realize that the HJ constraint in Eq. (26) proposes that we replace the bulk action S_b, induced on the timelike slice due the foliation of the bulk spacetime, with that of an unknown effective $(D-1)$-dimensional boundary theory, whose action we denote by action S^{eff}. With this in view, the HJ constraint now plays the role of a functional differential equation allowing for the determination of the functional form of the local terms in the boundary action S^{eff}, as we shall see below.

It is well know that the RG flow of quantum field theory in a curved background induces, in the effective action, an Einstein gravity term plus a cosmological constant. Indeed, a computation of the $\langle T_{\mu\nu}\rangle$ for the quantum field and its subsequent regularization is found to renormalize both the Einstein tensor and the cosmological constant. Therefore, at the cutoff scale μ, a general form for the effective action S^{eff} is given by

$$\mathcal{S}^{\mathrm{eff}}(g,\phi)=\mathcal{S}_{\mathrm{l}}(g,\phi)+\mathcal{S}_{\mathrm{nl}}(g,\phi), \tag{28}$$

where $\mathcal{S}_l$ represent the local part of the effective action whose form is

$$\begin{aligned}\mathcal{S}_l(g,\phi)=\int_{D-1}\sqrt{g}d^{D-1}x\bigg[&\kappa(\phi)\left(\mathcal{R}-\frac{1}{2}\nabla^{\mu}\phi^I\nabla_{\mu}\phi^I\right)\\&+U(\phi)-\mathcal{A}(\phi)\mathcal{R}^2-\mathcal{B}(\phi)\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}\\&-\mathcal{C}(\phi)\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}\bigg],\end{aligned} \tag{29}$$

where $\mathcal{R}$, $\mathcal{R}^2$, $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$, and $\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$ denote the $(D-1)$-dimensional curvature terms constructed form the boundary metric $g_{\mu\nu}$ in Eq. (14). The boundary values of the scalar fields ϕ^I are to be equated with the dimensionless coupling constants of the boundary theory, and $U(\phi)$, $\kappa(\phi)$, $\mathcal{A}(\phi)$, $\mathcal{B}(\phi)$, and $\mathcal{C}(\phi)$ are local functions of these couplings. $\mathcal{S}_{\mathrm{nl}}$ contains, on the other hand, all higher derivative and nonlocal terms subject to the symmetries inherited from the bulk.[9] In terms of the nonlocal action $\mathcal{S}_{\mathrm{nl}}$, the boundary theory operators $\langle\mathcal{O}_I\rangle$ and energy-momentum tensor $\langle T_{\mu\nu}\rangle$, are given by

$$\frac{1}{\sqrt{g}}\frac{\delta\mathcal{S}_{\mathrm{nl}}}{\delta\phi^I}\equiv\langle\mathcal{O}_I\rangle, \tag{30}$$

$$\frac{1}{\sqrt{g}}\frac{\delta\mathcal{S}_{\mathrm{nl}}}{\delta g^{\mu\nu}}\equiv\langle T_{\mu\nu}\rangle. \tag{31}$$

Our goal now is to determine the local boundary terms in $\mathcal{S}^{\mathrm{eff}}$. For this, we need to insert the effective action $\mathcal{S}^{\mathrm{eff}}$ into the HJ constraint (26), equating contributions from the left hand side with terms on the right hand side that have the same functional form. By treating the metric $g_{\mu\nu}$ and the scalars ϕ^I as arbitrary classical fields, this procedure generates a set of bulk-boundary relations for the unknown functions in the local action $\mathcal{S}_{\mathrm{l}}$, which are

$$2\Lambda+V=\left[\frac{1}{4}\frac{D-1}{D-2}U^2-\frac{1}{2}(\partial_I U)^2\right]+\left[-\frac{e_1}{4}(D^2-1)U-\frac{e_2}{4}(D-1)^2U+e_3\partial^I\partial_I U\right], \tag{32}$$

$$\begin{aligned}1=&\left[\frac{1}{2}\frac{D-3}{D-2}\kappa U-\partial^I\kappa\partial_I U\right]+\left[-\frac{a}{2}\frac{D-1}{D-2}U^2-\frac{b}{2}\frac{1}{D-2}U^2-c\frac{1}{(D-1)^2}U^2\right]\\&+\left[-\frac{e_1}{4}(D^2-5)\kappa-\frac{e_2}{4}(D-3)^2\kappa+e_3\partial^I\partial_I U\right],\end{aligned} \tag{33}$$

$$\begin{aligned}a=&\left[\frac{1}{4}\frac{D-1}{D-2}\kappa^2-\frac{1}{2}\frac{D-5}{D-2}U\mathcal{A}-\frac{1}{2}\partial^I\kappa\partial_I\kappa+\partial^I U\partial_I\mathcal{A}\right]+\left[-a\frac{D-3}{D-2}\kappa U-2c\frac{(D-1)(D-3)}{(D-2)^2}\kappa U\right]\\&+\left[e_1\left(\frac{D^2-9}{4}\mathcal{A}+\mathcal{B}\right)+\frac{e_2}{4}(D-5)^2\mathcal{A}-e_3\partial^I\partial_I\mathcal{A}\right],\end{aligned} \tag{34}$$

[9] The usual quartic, quadratic and logarithmic divergences for quantum fields coupled to curved spacetime are contained in the local action $\mathcal{S}_{\mathrm{l}}$ through U, κ and $(\mathcal{A},\mathcal{B},\mathcal{C})$, respectively. The nonlocal action $\mathcal{S}_{\mathrm{nl}}$ may also contain extra logarithmic divergences.

$$b=\left[-\kappa^2-\frac{1}{2}\frac{D-5}{D-2}U\mathcal{B}+\partial^I U\partial_I\mathcal{B}\right]+\left[-b\frac{D-3}{D-2}\kappa U-4c\kappa U\right]+\left[e_1\left(\frac{D^2-5}{4}\mathcal{B}+2\mathcal{A}+4\mathcal{C}\right)+\frac{e_2}{4}(D-5)^2\mathcal{B}-e_3\partial^I\partial_I\mathcal{B}\right], \tag{35}$$

$$c=\left[-\frac{1}{2}\frac{D-9}{D-2}U\mathcal{C}+\partial^I U\partial_I\mathcal{C}\right]+\left[\frac{e_1}{4}(D^2-17)\mathcal{C}+\frac{e_2}{4}(D-9)^2\mathcal{C}\right], \tag{36}$$

$$\beta^I\partial_I\kappa=\left[-(D-1)\kappa+\frac{2(D-2)}{U}\right]+\left[\frac{e_1}{2}\frac{(D^2-1)(D-2)}{U}\kappa+\frac{e_2}{2}\frac{(D-2)(D-1)^2}{U}\kappa-2e_3\frac{(D-2)}{U}\partial^I\partial_I\kappa\right], \tag{37}$$

where the beta functions β^I's are defined by

$$\beta^I(\phi)=-2\frac{D-2}{U}\partial_I U. \tag{38}$$

In addition, we have to this order in the expansion, terms involving the functional derivatives of the nonlocal action $S_{\rm nl}$. The bulk-boundary relations for them are

$$\langle T^\mu{}_\mu\rangle\equiv\langle T\rangle=\frac{\beta^I}{2}\langle\mathcal{O}_I\rangle, \tag{39}$$

$$V=\left[\frac{\langle T\rangle^2}{D-2}-\langle T^{\mu\nu}\rangle\langle T_{\mu\nu}\rangle-\frac{1}{2}\langle\mathcal{O}^I\rangle\langle\mathcal{O}_I\rangle\right]+[e_1\langle T^{\mu\nu}T_{\mu\nu}\rangle+e_2\langle T^2\rangle+e_3\langle\mathcal{O}^I\mathcal{O}_I\rangle]. \tag{40}$$

In the next section, we shall use these new bulk/boundary relations to study the cosmological constant problem. In particular, we are interested to see whether the solution proposed in Ref. [15], for the vanishing of Λ, continues to hold in the presence of the α' and the $1/N$ corrections.

VI. WHAT IS NEW ON THE COSMOLOGICAL CONSTANT PROBLEM?

The problem of the cosmological constant is why the vacuum energy density is zero or extremely small by particle physics standards. It is a hard problem because it involves not only the high-energy but the low-energy physics as well. It is not sufficient, for example, to find a cosmological constant that is zero at high energies (near the Planck scale), one must also explain the absence of the vacuum contributions as the scales run to low energies. This low-energy aspect of the cosmological constant is, in fact, the most puzzling, and seems to require some fundamental new ideas in the basic principles of low-energy effective field theories, RG flow, and gravity. But the low-energy physics in the standard framework of four-dimensional effective field theory does not seem to offer a solution to the problem.[10] On the other hand, it is very hard to change the low-energy theory in a sensible way, given all of the well known theoretical and experimental success. Faced with this riddle, one way out would be to imagine a scenario in which the observed four-dimensional universe, where the problem is severely posed, is related to a world of a higher dimension. If the higher-dimensional world does not obey the usual assumptions of four-dimensional low-energy effective field theories, which lead to the cosmological constant problem, one may then find a solution to this problem within in this scenario.

In the following we will reexamine the cosmological constant problem using a scenario in which the observed four-dimensional universe is embedded into a higher-dimensional background of dimension $D=5$. Our approach is directly motivated by the new insights from string theory through the AdS_5-CFT_4 correspondence, as well as by recent ideas that have appeared in the study of warped string compactification scenarios along the lines of Randall and Sundrum[11] in Refs. [21,15,22,16], reviving earlier work by Rubakov and Shaposhnikov [22]. The starting point of our discussion is the holographic formulation of the RG equations in which the RG scale is treated as a physical extra dimension. We also assume the warp geometry for the five-dimensional bulk spacetime,[12] which generalizes the AdS_5-CFT_4 duality to four-dimensional boundary theories with dynamical gravity, as our world. Following Ref. [15], and applying the results of Sec. V to a five-dimensional bulk spacetime of warp geometry and $\Lambda=0$, one finds that the HJ evolution equations in the bulk can also be reformulated as an RG-flow equations[13] for the four-dimensional boundary effective action, even after the inclusion of the α' and $1/N$ corrections. Our calculations, therefore, extends previous results found within the context of classical five-dimensional supergravity [15], and thus within the large N and large 't Hooft coupling limit, to the regime where these limits are relaxed. In particular, new interesting bulk-boundary relations were found, suggesting

[10]For a complete review on these issues see the paper by Weinberg in Ref. [22]

[11]Despite recent attempts in, Ref. [15] it does not exist yet a complete and consistent embedding of the Randall-Sundrum scenario within string or M-theory.

[12]Such backgrounds could be obtained, for example, via F-theory compactification on Calabi-Yau fourfolds [31,32]

[13]To find the RG-flow equations of the boundary effective theory, one solves for the evolution equations in Eqs. (21) and (16) using the warp geometry ansatz for the bulk, after replacing by the constraints from the HJ constraint in Eq. (26).

an intimate connection between the RG-flow symmetry of the boundary effective action and the bulk Einstein's equations.

Let us now address the consequences of the RG-flow symmetry of the boundary effective action, in the presence of the leading order corrections in α' and $1/N$, on the four-dimensional cosmological constant. Using the same line of reasoning as in Ref. [15], our RG-flow equations also imply that once we have a solution for the gravitational part of the boundary effective action at one scale, there is a solution along the whole RG trajectory. As a result, assuming that the boundary cosmological constant is canceled at high energies (due to extended supersymmetry, for example), it will naturally remain zero under the RG flow. So it appears as if the boundary cosmological constant continues to decouple from the RG-induced vacuum energy of the matter fluctuations, even after relaxing the large N and the large 't Hooft coupling limit. As we will show now, this decoupling arises due to a cancellation between the contraction rate of the warp factor and any variation in the matter induced vacuum energy, in close similarity with with the mechanism proposed in Ref. [15]. Using a five-dimensional background of warp geometry with vanishing Λ as our bulk spacetime, the field equations that follow from the effective action (28) are then the four-dimensional Einstein equation and the scalar field equations

$$\kappa\left(\mathcal{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)-\frac{1}{2}g_{\mu\nu}U(\phi) =(\mathcal{A}^{(1)}H_{\mu\nu}+\mathcal{B}^{(2)}H_{\mu\nu}+\mathcal{C}H_{\mu\nu}) +\mathcal{T}^{\phi}_{\mu\nu}(\kappa,\mathcal{A},\mathcal{B},\mathcal{C},\phi,g_{\mu\nu})-\langle T_{\mu\nu}\rangle, \tag{41}$$

$$\nabla_{\mu}(\kappa\nabla^{\mu}\phi^{I})+\partial_{I}\kappa\left(R-\frac{1}{2}\nabla^{\lambda}\phi^{I}\nabla_{\lambda}\phi^{I}\right) =(\partial_{I}\mathcal{A}\mathcal{R}^{2}+\partial_{I}\mathcal{B}\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}+\partial_{I}\mathcal{C}\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}) -\partial_{I}U-\langle\mathcal{O}_{I}\rangle, \tag{42}$$

where ${}^{(1)}H_{\mu\nu}$, ${}^{(2)}H_{\mu\nu}$, and $H_{\mu\nu}$ are the contributions to the field equations from the higher curvature terms, and are given by

$${}^{(1)}H_{\mu\nu}=2\nabla_{\mu}\nabla_{\nu}\mathcal{R}-2g_{\mu\nu}\Box\mathcal{R}-\frac{1}{2}g_{\mu\nu}\mathcal{R}^{2}+2\mathcal{R}\mathcal{R}_{\mu\nu}, \tag{43}$$

$${}^{(2)}H_{\mu\nu}=2\nabla_{\alpha}\nabla_{\nu}\mathcal{R}^{\alpha}{}_{\mu}-\Box\mathcal{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\Box\mathcal{R}+2\mathcal{R}^{\alpha}_{\mu}\mathcal{R}_{\alpha\nu} -\frac{1}{2}g_{\mu\nu}\mathcal{R}^{\alpha\beta}\mathcal{R}_{\alpha\beta}, \tag{44}$$

$$H_{\mu\nu}=2\nabla_{\mu}\nabla_{\nu}\mathcal{R}-4\Box\mathcal{R}_{\mu\nu}+2\mathcal{R}_{\mu\alpha\beta\gamma}\mathcal{R}_{\nu}^{\ \alpha\beta\gamma} -\frac{1}{2}g_{\mu\nu}\mathcal{R}^{\alpha\beta\gamma\delta}\mathcal{R}_{\alpha\beta\gamma\delta}-4\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu} +4\mathcal{R}^{\alpha\beta}\mathcal{R}_{\alpha\mu\beta\nu}. \tag{45}$$

$\mathcal{T}^{\phi}_{\mu\nu}$ represents the stress energy-momentum tensor of the scalar fields ϕ^{I}. In addition, ϕ^{I}, $\mathcal{T}^{\phi}_{\mu\nu}$ also depend on the functions $\kappa,\mathcal{A},\mathcal{B},\mathcal{C}$, their covariant derivatives and the various curvature terms of the metric $g_{\mu\nu}$. $\langle T_{\mu\nu}\rangle$ and $\langle\mathcal{O}_{I}\rangle$ were defined earlier in Eqs. (30) and (31), and they represent the boundary expectation values to which the metric $g_{\mu\nu}$ and the scalar fields ϕ^{I} couple, respectively.

At this point, one could make use of the RG-flow equations of the boundary effective theory to deduce the RG trajectories of all the quantities appearing in the field equations (41) and (42), and show the decoupling mechanism that is claimed to arise for the cosmological constant. Since this approach has already been used in the previous literature such as in Ref. [15], what we propose here is a much simpler and direct method making use of the bulk-boundary relations derived in Sec. V. To address the consequences of the bulk-boundary relations on the boundary cosmological constant, let us take the trace of the four-dimensional Einstein's equations in Eq. (41), yielding

$$\kappa R=\langle T\rangle-2U+2(3\mathcal{A}+\mathcal{B}+\mathcal{C})\Box\mathcal{R}=\kappa\Lambda^{(4)}, \tag{46}$$

where we have assumed the boundary theory to be at an energy scale much less than the cutoff scale μ, so that the scalar fields are practically independent of the four-dimensional boundary coordinates, i.e., $\nabla_{\mu}\phi^{I}(x)=0$. Clearly, the terms on the right-hand side of Eq. (46) represents the effective cosmological constant on the boundary. We would have $\Lambda^{(4)}=0$ if the first two terms on the right-hand side of Eq. (46) cancel each other, and the third term is zero. First how do we make the third term vanish? Since the HJ constraint, and the hence the bulk-boundary relations derived from it, are nothing more than constraints on the variations of both $\mathcal{S}_{\rm l}$, and $\mathcal{S}_{\rm nl}$ in $\mathcal{S}^{\rm eff}$, one may consider these constraints for any boundary field configuration, including a preferred one, such that $3\mathcal{A}+\mathcal{B}+\mathcal{C}=0$. Using this condition, the trace of the Einstein equation in Eq. (46) becomes

$$\kappa R=\langle T\rangle-2U=\kappa\Lambda^{(4)}. \tag{47}$$

The condition $3\mathcal{A}+\mathcal{B}+\mathcal{C}=0$ is easily seen to be satisfied if the higher-curvature contributions entered the local effective S_{1} in Eq. (29) as a Gauss-Bonnet term ($\mathcal{R}^{2}-4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}+\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$). This Gauss-Bonnet term was considered in Ref. [16] and [18] in the study of naked singularities within the context of brane world scenarios. This is not the point of view we take here. We consider, instead, the situation where the condition $3\mathcal{A}+\mathcal{B}+\mathcal{C}=0$ is satisfied for arbitrary coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. But since the Gauss-Bonnet term is a topological invariant on the four-dimensional boundary, only two of them are independent, so me may choose $\mathcal{C}=0$. The coefficients $\mathcal{A}$ and $\mathcal{B}$ satisfy then the condition $3\mathcal{A}+\mathcal{B}=0$. Using the bulk-boundary relations (34), (35), and (36), the conditions $\mathcal{C}=0$ and $3\mathcal{A}+\mathcal{B}=0$ translate in thus into conditions on the bulk parameters (a,b,c), where $c=0$ and a and b being related to each other.

Now, let us turn to the remaining two terms on the right-hand side of Eq. (47). At first sight it is not obvious why $\langle T\rangle$ and $2U$ should cancel each other. However, by invoking again the fact that the HJ constraint is simply a condition on the variations of $\mathcal{S}_l$ and $\mathcal{S}_{\rm nl}$ which hold for an arbitrary field configuration, one may consider it for a flat boundary spacetime with constant scalars. In this case, using the bulk-boundary relations in Eqs. (32) and (40), after setting $D=5$ and $\Lambda=0$, we find that $\langle T\rangle$ and $2U$ are given by the following expressions:

$$(2U)^2=12V+6(\partial_I U)^2+24(3e_1+2e_2)U-12e_3\partial^I\partial_I U, \tag{48}$$

$$\langle T\rangle^2=12V+6\langle\mathcal{O}_I\rangle^2-3(e_1+4e_2)\langle T^2\rangle-12e_3\langle\mathcal{O}_I^2\rangle. \tag{49}$$

So far, only the trace of the Einstein equation in Eq. (41) and the identities (48) and (49) (from the HJ constraint) did enter our analysis of the boundary cosmological constant. To progress further we make of the equation of motion for ϕ^I, which for $\nabla_\mu\phi^I=0$ and flat boundary spacetime reads

$$\partial_I U+\langle\mathcal{O}_I\rangle=0. \tag{50}$$

Now, inserting Eq. (50) into both $(2U)^2$ and $\langle T\rangle^2$, and evaluating their difference afterwards, we find

$$(2U)^2-\langle T\rangle^2=12e_3[\langle\mathcal{O}_I^2\rangle-\partial^I\partial_I U]+3e_1[24U+\langle T^2\rangle]+12e_2[4U+\langle T^2\rangle]. \tag{51}$$

The above relation cannot be simplified further since we have already made used of all the equations that are available to us (which are the equations of motion and the HJ constraint). The consequences of this relation on the cosmological constant problem within the holographic RG-flow approach are the topic of the next section.

VII. DISCUSSION

It appears from Eq. (51) that $2U$ and $\langle T\rangle$ would not cancel each other in the presence of the leading $1/N$ corrections, parametrized by the e_1, e_2, and e_3 coefficients. From Eq. (47), we see that this mismatch between $2U$ and $\langle T\rangle$ implies a nonzero effective cosmological constant $\Lambda^{(4)}$ on the boundary, which is in clear distinction from the results of Ref. [15]. In Ref. [15], since the authors were only considering the large N limit, for them $e_1=e_2=e_3=0$, and thus they obtained the cancellation between $2U$ and $\langle T\rangle$, necessary for the vanishing of the boundary cosmological constant. In geometric terms, this result was interpreted as meaning that there exist a natural mechanism in which the vacuum energy that is generated on the four-dimensional brane world, as we flow towards the IR, is canceled by the ever decreasing warp factor of the five-dimensional geometry. From the holographic RG-flow perspective (based on the HJ formalism), this result shows that, in the strong 't Hooft coupling and large N regime, the potential energy U is canceled by the trace of the stress energy tensor at all scales, once this is achieved at one particular scale,[14] yielding thus the RG stability of the cosmological constant. Given the usual difficulties in reconciling the RG-flow intuition and the observational evidence for a small cosmological constant, this is certainly a useful progress towards the final solution. It is important to notice that the RG stability of the cosmological constant $\Lambda^{(4)}$ established in the strong 't Hooft coupling and large N regime of the boundary theory, is not restricted to any preferred value for $\Lambda^{(4)}$. This leaves, of course, the question of whether it is possible to pick up naturally initial conditions in the UV for which $\Lambda^{(4)}=0$.

In our actual calculation, we have not addressed at all this question, rather what we were interested in is to extend the RG stability of the cosmological constant to the regimes where the strong 't Hooft coupling and the large N limits are relaxed. What we found, in this case, is that the fate of the RG-stability mechanism, of Ref. [15], is not sensitive to the α' corrections, which were introduced to account for the relaxation of the strong 't Hooft coupling limit. However, one sees from Eq. (51) that the $1/N$ corrections do seem, on the other hand, to ruin the RG stability of the cosmological constant if no other equations are supplemented at this order to Eq. (51). As we have seen in Sec. IV, the derivation of $1/N$ corrections are less systematic and much harder to implement in the HJ formulation than the α' corrections. Using the analogy with the transition from the HJ equation to the Schrödinger equation, and treating $1/N$ as $\sqrt{\hbar}$, the $1/N$ corrections are expressed as the second order variation of the boundary action. Although this is a good starting point to probe the effects of the $1/N$ corrections, it is clear that one needs further information and better knowledge, especially on the side of the boundary matter sector[15] to remove the arbitrariness left in the coefficients e_1, e_2, and e_3 parametrizing the $1/N$ corrections. It is very plausible that when more systematic methods become available[16] further relations could be found between the potential U and the boundary operators such as $\langle\mathcal{O}_I^2\rangle$ and $\langle T^2\rangle$, leading to the cancellation among the terms on the right-hand side of Eq. (51). So instead of using Eq. (51) to declare the failure of the RG stability of the cosmological constant, outside the regime of strong 't Hooft coupling and large N limits, we take the point of view that it calls for a better understanding of the $1/N$ corrections beyond the simple addition of the second order variation of the boundary effective action to the HJ constraint.

Going now back to Eq. (51), it is very plausible just from the CFT point of view, to have a theory where

[14]Both interpretations hold only in the case of a five-dimensional background of warp geometry.

[15]After all it is the matter fields on the boundary that form representations of the boundary gauge group, which makes them sensitive to the choice of N.

[16]According to suggestions made in [15], systematic methods for deriving the $1/N$ corrections could be found using the nonlocal loop equations in Ref. [33] or string field theory.

$$\langle \mathcal{O}_I^2 \rangle \propto \partial^I \partial_I U, \tag{52}$$

$$\langle T^2 \rangle \propto \langle T \rangle \sim U. \tag{53}$$

Furthermore, in Eq. (26) since both of the coefficients e_1 and e_2 multiply the second order variation of the boundary effective action, with the respect to the metric $g_{\mu\nu}$, we expect that they are not independent, and hence $e_1 \propto e_2$. Combining this relation with the relations from Eq. (52) and (53), we see that there is much room for the right-hand side of (51) to vanish, allowing us to recover the RG stability of the cosmological constant in the presence of the leading $1/N$ corrections. Hopefully, we will come back in future work to prove the additional relations (52) and (53) needed to preserve the cancellation between the potential energy U and the and the trace of the stress-energy tensor $\langle T \rangle$ in Eq. (51).

Finally, it would be interesting to use the new bulk-boundary relations derived in Sec. V to study the Randall-Sundrum scenario. We treat this question in Ref. [34].

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APPENDIX

The purpose of this section is to give a Hamiltonian formulation of the higher-curvature theory considered in Sec. III, which is represented by the D-dimensional bulk action

$$S_T = S_1 + S_2 + S_3, \tag{A1}$$

where S_1, S_2, and S_3 are given by

$$S_1 = \int_D \sqrt{\mathcal{G}} dr d^{D-1}x [\,{}^{\{D\}}R + 2\Lambda + a_0 {}^{\{D\}}R^2 + b_0 {}^{\{D\}}R_{\mu\nu} {}^{\{D\}}R^{\mu\nu} + c_0 {}^{\{D\}}R_{\mu\nu\rho\sigma} {}^{\{D\}}R^{\mu\nu\rho\sigma}], \tag{A2}$$

$$S_2 = 2\int_{D-1} \sqrt{g} d^{D-1}x [K + \mathcal{K}(K, \nabla K; a_0, b_0, c_0)], \tag{A3}$$

$$S_3 = \int_D \sqrt{\mathcal{G}} dr d^{D-1}x \left[V(\phi) - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right]. \tag{A4}$$

All the terms appearing in S_1, S_2, and S_3 were introduced and defined in Sec. III.

To obtain a Hamiltonian formulation of the bulk action S_T, it will be useful to resort to the well-known technique in general relativity, which consists of slicing the D-dimensional bulk spacetime $\mathcal{M}$, with metric $\mathcal{G}_{AB}$, into an arbitrary foliation defined by the isosurfaces $\{\Sigma\}$ [32]. For the purpose of studying of the holographic RG flow of theories induced on *timelike* boundaries sitting at different locations in the radial direction of the bulk spacetime $\mathcal{M}$, we choose to foliate $\mathcal{M}$ along timelike isosurfaces. Because of this, there will be some sign flips between our formulas and the ones that we would have obtained had we chosen a foliation along spacelike slices. So, given that $(\mathcal{M}, \mathcal{G}_{AB})$ is the D-dimensional bulk spacetime,[17] we can foliate it by a family of $(D-1)$-dimensional timelike hyper surfaces, $\{\Sigma_r\}$, parametrized by the scalar function $r=$const. Thus, we can write the bulk metric $\mathcal{G}_{AB}$ as

$$ds^2 = \mathcal{G}_{AB} dx^A dx^B = \mathcal{G}_{rr} dr^2 + 2\mathcal{G}_{r\mu} dr dx^\mu + \mathcal{G}_{\mu\nu} dx^\mu dx^\nu. \tag{A5}$$

Here and throughout all the paper our notational conventions will be to take the upper case Latin letters such as A and B to denote the D-dimensional indices $(0,1,\ldots,D-2,r)$ over $\mathcal{M}$, and the lower case Greek indices such as μ and ν to denote the $(D-1)$-dimensional indices $(0,1,\ldots,D-2)$ spanning the Σ_r hypersurface.

Let r^μ be a vector field on $\mathcal{M}$ satisfying $r^\mu \nabla_\mu r = +1$, and let n^μ be the spacelike inward pointing vector fields normal to the timelike hypersurface with normalization $\mathcal{G}_{\mu\nu} n^\mu n^\nu = +1$. By introducing the *lapse* function N and *shift* vector N^μ, r^μ admits a decomposition in terms of its normal and tangential components with respect to Σ_r, as follows:

$$r^\mu = N^\mu - N n^\mu, \tag{A6}$$

where N and N^μ are given by

$$N = -r^\mu n_\mu = -(n^\mu \nabla_\mu r)^{-1}, \tag{A7}$$

$$N_\mu = g_{\mu\nu} r^\nu. \tag{A8}$$

In terms of these definitions the metric in Eq. (A5) can be rewritten as

$$ds^2 = (N^2 + N_\mu N^\mu) dr^2 + 2N_\mu dx^\mu dr + g_{\mu\nu}(x,r) dx^\mu dx^\nu, \tag{A9}$$

where the boundary metric on Σ_r is related to the bulk metric by the formula $g_{\mu\nu} = \mathcal{G}_{\mu\nu} - n_\mu n_\nu$. Using the Gaussian normal coordinates, corresponding to the gauge choice $N^\mu = 0$ and $N = -1$, the metric in (A9) takes on the simple form

[17] In general, even though $\mathcal{M}$ could be geometrically different from the pure AdS_D form (due to the possible bulk-matter stress-energy momentum tensor back reaction), it still has the same topology, allowing, therefore, the derivation of the gauge invariant correlators on the CFT_{D-1} boundary from AdS_D bulk action [11].

$$ds^2 = dr^2 + g_{\mu\nu}(x,r)dx^\mu dx^\nu. \tag{A10}$$

Another concept entering the description of the bulk spacetime $\mathcal{M}$ in terms of its foliations $\{\Sigma_r\}$, is the notion of extrinsic curvature $K_{\mu\nu}$, which is defined by

$$\begin{aligned} K_{\mu\nu} &= g_\mu{}^\rho \nabla_\rho n_\nu, \\ &= \frac{1}{2}\mathcal{L}_r g_{\mu\nu}. \end{aligned} \tag{A11}$$

The meaning given to $K_{\mu\nu}$ is that it accounts for the ''bending'' of Σ_r in $\mathcal{M}$. Finally, to obtain a Hamiltonian functional for general relativity, we need to express the gravitaional action in Eq. (A1) in terms of the quantities $(g_{\mu\nu}, K_{\mu\nu}; N^\mu = 0,\ N = -1)$, and their time and space derivatives. Splitting S_T along the timelike foliations, we find the following Lagrangian:

$$\begin{aligned} \mathcal{L}_T = \sqrt{g}[&\mathcal{R} + 2\Lambda + a_0\mathcal{R}^2 + b_0\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + c_0\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} \\ &+ (K^2 - K_{\mu\nu}K^{\mu\nu}) + 2a_0\mathcal{R}(K^2 - K_{\mu\nu}K^{\mu\nu}) \\ &+ 2b_0\mathcal{R}_{\mu\nu}(KK^{\mu\nu} - K^{\mu\rho}K^\nu_\rho) \\ &+ 4c_0\mathcal{R}_{\mu\nu\rho\sigma}K^{\mu\rho}K^{\nu\sigma} + \mathcal{O}(K^4)] \\ &+ \sqrt{g}\left[V(\phi) - \frac{1}{2}\nabla_\mu\phi^I\nabla^\mu\phi^I - \frac{1}{2}(\dot{\phi}^I)^2\right], \end{aligned} \tag{A12}$$

where $\mathcal{R}_{\mu\nu\rho\sigma}$, $\mathcal{R}_{\mu\nu}$, and $\mathcal{R}$ denote the $(D-1)$-dimensional Riemann tensor, Ricci tensor, and Ricci scalar, respectively. The sum over the scalar field index I is understood in the text and hereafter. Using this Lagrangian, the canonical momenta conjugate to ϕ^I and $g_{\mu\nu}$ are

$$\pi_I = \frac{1}{\sqrt{g}}\frac{\partial L_T}{\partial\dot{\phi}^I} = -\dot{\phi}^I,$$

$$\begin{aligned} \pi_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\partial L_T}{\partial g}{}_{\mu\nu} &= (K_{\mu\nu} - Kg_{\mu\nu}) + 2a_0\mathcal{R}(K_{\mu\nu} - Kg_{\mu\nu}) \\ &- b_0(K\mathcal{R}_{\mu\nu} + \mathcal{R}_{\rho\sigma}K^{\rho\sigma}g_{\mu\nu}) \\ &+ b_0(\mathcal{R}^\rho_\mu K_{\rho\nu} + \mathcal{R}^\rho_\nu K_{\rho\mu}) \\ &- 4c_0\mathcal{R}_{\mu\rho\nu\sigma}K^{\rho\sigma} + \mathcal{O}(K^3), \end{aligned}$$

$$\begin{aligned} \pi = g^{\mu\nu}\pi_{\mu\nu} = \pi^\mu{}_\mu \\ = -(D-2)K - (2a_0D - 4a_0 + b_0)\mathcal{R}K \\ -(b_0D - 3b_0 + 4c_0)\mathcal{R}_{\mu\nu}K^{\mu\nu} + \mathcal{O}(K^3), \end{aligned} \tag{A13}$$

where

$$\dot{\phi}^I = \mathcal{L}_r\phi^I = \frac{d\phi^I}{dr},$$

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_r g_{\mu\nu} = -\frac{1}{2}\dot{g}_{\mu\nu} = -\frac{1}{2}\frac{dg_{\mu\nu}}{dr}. \tag{A14}$$

Replacing $\dot{\phi}^I$ and $\dot{g}_{\mu\nu}$ in $\mathcal{L}_T$ by their canonical momenta, and performing the Legendre transformation, we find the following expression for the total Hamiltonian:

$$\begin{aligned} -\frac{1}{\sqrt{g}}\mathcal{H}_T = (&\mathcal{R} + 2\Lambda + a_0\mathcal{R}^2 + b_0\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \\ &+ c_0\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}) + \left(V(\phi) - \frac{1}{2}\nabla_\mu\phi^I\nabla^\mu\phi^I\right) \\ &+ \left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{\pi^2}{D-2}\right) + \frac{1}{2}\pi_I\pi_I - 2a_0\mathcal{R} \\ &\times\left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{\pi^2}{D-2}\right) - 2b_0\mathcal{R}_{\mu\nu} \\ &\times\left(\pi^\mu{}_\rho\pi^{\rho\nu} - \frac{\pi\pi^{\mu\nu}}{D-2}\right) + 4c_0\mathcal{R}_{\mu\nu\rho\sigma}\pi^{\mu\rho}\pi^{\nu\sigma} \\ &+ \frac{4c_0}{(D-2)^2}\mathcal{R}\pi^2 - \frac{8c_0}{D-2}\mathcal{R}_{\mu\nu}\pi\pi^{\mu\nu}. \end{aligned} \tag{A15}$$

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