Superhydrodynamics

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We present a covariant and supersymmetric theory of relativistic hydrodynamics in four-dimensional Minkowski space.

Relativistic fluid mechanics [1] is generally simpler to formulate than its nonrelativistic counterpart. As it is also believed to provide a more accurate description of hydrodynamical phenomena, much work has been invested in its development [2,3]. Recently, an interesting extension of the theory to include non-Abelian charges and currents has been proposed [4]. One of the important aspects of this formalism is that it includes vorticity consistently at the Lagrangian level, by developing a non-Abelian generalization of the Clebsch decomposition of the vector conjugate to the current; for a review with many references, see [5].

In a related development, Jackiw and Polychronakos [6] have presented a supersymmetric theory of fluid mechanics in $(2+1)$-dimensional space-time. This model is rather special, as it descends from a supermembrane theory in $3+1$ dimensions [7–9]. It results in a supersymmetric generalization of the nonrelativistic planar Chaplygin gas [10]. An interesting result obtained in [6] is that the vorticity in the theory is generated by the fermion fields, rather than by the bosonic component of the fluid. In spite of these advances, so far a relativistic and supersymmetric theory of fluid mechanics in $3+1$ dimensions is lacking. The present work contributes to filling this gap.

The main result of this work is a supersymmetric component action, which we present both in its Lagrangian and Hamiltonian form, for a pseudo-classical conserved current V_μ made up of bosonic as well as fermionic contributions. To be more precise, the current is constructed from two real scalar fields (C,N) and a chiral fermion field $\psi_\pm$:[1]

$$V_\mu = \frac{1}{G(C)}\left(\partial_\mu N + \frac{i}{2}G'(C)\bar\psi_+\gamma_\mu\psi_-\right), \quad \partial\cdot V = 0. \quad (1)$$

Here $G(C)$ is some function of the real scalar field C which, for the purpose of constructing models, needs no further

specification. However, under particular conditions discussed below the current allows an elegant and straightforward interpretation in terms of fluid flow.

The fields $(C,N,\psi_\pm)$ and V_μ representing the conserved current are described by the supersymmetric Lagrangian:

$$\mathcal{L} = -\frac{1}{2}G(C)[(\partial_\mu C)^2 - V_\mu^2 + \bar\psi_+\overleftrightarrow{\slashed\partial}\psi_-]$$

$$- V^\mu\left(\partial_\mu N + \frac{i}{2}G'(C)\bar\psi_+\gamma_\mu\psi_-\right)$$

$$- \frac{1}{8}G''(C)\bar\psi_+\psi_+\bar\psi_-\psi_-. \quad (2)$$

The infinitesimal supersymmetry transformations leaving the action invariant, parametrized by anti-commuting spinor parameters $\epsilon_\pm$, are

$$\delta C = \bar\epsilon_+\psi_+ + \bar\epsilon_-\psi_-, \quad \delta N = iG(C)(\bar\epsilon_-\psi_- - \bar\epsilon_+\psi_+),$$

$$\delta\psi_+ = (\slashed\partial C + i\slashed V)\epsilon_-, \quad \delta\psi_- = (\slashed\partial C - i\slashed V)\epsilon_+, \quad (3)$$

$$\delta V_\mu = 2i\bar\epsilon_+\sigma_{\mu\nu}\partial^\nu\psi_+ - 2i\bar\epsilon_-\sigma_{\mu\nu}\partial^\nu\psi_-.$$

Under these transformations the variation of the Lagrangian is a total divergence:

$$\delta\mathcal{L} = \partial^\mu(\bar\epsilon_+ B_{+\mu} + \bar\epsilon_- B_{-\mu}), \quad (4)$$

where the vector-spinor fields $B_{\pm\mu}$ are given, modulo equations of motion, by

$$B_{+\mu} \simeq -\frac{1}{2}G(C)\gamma_\mu(\slashed\partial C - i\slashed V)\psi_+ - \frac{1}{2}G'(C)\gamma_\mu\psi_-\bar\psi_+\psi_+, \quad (5)$$

$$B_{-\mu} \simeq -\frac{1}{2}G(C)\gamma_\mu(\slashed\partial C + i\slashed V)\psi_- - \frac{1}{2}G'(C)\gamma_\mu\psi_+\bar\psi_-\psi_-.$$

The commutator of two supersymmetry transformations (3) closes as usual, modulo field equations. A complete and

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[1]Our conventions for chiral spinors are such that $\gamma_5\psi_\pm = \pm\psi_\pm$ and $\bar\psi_\pm\gamma_5 = \pm\bar\psi_\pm$; charge conjugation acts as $\psi_\pm = C\bar\psi_\pm^T$, where $\bar\psi_\pm = i\psi_\pm^\dagger\gamma_0$. It should also be noted that the Euclidean $\gamma_4 = i\gamma_0$ is Hermitian, hence γ_0 is anti-Hermitian.

manifest off-shell formulation can be obtained from a superspace treatment. The relevant superspace action is defined in terms of a real vector superfield V with lowest components $(C, \psi_\pm, V_\mu)$, and a pair of conjugate chiral superfields (Λ_+, Λ_-) with lowest component z such that $\mathrm{Re}\, z = N$. The superfield expression for the Lagrangian is

$$S = \int d^4x \int d^2\theta_+ \int d^2\theta_- \left(\frac{1}{2} V(\Lambda_+ + \Lambda_-) - F(V) \right). \tag{6}$$

Here $F(V)$ is an arbitrary real function of the superfield V, with lowest scalar component $F(V)|_{\theta_\pm = 0} = F(C)$. Reduction of the superspace expression (6) in terms of components leads, after elimination of a number of auxiliary fields and some rescaling of the scalars, to the expression (2) with the identification $G(C) = F''(C)$.

The field equations derived from the Lagrangian (2) are

$$\partial \cdot V = 0, \quad \partial_\mu N = G(C) V_\mu - \frac{i}{2} G'(C) \bar{\psi}_+ \gamma_\mu \psi_-, \tag{7}$$

$$G(C) \Box C = -\frac{1}{2} G'(C) [(\partial C)^2 + V^2 - \bar{\psi}_+ \overset{\leftrightarrow}{\partial} \psi_-]$$

$$+ \frac{i}{2} G''(C) \bar{\psi}_+ \rlap{/}{V} \psi_- + \frac{1}{8} G'''(C) \bar{\psi}_+ \psi_+ \bar{\psi}_- \psi_-$$

for the bosonic fields, and

$$G(C) \rlap{/}{\partial} \psi_\pm = -\frac{1}{2} G'(C) (\rlap{/}{\partial} C \mp i \rlap{/}{V}) \psi_\pm - \frac{1}{4} G''(C) \psi_\mp \bar{\psi}_\pm \psi_\pm \tag{8}$$

for the fermionic ones. The first two equations (7) indeed reproduce Eq. (1).

Because of translation invariance and supersymmetry, the theory described by $\mathcal{L}$ conserves the four-momentum and the supercharges. The corresponding currents are provided by the energy-momentum tensor and the supercurrents. For the energy-momentum tensor we use the symmetrized version

$$T_{\mu\nu} = G(C) \left[\partial_\mu C \partial_\nu C + V_\mu V_\nu + \frac{1}{4} \bar{\psi}_+ (\gamma_\mu \overset{\leftrightarrow}{\partial}_\nu + \gamma_\nu \overset{\leftrightarrow}{\partial}_\mu) \psi_- \right]$$

$$- \frac{i}{4} G'(C) \bar{\psi}_+ (\gamma_\mu V_\nu + \gamma_\nu V_\mu) \psi_-$$

$$- g_{\mu\nu} \left[\frac{1}{2} G(C) ((\partial C)^2 + V^2 + \bar{\psi}_+ \overset{\leftrightarrow}{\partial} \psi_-) \right.$$

$$\left. + \frac{1}{8} G''(C) \bar{\psi}_+ \psi_+ \bar{\psi}_- \psi_- \right]. \tag{9}$$

It is obtained from the non-symmetric translational Noether currents $\Theta_{\mu\nu}$ by the addition of an improvement term: $T_{\mu\nu} = \Theta_{\mu\nu} + \Omega_{\mu\nu}$, where

$$\Omega_{\mu\nu} = \frac{1}{4} \partial^\lambda (\varepsilon_{\mu\nu\kappa\lambda} G(C) \bar{\psi}_+ \gamma_\kappa \psi_-), \quad \partial^\mu \Omega_{\mu\nu} = 0. \tag{10}$$

The field equations (7) and (8) imply $\partial^\mu T_{\mu\nu} = 0$, from which the conservation of four-momentum $P_\mu = \int d^3x\, T_{\mu 0}$ follows. The supercurrents are obtained directly by Noether's procedure and read (in an on-shell version)

$$S_{+\mu} = G(C)(\rlap{/}{\partial} C - i \rlap{/}{V}) \gamma_\mu \psi_+ - \frac{1}{2} G'(C) \gamma_\mu \psi_- \bar{\psi}_+ \psi_+, \tag{11}$$

$$S_{-\mu} = G(C)(\rlap{/}{\partial} C + i \rlap{/}{V}) \gamma_\mu \psi_- - \frac{1}{2} G'(C) \gamma_\mu \psi_+ \bar{\psi}_- \psi_-.$$

As on shell $\partial \cdot S_\pm = 0$, the conservation of the supercharges $Q_\pm = \int d^3x\, S_{\pm 0}$ follows.

The manifestly covariant Lagrangian description of the theory has an equivalent canonical formulation in terms of a Hamiltonian with a corresponding bracket structure. First, the canonical momenta are defined by

$$\pi_C = \frac{\delta \mathcal{L}_{\mathrm{eff}}}{\delta \dot{C}} = G(C) \dot{C}, \quad \pi_N = \frac{\delta \mathcal{L}_{\mathrm{eff}}}{\delta \dot{N}} = V_0,$$

$$\pi_+ = \gamma_0 \frac{\delta \mathcal{L}_{\mathrm{eff}}}{\delta \dot{\bar{\psi}}_-} = \frac{1}{2} G(C) \psi_+, \tag{12}$$

$$\pi_- = \gamma_0 \frac{\delta \mathcal{L}_{\mathrm{eff}}}{\delta \dot{\bar{\psi}}_+} = \frac{1}{2} G(C) \psi_-.$$

The inclusion of the γ_0 in the definition of the fermionic momenta is motivated by the general requirement that the charge conjugation properties of the momenta reflect those of the fermion variables themselves. Furthermore, note that V_0 is a canonical momentum, whereas the 3-vector field $\vec{V}$ is an auxiliary field, which can be eliminated by its algebraic field equation; in particular in the following we use the identifications

$$V_0 = \pi_N, \quad \vec{V} = \frac{1}{G(C)} \left(\vec{\nabla} N + \frac{i}{2} G'(C) \bar{\psi}_+ \vec{\gamma} \psi_- \right). \tag{13}$$

As usual, the fermionic momenta are not independent, and the theory possesses the second-class constraints

$$\chi_\pm = \pi_\pm - \frac{1}{2} G(C) \psi_\pm \simeq 0. \tag{14}$$

As a result, the dynamics of the theory is generated by a Hamiltonian—obtained from $\mathcal{L}$ by a straightforward Legendre transformation—and a set of Dirac-Poisson brackets derived by reduction of the standard Poisson brackets to the

physical shell (14) in the full phase-space. In summary, one first constructs a Hamiltonian density

$$\mathcal{H} = \frac{1}{2G(C)}\pi_C^2 + \frac{1}{2}G(C)\pi_N^2 - \frac{i}{2}G'(C)\pi_N\bar{\psi}_+\gamma_0\psi_-$$

$$+ \frac{1}{2}G(C)((\vec{\nabla}C)^2 + \bar{\psi}_+\overleftrightarrow{\vec{\nabla}}\psi_-)$$

$$+ \frac{1}{2G(C)}\left(\vec{\nabla}N + \frac{i}{2}G'(C)\bar{\psi}_+\vec{\gamma}\psi_-\right)^2$$

$$+ \frac{1}{8}G''(C)\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-, \tag{15}$$

where it is to be noted that $\vec{\nabla} = \vec{\gamma}\cdot\vec{\nabla}$ now denotes a three-dimensional contraction. Subsequently the field equations are obtained from the Hamiltonian $H = \int d^3x\mathcal{H}$ by the Dirac-Poisson brackets

$$\partial_0 A = \{A, H\}^*, \tag{16}$$

defined in terms of the elementary brackets or antibrackets

$$\{C(\mathbf{r},t), \pi_C(\mathbf{r}',t)\}^* = \delta^3(\mathbf{r}-\mathbf{r}'),$$

$$\{N(\mathbf{r},t), \pi_N(\mathbf{r}',t)\}^* = \delta^3(\mathbf{r}-\mathbf{r}'),$$

$$\{\pi_C(\mathbf{r},t), \bar{\psi}_\pm(\mathbf{r}',t)\}^* = \frac{G'(C)}{2G(C)}\bar{\psi}_\pm(\mathbf{r},t)\delta^3(\mathbf{r}-\mathbf{r}'), \tag{17}$$

$$\{\psi_\pm(\mathbf{r},t), \pi_C(\mathbf{r}',t)\}^* = -\frac{G'(C)}{2G(C)}\psi_\pm(\mathbf{r},t)\delta^3(\mathbf{r}-\mathbf{r}'),$$

$$\{\psi_\pm(\mathbf{r},t), \bar{\psi}_\mp(\mathbf{r}',t)\}^* = \frac{\gamma^0(1\mp\gamma_5)}{2G(C)}\delta^3(\mathbf{r}-\mathbf{r}').$$

In particular we note that the equation

$$\dot{\pi}_N = \{\pi_N, H_{\text{eff}}\}^* = \vec{\nabla}\cdot\left[\frac{1}{G(C)}\left(\vec{\nabla}N + \frac{i}{2}G'(C)\bar{\psi}_+\vec{\gamma}\psi_-\right)\right], \tag{18}$$

after the identification (13) is equivalent with $\dot{V}_0 = \vec{\nabla}\cdot\vec{V}$, or $\partial\cdot V = 0$. A somewhat tedious calculation shows that Eqs. (16) and (17) indeed reproduce all covariant field equations (7) and (8). We can now construct the canonical expressions for the conserved four-momentum and the supercharges; for the energy-momentum vector we find the results

$$P_0 = H = \int d^3\mathbf{r}\mathcal{H}, \tag{19}$$

$$P_i = \int d^3\mathbf{r}\left[\pi_C\nabla_i C + \pi_N\nabla_i N + \frac{1}{2}G(C)\bar{\psi}_+\gamma_0\overleftrightarrow{\nabla}_i\psi_-\right].$$

The canonical expressions for the supercharges are

$$Q_+ = \int d^3\mathbf{r}\Bigg[(\pi_C - iG(C)\pi_N)\psi_+$$

$$+ (G(C)\vec{\nabla}C - i\vec{\nabla}N)\gamma_0\psi_+ + \frac{1}{4}G'(C)\gamma_0\psi_-\bar{\psi}_+\psi_+\Bigg], \tag{20}$$

$$Q_- = \int d^3\mathbf{r}\Bigg[(\pi_C + iG(C)\pi_N)\psi_- + (G(C)\vec{\nabla}C + i\vec{\nabla}N)\gamma_0\psi_-$$

$$+ \frac{1}{4}G'(C)\gamma_0\psi_+\bar{\psi}_-\psi_-\Bigg].$$

Like the Hamiltonian generates the time-evolution, the supercharges generate the supersymmetry transformations: the results (3) are reproduced by the brackets

$$\delta(\epsilon_\pm)A = \{A, \bar{\epsilon}_\pm Q_\pm\}^*. \tag{21}$$

The supercharges satisfy the standard super-Poincaré algebra, in particular $\{Q_\pm, \bar{Q}_\mp\}^* = 2\not{P}$. Note that the supersymmetry transformation rule implies $\omega_\mu \equiv \{V_\mu, Q_\pm\}^* = 2i\sigma_{\mu\nu}\partial^\nu\psi_\pm$, where ω_μ is a conserved fermionic current: $\partial\cdot\omega = 0$, as expected from supersymmetry and the conservation law of V_μ. It is also of interest to discuss the brackets of the current components among themselves. By applying the identifications (13) one finds the non-trivial results

$$\{V_0(\mathbf{r},t), \vec{V}(\mathbf{r}',t)\}^* = \frac{1}{G(C)}\vec{\nabla}_\mathbf{r}\delta^3(\mathbf{r}-\mathbf{r}'),$$

and

$$\{V_i(\mathbf{r},t), V_j(\mathbf{r}',t)\}^* = \frac{(G'(C))^2}{(G(C))^3}\bar{\psi}_+\gamma^0\sigma_{ij}\psi_-$$

$$\times \delta^3(\mathbf{r}-\mathbf{r}'). \tag{22}$$

Note that the charge-conjugation properties of the spinors guarantee that $\bar{\psi}_+\gamma^0\sigma_{ij}\psi_- = i\psi_-^\dagger\sigma_{ij}\psi_- = i\psi_+^\dagger\sigma_{ij}\psi_+$, as in our conventions σ_{ij} is anti-Hermitian.

For easy comparison we have performed the canonical analysis in terms of the fields related by the linear supersymmetry transformations (3), at the price of dealing with off-diagonal terms in the Dirac-Poisson brackets (17). Observe, however, that the brackets can be diagonalized by a field redefinition

$$\Phi_\pm = \sqrt{G(C)}\psi_\pm. \tag{23}$$

Indeed, in terms of the new fermion fields the brackets read

$$\{C(\mathbf{r},t), \pi_C(\mathbf{r}',t)\}^* = \delta^3(\mathbf{r}-\mathbf{r}'),$$

$$\{N(\mathbf{r},t),\pi_N(\mathbf{r}',t)\}^* = \delta^3(\mathbf{r}-\mathbf{r}'), \tag{24}$$

$$\{\pi_C(\mathbf{r},t),\bar{\Phi}_\pm(\mathbf{r}',t)\}^* = \{\Phi_\pm(\mathbf{r},t),\pi_C(\mathbf{r}',t)\}^* = 0,$$

$$\{\Phi_\pm(\mathbf{r},t),\bar{\Phi}_\mp(\mathbf{r}',t)\}^* = \frac{1}{2}\gamma^0(1\mp\gamma_5)\delta^3(\mathbf{r}-\mathbf{r}'),$$

but of course the supersymmetry transformations (3) now become non-linear. We do not present the new expressions here for the energy-momentum and supercurrents; however, the conserved current (1) is of particular interest and we observe that after the field redefinition (23) its space components read

$$\vec{V} = \frac{1}{G(C)}\vec{\nabla}N + \frac{iG'(C)}{2(G(C))^2}\bar{\Phi}_+\vec{\gamma}\Phi_-. \tag{25}$$

To complete its hydrodynamical interpretation, we relate the fields in our model to the fluid density ρ and velocity u_μ. First we consider the bosonic reduction, obtained by requiring the fermion field to vanish: $\Phi_\pm = 0$. The field equations (7) then reduce to

$$V_\mu = \rho u_\mu = \frac{1}{G(C)}\partial_\mu N, \quad V_\mu^2 = -\rho^2, \quad \partial\cdot V = 0, \tag{26}$$

$$G(C)\square C = \frac{1}{2}G'(C)(-(\partial C)^2+\rho^2).$$

In the present context the current-conservation condition is to be interpreted as the equation of continuity in hydrodynamics, while the last equation in Eq. (26) can be used to express the density ρ in terms of the field C. The first equation in Eq. (26) then implies that the velocity field for fixed ρ (i.e., for fixed C) depends only on one degree of freedom, N. This is also the situation encountered in potential flow, when $u_\mu = \partial_\mu\theta$, $(\partial\theta)^2 = -1$. Therefore our bosonic model describes potential flow with vanishing vorticity, provided

$$\partial_\mu\theta = \frac{1}{\rho G(C)}\partial_\mu N. \tag{27}$$

Such a relation is consistently realized if either $\partial_\mu\rho \sim \partial_\mu\theta \sim V_\mu$, or else if N can be expressed entirely in terms of θ, independent of ρ: $\partial N/\partial\rho = 0$. In the first case $\partial_\mu C \sim V_\mu$, and the bosonic part of the energy-momentum tensor manifestly takes the perfect fluid form; indeed, if $\partial_\mu C = \kappa u_\mu$, then

$$T_{\mu\nu}^{(B)} = pg_{\mu\nu} + (p+\varepsilon)u_\mu u_\nu, \tag{28}$$

where the pressure and energy density are given by[2] $p = \varepsilon = \frac{1}{2}G(C)(\kappa^2+\rho^2)$. The case $\partial N/\partial\rho = 0$ is realized if $N = 2m\theta$, $\rho = 2m/G(C)$, with m a constant of proportionality. It follows that N must satisfy the condition $(\partial N)^2 = -4m^2$ = const, while the combination of these conditions with Eq. (26) gives

$$\frac{[G(C)]^{5/2}}{G'(C)}\partial^\mu(\sqrt{G(C)}\partial_\mu C) = 2m^2. \tag{29}$$

For example, if $G(C) = 1/C$, we find the imaginary-mass Klein-Gordon equation

$$(\square + m^2)\sqrt{C} = 0, \tag{30}$$

which allows non-trivial stationary solutions for the density $\rho = 2mC$. The more general Ansatz $G(C) = C^p$ leads to the non-linear relation

$$\square C^{(p+2)/2} = m^2 p(p+2)C^{-(3p+2)/2}. \tag{31}$$

If the fermion fields are switched on, the potential character of the flow disappears; indeed, as

$$V_\mu = \rho u_\mu = \frac{1}{G(C)}\partial_\mu N + \frac{iG'(C)}{2(G(C))^2}\bar{\Phi}_+\gamma_\mu\Phi_-, \tag{32}$$

the velocity is now parametrized in terms of one bosonic and two fermionic degrees of freedom. As Eq. (22) shows, the vorticity is non-zero as a result of the fermionic contribution indeed. In terms of the redefined fields Φ these bracket relations take the form

$$\{V_i(\mathbf{r},t),V_j(\mathbf{r}',t)\}^* = \left(\frac{G'(C)}{[G(C)]^2}\right)^2\bar{\Phi}_+\gamma^0\sigma_{ij}\Phi_-\delta^3(\mathbf{r}-\mathbf{r}'). \tag{33}$$

In particular, for the case $G(C) = 1/C$ the coefficients of the bi-fermion term on the right-hand side of Eqs. (32) and (33) reduce to constants, and the current is a linear combination of free boson and fermion currents.

The models constructed here can be generalized to include vorticity from bosonic potentials, by a supersymmetric extension of the Clebsch decomposition of V_μ; this is explained elsewhere [11]. Consistent extensions of the models at the quantum level are of potential interest in cosmology, where they could provide an effective description of an early supersymmetric phase of the universe, and in condensed matter physics, where they might apply to quantum fluids like ^{3}He-^{4}He mixtures, up to effects proportional to the mass-differences of these isotopes.

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[2]It is possible to modify the action (6) so as to add a constant term to the energy; then the equation of state is changed to $p+\mu^2 = \varepsilon - \mu^2 = \frac{1}{2}G(C)(\kappa^2+\rho^2)$; details are given in [11].

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