Kinematic constraints to the key inflationary observables

Mark B. Hoffman
Department of Physics, University of Chicago, Chicago, Illinois 60637-1433

Michael S. Turner
Department of Physics, University of Chicago, Chicago, Illinois 60637-1433,
Department of Astronomy and Astrophysics, Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637-1433,
and NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, Illinois 60510-0500
(Received 11 July 2000; revised manuscript received 14 November 2000; published 11 June 2001)

The observables T/S and $n-1$ are key to testing and understanding inflation. (T, S, and $n-1$ respectively quantify the gravity-wave and density-perturbation contributions to CMB anisotropy and the deviation of the density perturbations from the scale-invariant form.) Absent a standard model, there is no definite prediction for, or relation between, T/S and $n-1$. By reformulating the equations for slow-roll inflation, we show that in the $T/S-(n-1)$ plane there are excluded regions, regions in which the density perturbations are not well approximated by a power law, and regions in which models with a ''featureless'' potential must lie.

DOI: 10.1103/PhysRevD.64.023506 PACS number(s): 98.80.Es, 98.70.Vc, 98.80.Cq

INTRODUCTION

Cosmic microwave background (CMB) anisotropy measurements have begun to test inflation, the leading paradigm to extend the standard big-bang cosmology. Within a decade they should test it decisively and even probe the underlying physics [1–3]. Recent results from the BOOMERanG and MAXIMA CMB experiments [4,5] (as well as results from earlier experiments [6]) are consistent with the flat universe predicted by inflation and are beginning to address its second basic prediction: almost scale-invariant adiabatic, Gaussian density perturbations produced by quantum fluctuations during inflation [7]. The third prediction, a nearly scale-invariant spectrum of gravity waves, will be more difficult to confirm, but is a critical probe of inflation [8].

The key inflationary observables are the level of anisotropy arising from density (scalar) perturbations (quantified by the contribution to the CMB quadrupole anisotropy, S), the level of anisotropy arising from gravity-wave (tensor) perturbations (T), and the power-law index n that characterizes the density perturbations (scale invariance refers to equal amplitude fluctuations in the gravitational potential on all length scales and corresponds to $n=1$). If T, S and $n-1$ can be measured, then the scalar-field potential that drove inflation can be partially reconstructed [9]. The most promising means of measuring T is its unique signature in the polarization of CMB anisotropy [10] (however, direct detection by a future space-based experiment should not be dismissed).

While there is no standard model of inflation, most models can be cast in terms of the classical evolution of a single, new scalar field ϕ (dubbed the inflaton) [11]. Predictions for S, T and $n-1$ can be expressed in terms of the scalar-field potential $V(\phi)$ and its first two derivatives. While there is a model-independent relation between T/S and the power-law index n_T that characterizes the gravity-wave spectrum, $T/S=-5n_T$ [12,13], no such relation for n and T/S exists [14].

This is unfortunate because n_T is very difficult to measure, while n will be measured to a precision of better than 1% by the Microwave Anisotropy Probe (MAP) and Planck experiments (BOOMERanG and MAXIMA have already determined that $n\simeq 1.01^{+0.09}_{-0.07}$ [15]). Even an approximate or generic relation between $(n-1)$ and T/S would be valuable, both as a test of inflation and as a guide for the expected level of gravity waves when n is measured.

The formation of large-scale structure and CMB measurements already indicate that a significant part of CMB anisotropy arises from scalar perturbations, i.e. T/S cannot be $\gg 1$. On the other hand, nothing precludes $T/S \ll 1$, and if T/S is much less than 10^{-3}, the prospects for measuring T are poor [10]. One inflation theorist has opined that $T/S \ll 1$ for all reasonable models [16].

The goal of this work is to provide objective theoretical guidance. By casting the equations governing inflation in a form that is essentially independent of the inflaton potential (''flow equations'' for T/S and $n-1$), we show that the $T/S-(n-1)$ plane is not uniformly populated by models of inflation: For $n<1$, models that are consistent with the equations governing inflation generally lie near the lines $T/S \approx 0$ and $T/S \approx -5(n-1)$, and there is an excluded region between these two lines. For $n>1$, models lie either at $T/S \approx 0$ or $T/S \approx 0.5$. Other values for T/S and $n-1$ are possible, but at the expense of a spectrum of density perturbations that is poorly represented by a power law. (The CMB will be able to test how well a power law describes the density perturbations.)

FLOW EQUATIONS

The kinematic equations that govern inflation are well known [17,18]

$$\ddot{\phi}+3H\dot{\phi}+V'(\phi)=0 \tag{1}$$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_{\rm Pl}{}^2}\left[V(\phi)+\frac{1}{2}\dot{\phi}^2\right] \tag{2}$$

where $a(t)$ is the cosmic scale factor, prime denotes $d/d\phi$, and overdot denotes d/dt. During inflation ϕ rolls slowly and the $\ddot{\phi}$ term in its equation of motion and its kinetic term in the Friedmann equation can be neglected [17,19], so that

$$\dot{\phi} \simeq \frac{-V'}{3H} \tag{3}$$

$$N(\phi) \equiv \int_{\phi}^{\phi_{\rm end}} H dt \simeq -\frac{8\pi}{m_{\rm Pl}^2} \int_{\phi}^{\phi_{\rm end}} \frac{d\phi}{x(\phi)} \tag{4}$$

where $x(\phi) \equiv V'(\phi)/V(\phi)$ measures the steepness of the potential and $N(\phi)$, the number of e-folds before the end of inflation, is the natural time variable. Inflation ends when the slow-roll conditions

$$m_{\rm Pl}|V'/V| = m_{\rm Pl}|x| < \sqrt{48\pi}, \tag{5}$$

$$m_{\rm Pl}^2|V''/V| = m_{\rm Pl}^2|x' + x^2| < 24\pi \tag{6}$$

are violated (at $\phi = \phi_{\rm end}$) [17,19].

The inflationary observables are related to the same quantities that govern the kinematics of inflation [13]

$$(n-1) = \frac{m_{\rm Pl}^2}{8\pi}[2x' - x^2] \tag{7}$$

$$T/S = \frac{5 m_{\rm Pl}^2}{8\pi} x^2 \tag{8}$$

$$T = 0.6 V/m_{\rm Pl}^4. \tag{9}$$

These expressions are given to lowest order in x^2 and x' (see Ref. [20] for higher-order corrections). Note, $n-1$ is only equal to $n_T = -5(T/S)$ if $x' = 0$.

By combining the slow-roll equations with those governing $(n-1)$ and T/S, we can write equations that govern the inflationary observables (almost) without reference to a model,

$$\frac{d(T/S)}{dN} = (n-1)\frac{T}{S} + \frac{1}{5}\left(\frac{T}{S}\right)^2 \tag{10}$$

$$\frac{d(n-1)}{dN} = -\frac{1}{5}(n-1)\frac{T}{S} - \frac{1}{25}\left(\frac{T}{S}\right)^2 \pm \frac{m_{\rm Pl}^3}{16\pi^2}\sqrt{\frac{2\pi}{5}\frac{T}{S}}\, x'' \tag{11}$$

where the sign of the last term matches that of V'.

We call these "flow equations" as they describe the trajectory in the T/S – $(n-1)$ plane during inflation. Because of the x'' term they are not completely independent of the potential. To "close" the flow equations we will assume that the potential is smooth enough so that we can treat x'' as being approximately constant. For sufficiently smooth and featureless potentials x'' should also be small.

Finally, one might wonder what happened to the most stringent constraint on inflation: achieving density perturbations of amplitude 10^{-5} or so ($S \sim 10^{-10}$). The flow equations involve the quantities T/S, $(n-1)$ and $dN/d\phi$ which are unaffected by a rescaling of the potential, $V \rightarrow aV$. This rescaling changes S: $S \rightarrow aS$. Thus, any potential can be rescaled to give proper size density perturbations without affecting the flow equations.

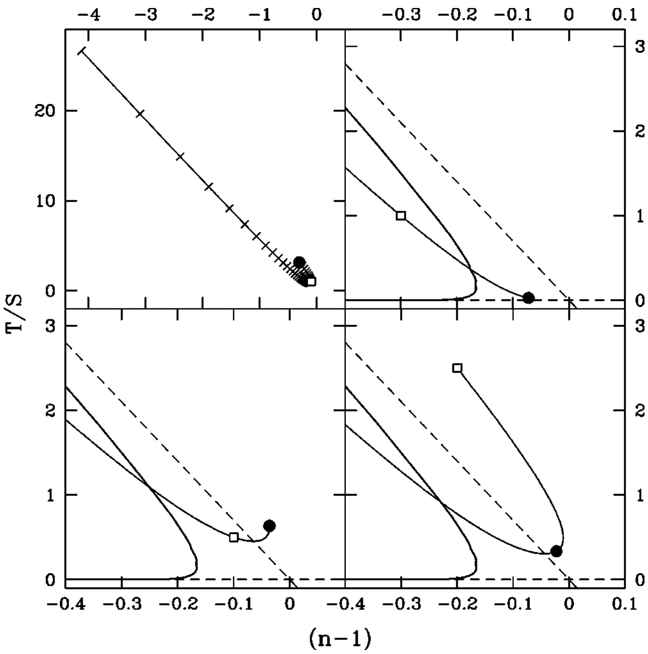

FIG. 1. Trajectories in the T/S – $(n-1)$ plane. Squares indicate the initial choices for T/S and $(n-1)$; circles indicate the values 50 e-folds before the end of inflation. A trajectory ends when T/S and/or $|n-1|$ become large; most of inflation occurs when T/S and $|n-1|$ are small. The upper left panel shows a complete trajectory, with ticks indicating e-folds before the end of inflation (from the circle, 50,49, . . . ,1). The other three panels show trajectories in more detail. Note how T/S and $(n-1)$ are pulled toward the lines $T/S \approx -5(n-1)$ and $T/S \approx 0$ (these "attractors" are shown as broken lines and the boundary of the excluded region is a solid curve).

THE T/S – $(n-1)$ PLANE

The scales relevant for structure formation (1 Mpc to 10^4 Mpc) crossed outside the horizon roughly 50 e-folds before the end of inflation (i.e., when $N=50$) [17], and so it is T/S and $(n-1)$ at this time that can be measured by CMB experiments. We find them by evolving T/S and $(n-1)$ until inflation ends and counting back 50 e-folds. To determine when inflation ends, we recast the slow-roll conditions (5),(6):

$$T/S < 30 \tag{12}$$

$$\left|(n-1) + \frac{3}{5}\frac{T}{S}\right| < 6; \tag{13}$$

the violation of either indicates the end of inflation.

To be very specific about our procedure, we choose a starting point for our calculation in the range, $0 < T/S < 10$ and $-0.5 < (n-1) < 0.5$. Our results do not depend upon the range of these initial values. We then integrate with fixed x'' until one of the slow-roll conditions is violated, signaling the end of inflation, and count back 50 e-folds to find $(T/S)_{50}$ and $(n-1)_{50}$. Some trajectories are shown in Fig. 1.

Figures 2 and 3 summarize the $(T/S)_{50}$ – $(n-1)_{50}$ phase space generated from the range of initial conditions considered. It is not uniformly populated. For $x'' < \mathcal{O}(1)$, solutions cluster around two "attractors," $(T/S)_{50} \approx 0$ and $(T/S)_{50} \approx -5(n-1)_{50}$, and for $(n-1)_{50} < 0$, there is an excluded re-

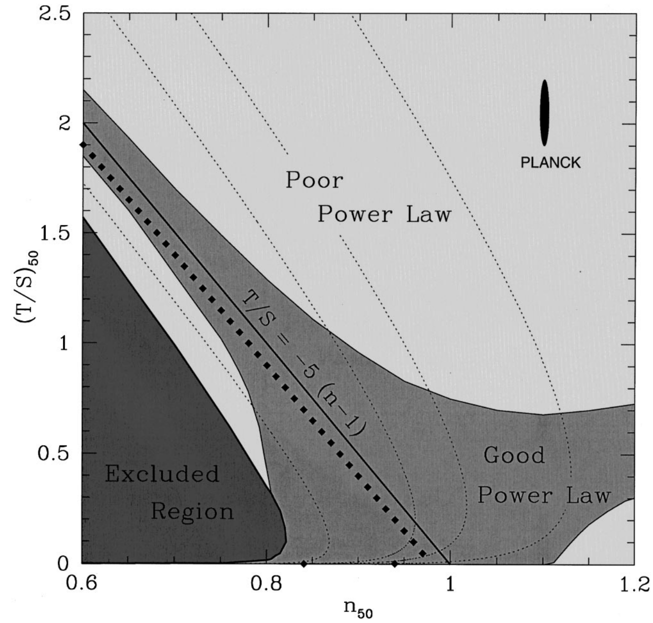

FIG. 2. Summary of our model search using the flow equations. The dotted curves correspond to $x''=0,1,2,5$ (from left to right). We found no models in the excluded region. The good power law region represents the models for which the density perturbation spectrum closely follows a power law. (Specifically, $|dn/d\ln k|<10^{-2}$.) Diamonds indicate various known inflationary models: chaotic, $V(\phi)=\lambda\phi^n$ for $n=2,3,\ldots$ (diamonds on the diagonal); new inflation ($n=0.94$) and natural inflation (with $n=0.84$). The ellipse is the 2σ error ellipse ''forecasted'' for the Planck satellite [21].

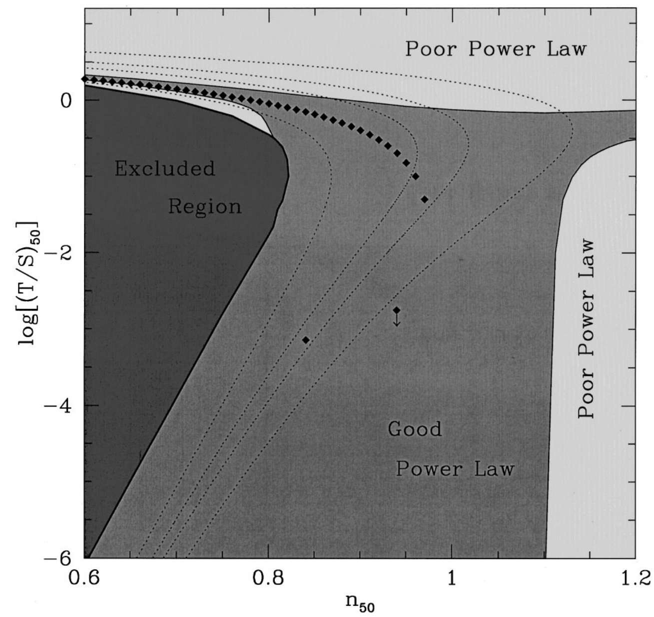

FIG. 3. Same as Fig. 2, with a logarithmic scale for T/S.

gion between these two lines, which cannot be reached for any value of x'', i.e. there are *no* models in this excluded region. Noting that $dn/d\ln k=-dn/dN$ and using Eq. (11), we can calculate the running of the scalar index. We define a good-power-law region, for which $|dn/d\ln k|<10^{-2}$; models outside of this region will have a poor power law, a prediction that can be tested by CMB measurements. The results shown in Figs. 2 and 3 do not depend strongly on the value of N chosen for our calculation. For values of N taken between 40 and 60, the borders of the regions shift slightly but not significantly.

Taking $x''=0$ it is simple to show why there are no models in the excluded region of the T/S – $(n-1)$ plane. In this limit, the flow equations are: $s\equiv(n-1)+\frac{1}{5}(T/S)=$ const and $r\propto\exp(sN)$, where $r=T/S$. Unless r and/or s are small, corresponding to the attractor solutions, r grows very rapidly and inflation does not last 50 e-folds.

Models outside of our good-power-law region are possible, but they come at the expense of a density-perturbation spectrum that is not well represented by a power law. Moreover, they are characterized by large x'', which typically signals that the potential has a feature, e.g. a ''bump'' or a ''kink.'' We note that it is possible to have large x'' but still have a good power law if $T/S\ll 1$. This explains the results of a recent paper [22] in which models with n as large as 2 were constructed. In particular, for the model with $n=2$, $x''\simeq 2000$, $T/S\simeq 3\times10^{-3}$ and $dn/d\ln k\simeq 0.3$.

So far, we have only considered one-field inflation. There are also models with more than one field, either implicitly or explicitly [23]. However, in essentially all models discussed in the literature only one field plays an ''active'' role during inflation [11]. The other field(s) are used to halt inflation and make a graceful exit to a radiation-dominated cosmology (e.g., by classical evolution in hybrid inflation or a phase

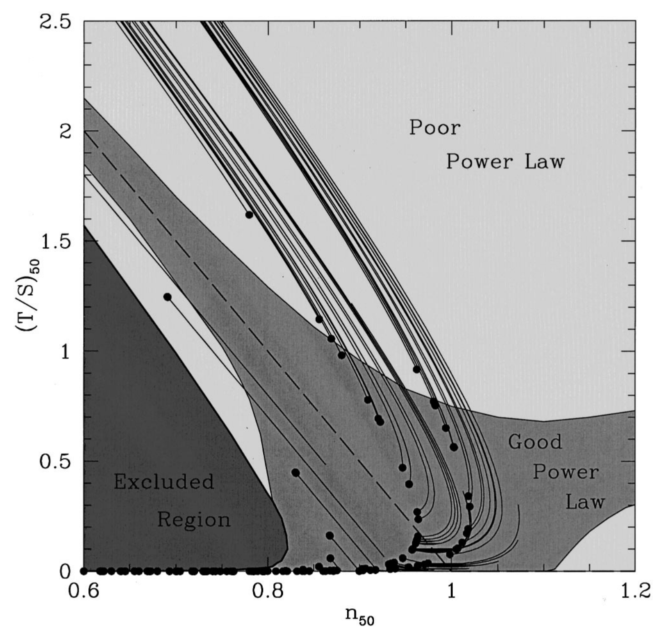

FIG. 4. Summary of two-field models. The filled circles represent the values of $(T/S)_{50}$ and n_{50} for the corresponding one-field models, and the attached curves are the values obtained for inflation ending early due to an auxiliary field. The dashed lines represent fixed points in the (T/S) – $(n-1)$ plane that result from models that do not end without an auxiliary field. In general, two-field models populate the same region as one-field models and extend the $T/S\approx 0$ part of the good-power-law region to $n>1$.

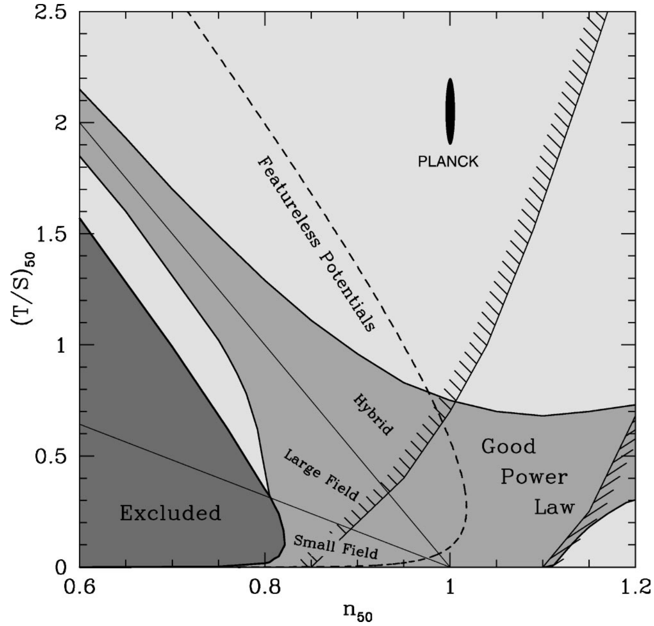

FIG. 5. Features of the $T/S - n$ plane. There is an excluded region in the lower left corner of the plane in which there are no models. Inflationary models that result in a scalar perturbation spectrum that is well represented by a power law are found in the good-power-law region. We have defined the borders of this region as $|dn/d\ln k|<10^{-2}$. The area to the left of the dashed line contains models that result from featureless potentials, i.e. potentials with small x'' (specifically $x''<2$). We have also shown the division of the $T/S - n$ plane into ''small field,'' ''large field,'' and ''hybrid'' theoretical models advocated in Ref. [24] along with the 99% confidence level constraints from recent CMB data [24]. The allowed region is within the hatched lines. The Planck ellipse is the same as in Fig. 2.

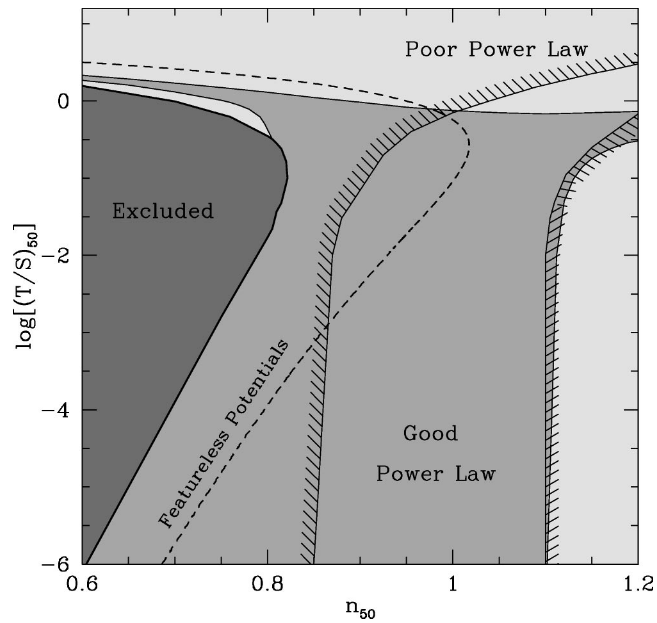

FIG. 6. Same as Fig. 5, with a logarithmic scale for T/S.

transition in extended inflation). The most well known of these is power-law inflation, $V(\phi)\propto\exp(-\beta\phi/m_{\rm Pl})$, a model in which inflation would not end in the absence of the action of another field. Our flow equations can also be applied to such multi-field models.

Models that require another field to end inflation show up when the right-hand sides of Eqs. (10),(11) vanish prior to violating the slow-roll conditions. In this case, there are fixed points in the $T/S - (n-1)$ plane, which are the most likely values for T/S and $(n-1)$ 50 e-folds prior to when the second field ends inflation. These points, shown in Fig. 4, populate the regions $T/S\approx 0$ for $n>1$ and the line $T/S=-5(n-1)$ for $n<1$.

It is also possible that a self-ending model has an auxiliary field that ends inflation ''early.'' We treat this possibility by populating the $(T/S)_{50} - (n-1)_{50}$ plane with the values of T/S and $(n-1)$ at $N>50$ for all one-field models. We find that the two-field models behave similarly to the one-field models. The only significant difference is that two-field models extend the $(T/S)_{50}\approx 0$ part of the good-power-law region to $(n-1)_{50}>0$ (see Fig. 4).

Finally, what about our taking $x''\simeq$const? It can affect the relationship between the initial and final values of $n-1$ and T/S if x'' is large, since x'' need not be constant (as is the case in some known models). Since we have covered a wide range of initial values we would expect that this fact would only slightly modify the $(n-1)_{50} - (T/S)_{50}$ phase space; indeed, we have also formulated the flow equations assuming $V'''/V=$const and obtain similar results.

DISCUSSION

Prior to this work there was one guiding relation for the inflationary observables: $T/S=-5n_T$. It has the virtue of exactitude and can test the consistency of the scalar-field inflationary framework, but it involves the power-law index of the gravity-wave perturbations, the most difficult observable to measure. By reformulating the equations governing inflation, we have found generic relations between T/S and $(n-1)$ which are summarized in Figs. 5 and 6 and below:

(i) For $n<1$ the good-power-law region in the $T/S - (n-1)$ plane has $T/S\approx -5(n-1)$ or 0.

(ii) For $n>1.1$ the good-power-law region in the $T/S - (n-1)$ plane has $0.25<T/S<0.7$ (one-field models) or 0 (two-field models).

(iii) For $0.85<n<1$ and $|x''|<2$ (featureless potentials), $T/S>10^{-3}$.

(iv) For $n<0.82$ there is a large excluded region in the $T/S - (n-1)$ plane in which absolutely *no* models are found.

(v) There is a correlation between the values of $dn/d\ln k$, T/S, and n.

Our results provide some guidance to CMB experimenters looking for tensor perturbations [cf. (ii) and (iii)] and additional consistency tests for inflation [cf. (iv) and (v)].

ACKNOWLEDGMENTS

We thank Andrew Liddle and David Spergel for valuable discussions and comments. This work was supported by the DoE (at Chicago and Fermilab) and by the NASA (at Fermilab by grant NAG 5-7092).

[1] http://map.gsfc.nasa.gov/
[2] http://astro.estec.esa.nl/SA-general/Projects/Planck/
[3] http://background.uchicago.edu/~whu
[4] P. de Bernardis *et al.*, Nature (London) **404**, 955 (2000); A. Lange *et al.*, Phys. Rev. D **63**, 042001 (2001).
[5] A. Balbi *et al.*, Astrophys. J. Lett. **545**, L1 (2000); S. Hanany *et al.*, *ibid.* **545**, L5 (2000).
[6] See, e.g., L. Knox and L. Page, Phys. Rev. Lett. **85**, 1366 (2000).
[7] A.H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S.W. Hawking, Phys. Lett. **115B**, 295 (1982); A.A. Starobinskii, *ibid.* **117B**, 175 (1982); J.M. Bardeen, P.J. Steinhardt, and M.S. Turner, Phys. Rev. D **28**, 697 (1983).
[8] V.A. Rubakov, M. Sazhin, and A. Veryaskin, Phys. Lett. **115B**, 189 (1982); R. Fabbri and M. Pollock, *ibid.* **125B**, 445 (1983); A.A. Starobinskii, Pis'ma Astron. Zh. **9**, 579 (1983) [Sov. Astron. Lett. **9**, 302 (1983)]; L. Abbott and M. Wise, Nucl. Phys. **B244**, 541 (1984).
[9] See, e.g., J. Lidsey *et al.*, Rev. Mod. Phys. **69**, 373 (1997); M.S. Turner, Phys. Rev. D **48**, 5539 (1993).
[10] M.S. Turner, Phys. Rev. D **55**, R435 (1997); M. Kamionkowski and A. Kosowsky, *ibid.* **57**, 685 (1998).
[11] See, e.g., D. Lyth and A. Riotto, Phys. Rep. **314**, 1 (1999).
[12] R. Davis, Phys. Rev. Lett. **69**, 1856 (1992); D. Lyth and A. Liddle, Phys. Lett. B **291**, 391 (1992).
[13] The precise relationship between S and T and the inflaton potential depends upon the background cosmological model; see M.S. Turner and M. White, Phys. Rev. D **53**, 6822 (1996). The values used in this paper are for $\Omega_M=0.35$ and $\Omega_\Lambda=0.65$. For $\Omega_M=1$, the relation is the more familiar $T/S=-7n_T$. The relationship between n, n_T and T/S and the actual quantities observed in CMB measurements is discussed in more detail in A. Kosowsky and M.S. Turner, *ibid.* **52**, 1739 (1995). In models where there is more than one active field during inflation or inflation is driven by non-standard kinetic terms, the relation is also modified; see E.D. Stewart and J.D. Cohn, *ibid.* **63**, 083519 (2001) and C. Armendariz-Picon, T. Damour, and V. Mukhanov, Phys. Lett. B **458**, 209 (1999).
[14] It had been conjectured that for many models $T/S\approx-5(n-1)$. There are, however, many exceptions to this; see R. Davis *et al.*, Phys. Rev. Lett. **69**, 1856 (1992); V.F. Mukhanov, L. Wang, and P.J. Steinhardt, Phys. Lett. B **414**, 18 (1997); and P.J. Steinhardt, astro-ph/9502024.
[15] A.H. Jaffe *et al.*, Phys. Rev. Lett. **86**, 3475 (2001).
[16] D. Lyth, Phys. Rev. Lett. **78**, 1861 (1997).
[17] E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990), Chap. 8.
[18] These equations apply to models where one field with a standard kinetic term (or one that can be put in canonical form) is dynamical during inflation, which includes the vast majority of the models discussed in the literature.
[19] P.J. Steinhardt and M.S. Turner, Phys. Rev. D **29**, 2162 (1984).
[20] A.R. Liddle and M.S. Turner, Phys. Rev. D **50**, 758 (1994).
[21] W.H. Kinney, Phys. Rev. D **58**, 123506 (1998).
[22] D. Huterer and M.S. Turner, Phys. Rev. D **62**, 063503 (2000).
[23] M. Sasaki and E.D. Stewart, Prog. Theor. Phys. **95**, 71 (1996).
[24] W.H. Kinney, A. Melchiorri, and A. Riotto, Phys. Rev. D **63**, 023505 (2001).