Chromomagnetic catalysis of color superconductivity in a (2+1)-dimensional Nambu–Jona-Lasinio model

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We investigate the influence of a constant uniform external chromomagnetic field H on the formation of color superconductivity. The consideration is made in the framework of a $(2+1)$-dimensional Nambu–Jona-Lasinio model with two different four-fermionic structures responsible for $\langle\bar{q}q\rangle$ and diquark $\langle qq\rangle$ condensates. In particular, it is shown that there exists a critical value H_c of the external chromomagnetic field such that at $H>H_c$ a nonvanishing diquark condensate is dynamically created (the so-called chromomagnetic catalysis effect of color superconductivity). Moreover, external chromomagnetic fields may in some cases enhance the diquark condensate of color superconductivity.

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I. INTRODUCTION

During the last two decades a great deal of attention was paid to the investigation of the QCD ground state at finite temperature and density (see, e.g., the recent review [1] and references therein). The main efforts were directed to the consideration of the quark-gluon plasma—a new state of matter which can exist at sufficiently high temperature and density. In addition, it was also realized [2] that at low (zero) temperature and high baryon density colored quarks, interacting via gluon exchange, can form Cooper pairs. Hence, the quark system would pass to the so-called color superconducting phase in which the color symmetry of the theory is spontaneously broken. However, since the corresponding value of the diquark condensate $\langle qq\rangle$ was estimated to be of order 1 MeV, one could not get any observable effects in this case.

Quite recently it was pointed out [3] that due to instantons there is a nonperturbative mechanism of forming a condensate $\langle qq\rangle\neq 0$. As a consequence, a rather large observable value of order 100 MeV for the diquark condensate was predicted and the color superconductivity (CSC) might possibly be detected in the future experiments on heavy ion collisions, i.e., at moderate baryon density, or realized in the interior of neutron stars. At present time, there exists a rich literature devoted to this new physical effect; the CSC phenomenon has been studied in the framework of an instanton model [3], in different versions of quark models [4] of the Nambu–Jona-Lasinio (NJL) type [5], some QCD-like theories with nonstandard color group and quark multiplets [6], and using lattice and $1/N$ approaches to four-fermion models [7]. CSC was also investigated in the frameworks of the renormalization group and variational as well as Dyson-Schwinger equation methods [8]. In all of the above cited papers [3–8] the nonperturbative feature $\langle\bar{q}q\rangle\neq 0$ of the QCD vacuum related to spontaneously broken chiral symmetry was taken into account. Then, the phase structure of the theory is the consequence of a competition between two dynamical order parameters $\langle\bar{q}q\rangle$ and $\langle qq\rangle$.

It is well known that the gluonic degrees of freedom influence the properties of the QCD vacuum, in which there is one more nonzero condensate $\langle F^a_{\mu\nu}F^{a\mu\nu}\rangle\equiv\langle FF\rangle$. Hence, in order to get a more adequate phase structure of the theory one should consider the competition of three dynamical parameters $\langle FF\rangle$, $\langle\bar{q}q\rangle$, and $\langle qq\rangle$. Of course, it is very hard to solve this situation within QCD itself. So, instead of this we shall incorporate the nonzero gluon condensate $\langle FF\rangle$ into a simpler NJL model consideration of the CSC phenomenon. The NJL model does not contain dynamical gluons, hence in this case the gluon condensate $\langle FF\rangle$ is rather an external parameter (similar to chemical potential, temperature, etc.), than a dynamical one. In the framework of the NJL model the condensate $\langle FF\rangle\neq 0$ can be realized in terms of an external (background) gauge field $A^a_\mu(x)$ [9].

The primary goal of the present paper is the investigation of the role which the gluon condensate will play in the formation of CSC. In the chosen NJL model approach we shall, in particular, consider a chromomagnetic gluon condensate, i.e., $\langle FF\rangle=H^2>0$, with H being a constant chromomagnetic background field. Let us first comment on the case of a vanishing diquark condensate. One can then imagine that the vacuum has a color ferromagnet-like domain structure. Inside each domain the chromomagnetic field H^a is homogeneous, but its direction is varying from one domain to another in such a way that space averaging of H^a is equal to zero. So color and Lorentz invariances are not broken [10]. On the other hand, when the system is in the color superconducting phase, the $SU_c(3)$ symmetry is broken spontaneously to $SU_c(2)$. Using pure symmetry arguments, it is easily shown that the three gluons living in the unbroken $SU_c(2)$ subgroup stay massless, whereas the remaining five gluons get masses by the Higgs mechanism. By analogy with ordinary superconductivity, it is expected that external chromomagnetic fields corresponding to massive gluons, i.e., external chromomagnetic fields of the form $H^a=(0,0,0,H^4,\ldots,H^8)$, should be expelled from the CSC phase (Meissner effect).[1] Moreover, sufficiently high values of such fields should destroy the CSC. However, our intuition tells us nothing about the action of external chromomagnetic fields, which in the color space look like $H^a=(H^1,H^2,H^3,0,\ldots,0)$, on the color superconducting state of the quark-gluon system.

In the present paper the influence of such external chromomagnetic fields of the form $H^a=(H^1,H^2,H^3,0,\ldots,0)$ on the phase structure of the NJL model is considered. Since we are mainly interested in clarifying the role of external chromomagnetic fields in the creation of a color diquark condensate, we put the chemical potential and temperature equal to zero. For simplicity, all the considerations are performed in $(2+1)$-dimensional space-time.[2]

The paper is organized as follows. In Sec. II the NJL model under consideration is presented and its effective potential at nonzero external chromomagnetic field is obtained in the one-loop approximation. This quantity contains all the information about the quark condensates of the theory. In the following Secs. III–V the phase structure of the model is investigated, first for zero background gauge field and then for nonvanishing vector potentials of two types (Abelian and non-Abelian), corresponding to the same external chromomagnetic field H, respectively. In Sec. VI it is shown that there exists a critical value H_c of the gluon condensate field at which the color diquark condensate is spontaneously generated (the chromomagnetic catalysis of CSC). Detailed investigations of global minimum points of the effective potential are relegated to the Appendix. Finally, a summary and discussion of the results are given in Sec. VII.

II. MODEL LAGRANGIAN AND EFFECTIVE POTENTIAL

In the present paper the influence of a constant external chromomagnetic field on the phase structure of a NJL-type model with quarks of two flavors and three colors is investigated at zero chemical potential and temperature. The Lagrangian of the model contains two different four-fermionic structures responsible for the dynamical appearance of $\langle\bar{q}q\rangle$ as well as $\langle qq\rangle$ condensates. (Earlier, similar considerations were done for the simplest NJL-type models in which only a chiral condensate could appear [9,15–18].) The $(2+1)$-dimensional model under consideration has the following Lagrangian:[3]

$$L=\bar{q}\gamma^\mu\left(i\partial_\mu+eA^a_\mu(x)\frac{\lambda_a}{2}\right)q+\frac{G_1}{6}[(\bar{q}q)^2+(\bar{q}i\gamma^5\vec{\tau}q)^2]+\frac{G_2}{3}[i\bar{q}^C\varepsilon\epsilon^b\gamma^5q][i\bar{q}\varepsilon\epsilon^b\gamma^5q_C]. \quad (1)$$

In Eq. (1) e denotes the gluon coupling constant, $q_C=C\bar{q}^T$, $\bar{q}^C=q^TC$ are charge-conjugated spinors, and $C=\gamma^2$ is the charge conjugation matrix (T denotes the transposition operation). Moreover, summation over repeated indices $a=1,\ldots,8$, $b=1,2,3$, $\mu=0,1,2$ is implied. The quark field $q\equiv q_{i\alpha}$ is a flavor doublet and color triplet as well as a four-component Dirac spinor, where $i=1,2$, $\alpha=1,2,3$. (Latin and Greek indices refer to flavor and color spaces, respectively; spinor indices are omitted.) Furthermore, we use the notations $\lambda_a/2$ for the generators of the color $SU_c(3)$ group as well as $\vec{\tau}\equiv(\tau^1,\tau^2,\tau^3)$ for Pauli matrices in the flavor space, and ε and ϵ^b are operators in the flavor and color spaces with matrix elements $(\varepsilon)^{ik}\equiv\varepsilon^{ik}$, $(\epsilon^b)^{\alpha\beta}\equiv\epsilon^{\alpha\beta b}$, where ε^{ik} and $\epsilon^{\alpha\beta b}$ are totally antisymmetric tensors. Next, let us at the moment suppose that in Eq. (1) $A^a_\mu(x)$ is an arbitrary external gauge field of the color group $SU_c(3)$. (The following investigations do not require the explicit inclusion of the kinetic gauge field part of the Lagrangian.) Below, the detailed structure of $A^a_\mu(x)$ corresponding to a constant chromomagnetic gluon condensate will be given.

In order to demonstrate the main ideas of our approach in a way as simple as possible (thereby obtaining relatively simple analytical expressions for the effective potential and allowing, in principle, for other applications in planar physics), we find it convenient to perform the following investigations in the $(2+1)$-dimensional space time.[4] We want to note that in this case the model (1) studied here and QCD_3 are not in the same universality class of theories, since they differ in their flavor symmetry groups; in fact, for the three-dimensional two-flavor case considered here, QCD_3 has a peculiar $U(2N_f)=U(4)$ flavor symmetry besides that of the color $SU_c(3)$ symmetry (see Appendix A). On the other

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[1]The Meissner effect for an ordinary external homogeneous magnetic field acting on a color superconductor was investigated by several authors. It was shown in [11] that, due to the presence of a massless combination of the photon and some gluon fields, an ordinary homogeneous magnetic field can penetrate into a color superconductor. The strength of the magnetic field inside the latter depends on details of the geometry, the relative strength of external electromagnetism, and the color forces. In some cases (e.g., for CSC in spherical regions), the applied static homogeneous magnetic field might also be completely expelled from CSC [12].

[2]Recall that in (2+1)-dimensional space-time NJL-type models are renormalizable in the framework of nonperturbative $1/N_c$ expansion techniques [13]. Moreover, these theories are used in order to describe different planar physical phenomena, including ordinary and high temperature superconductivity [14].

[3]In the $(2+1)$ dimension the four-component spinor representation of the Lorentz group is a reducible one. The corresponding algebra of γ matrices is given in Appendix A.

[4]At nonzero chemical potential μ and $A^a_\mu(x)=0$, models including diquark channels similar to Eq. (1) were considered in low dimensions [7] and in $(3+1)$-dimensional space-time as well [4,19]. In contrast, we consider here the case with $A^a_\mu(x)\neq0$ and chemical potential $\mu=0$.

hand, our Lagrangian (1), being invariant under (global) color symmetry, has been constructed in such a way that it mimics just chiral invariance of real QCD_4 under the ''chiral'' (flavor) group $SU(2)_L \times SU(2)_R$, which is here a subgroup of U(4).

By the above reasons, it is clear that the results obtained below in the framework of our $(2+1)$-dimensional model (1) cannot be considered to describe the symmetry breaking scenario of the full U(4) symmetry, nor are they applicable for modeling ''planar'' QCD_3. On the other hand, the important underlying mechanism of chromomagnetic catalysis of dynamical symmetry breaking resulting from ''dimensional reduction'' by external chromomagnetic fields has been shown to exist both in $D=(3+1)$ and $D=(2+1)$ [15–17]. On this basis, one might then expect that the mechanism of chromomagnetic catalysis of chiral symmetry breaking and CSC and its influence on the interplay of quark and diquark condensates of real two-flavor QCD_4 might be reasonably mimicked by our simpler $(2+1)$-dimensional model (1) (see also the discussion in the last section of the present paper).

After these necessary explanations, we mention that in order to obtain realistic estimates for masses of vector/axial-vector mesons and diquarks in extended NJL types of models [19], we have to allow for independent coupling constants G_1, G_2, rather than to consider them related by a Fierz transformation of a current-current interaction via gluon exchange. Clearly, such a procedure does not spoil chiral symmetry. For the general discussion of phase transitions below (see Secs. IV–VI) and also following the above arguments, we find it therefore convenient to treat the coupling constants in Eq. (1) as independent quantities instead of specifying them by additional model requirements.

The linearized version of the model (1) with auxiliary bosonic fields has the following form:

$$\begin{aligned}\tilde{L} = {} & \bar{q}\gamma^\mu\left(i\partial_\mu + eA^a_\mu(x)\frac{\lambda_a}{2}\right)q - \bar{q}(\sigma + i\gamma^5\vec{\tau}\vec{\pi})q \\ & - \frac{3}{2G_1}(\sigma^2+\vec{\pi}^2) - \frac{3}{G_2}\Delta^{*b}\Delta^b - \Delta^{*b}[iq^T C\varepsilon\epsilon^b\gamma^5 q] \\ & - \Delta^b[i\bar{q}\varepsilon\epsilon^b\gamma^5 C\bar{q}^T]. \end{aligned} \tag{2}$$

The Lagrangians (1) and (2) are equivalent, as can be seen by using the equations of motion for bosonic fields, from which it follows that

$$\Delta^b \sim iq^T C\varepsilon\epsilon^b\gamma^5 q, \quad \sigma \sim \bar{q}q, \quad \vec{\pi} \sim i\bar{q}\gamma^5\vec{\tau}q. \tag{3}$$

Obviously, the σ and $\vec{\pi}$ fields are color singlets. Besides, the (bosonic) diquark field Δ^b is a color antitriplet and an isospin singlet under the chiral $SU(2)_L \times SU(2)_R$ group. Note further that σ, Δ^b are scalars, but $\vec{\pi}$ is a pseudoscalar field (see Appendix A). Hence, if $\sigma \neq 0$, then chiral symmetry of the model is spontaneously broken. $\Delta^b \neq 0$ indicates the dynamical breaking of both the electromagnetic and the color symmetries of the theory.

In the one-fermion-loop approximation the color and chirally invariant effective action for the boson fields are expressed through the path integral over quark fields

$$\begin{aligned}&\exp(i3S_{\rm eff}(\sigma,\vec{\pi},\Delta^b,\Delta^{*b},A^a_\mu)) \\ &\quad = \int [d\bar{q}][dq]\exp\left(i\int \tilde{L}d^3x\right),\end{aligned}$$

where

$$S_{\rm eff}(\sigma,\vec{\pi},\Delta^b,\Delta^{*b},A^a_\mu) = -\int d^3x\left[\frac{\sigma^2+\vec{\pi}^2}{2G_1} + \frac{\Delta^b\Delta^{*b}}{G_2}\right] + \tilde{S},$$

and

$$\begin{aligned}\exp(i3\tilde{S}) &= \int [d\bar{q}][dq] \\ &\quad \times \exp\left(i\int [\bar{q}Dq + \overline{qMq^T} + q^T M q]d^3x\right) \\ &= \int [d\Psi]\exp\left(i\int [\Psi^T Z\Psi]d^3x\right).\end{aligned} \tag{4}$$

In Eq. (4) we have used the following notations:

$$\begin{aligned} D &= \gamma^\mu\left(i\partial_\mu + eA^a_\mu(x)\frac{\lambda_a}{2}\right) - \sigma - i\gamma^5\vec{\pi}\vec{\tau}, \\ M &= -i\Delta^{*b}C\varepsilon\epsilon^b\gamma^5, \\ \bar{M} &= -i\Delta^b\varepsilon\epsilon^b\gamma^5 C \end{aligned} \tag{5}$$

and

$$\Psi^T = (q^T, \bar{q}), \quad Z = \begin{pmatrix} M & -D^T/2 \\ D/2 & \bar{M}\end{pmatrix}, \tag{6}$$

where D^T denotes the transposed Dirac operator [see, e.g., Eq. (11) below]. Using the general formula

$$\begin{aligned}\det\begin{pmatrix} K & L \\ \bar{L} & \bar{K}\end{pmatrix} &= \det[-L\bar{L} + LKL^{-1}K] \\ &= \det[-\bar{L}L + \bar{L}K\bar{L}^{-1}\bar{K}],\end{aligned}$$

one can get from Eq. (4)

$$\begin{aligned}\exp(i3\tilde{S}) &= \det{}^{1/2}(Z) \\ &= {\rm const}\times\det{}^{1/2}(D)\det{}^{1/2}[D^T + 4MD^{-1}\bar{M}]. \end{aligned} \tag{7}$$

Let us assume that in the ground state of our model

$\langle\Delta^1\rangle=\langle\Delta^2\rangle=\langle\vec{\pi}\rangle=0$ and $\langle\sigma\rangle$, $\langle\Delta^3\rangle\neq 0$.[5] Obviously, the residual symmetry group of such a vacuum is $SU_c(2)$, whose generators are the first three generators of initial $SU_c(3)$. Now suppose that the constant external chromomagnetic field, simulating the presence of a gluon condensate $\langle FF\rangle=H^2$, has the following form: $H^a=(H^1,H^2,H^3,0,\ldots,0)$. Clearly, due to the residual $SU_c(2)$ invariance of the vacuum, one can consider the diquark condensate field $\Delta^a=(0,0,\Delta^3)$, putting $H^1=H^2=0$ and $H^3\equiv H$.

Some remarks about the structure of the external condensate field $A^a_\mu(x)$, used in Eq. (1), are needed. From this moment on, we select the $A^a_\mu(x)$ in such a form that the only nonvanishing components of the corresponding field strength tensor $F^a_{\mu\nu}$ are $F^3_{12}=-F^3_{21}=H=\text{const}$. It is well-known that in non-Abelian gauge theories a given external chromomagnetic field does not fix the type of corresponding vector potential uniquely. In other words, there can exist several physically (gauge) nonequivalent vector potentials, which produce the same chromomagnetic field [21]. For example, in the case under consideration, i.e., in three dimensions, the above homogeneous chromomagnetic field can be generated by two qualitatively different vector potentials

$$\begin{aligned}&A^1_\mu(x)=(0,\sqrt{H/e},0),\\&A^2_\mu(x)=(0,0,\sqrt{H/e}),\\&A^a_\mu(x)=0\quad(a=3,\ldots,8),\end{aligned}\tag{8}$$

or

$$A^a_\mu(x)=H\delta_{\mu2}x^1\delta^{a3}.\tag{9}$$

The second of these vector potentials defines the well-known Matinyan-Savvidy model of the gluon condensate in QCD [22]. In the following we shall denote expressions (8) and (9) as vector-potentials I and II, correspondingly.[6]

There exists an attractive picture of a domain structure of the physical vacuum of QCD which assumes that the space is split into an infinite number of macroscopic domains, each of which contain a homogeneous chromomagnetic background field generating a nonzero gluon condensate $\langle FF\rangle\neq 0$ [10]. Averaging over all domains results in a zero background chromomagnetic field, hence color as well as Lorentz symmetries are not broken. (Strictly speaking, our following calculations refer to some given macroscopic domain. The obtained results turn out to depend on color and Lorentz invariant quantities only, and are independent of the concrete domain.) Note also that it was pointed out in [23] that, at high temperature, Abelian-like vector-potentials of the form (9) may serve as a reasonable approximation to the true vacuum of the theory. However, at low temperature, the background gauge field may be essentially non-Abelian, having the form (8).

In order to find nonvanishing condensates $\langle\sigma\rangle$ and $\langle\Delta^3\rangle$, we should calculate the effective potential whose global minimum point provides us with these quantities. Suppose that [apart from the external vector-potential $A^a_\mu(x)$ (9)] all boson fields in $S_{\rm eff}$ do not depend on space-time. In this case, by definition, $S_{\rm eff}=-V_{\rm eff}\int d^3x$, where

$$\begin{aligned}&V_{\rm eff}=\frac{\sigma^2+\vec{\pi}^2}{2G_1}+\frac{\Delta^b\Delta^{*b}}{G_2}+\tilde{V},\\&\tilde{V}=\frac{i}{6v}\ln[\det(D)\det(D^T+4MD^{-1}\bar{M})],\\&v=\int d^3x.\end{aligned}\tag{10}$$

Due to our assumptions on the vacuum structure, in formulas (10) we should put $\Delta^{1,2}\equiv 0$ as well as $\vec{\pi}=0$. Then, taking into account the relations

$$\gamma^{\mu T}C=-C\gamma^\mu\quad(\mu=0,1,2),\quad \lambda^{aT}\epsilon^3=-\epsilon^3\lambda^a\quad(a=1,2,3),$$

we have for the operator D [cf. Eq. (5)] with vector potentials (8),(9) the following identity:

$$D^TC\epsilon^3\equiv[\gamma^{\mu T}(-i\partial_\mu+eA^a_\mu(x)\lambda^T_a/2)-\sigma]C\epsilon^3=C\epsilon^3D.\tag{11}$$

Now, using this formula as well as the relations $\det D=\det(\gamma^5D^T\gamma^5)$, $\det AB=\det A\det B$, $(\varepsilon\varepsilon)_{ij}=-\delta_{ij}$, $(\epsilon^3\epsilon^3)_{\alpha\beta}=-\delta_{\alpha\beta}+\delta_{\alpha3}\delta_{\beta3}$ in expression (10), one can obtain after some evident transformations (recall that $i,j=1,2$ and $\alpha,\beta=1,2,3$),

$$\begin{aligned}\tilde{V}&=\frac{i}{6v}\ln\det[\gamma^5D^T\gamma^5D^T+4\Delta\Delta^*\varepsilon\varepsilon\epsilon^3\epsilon^3]\\&=\frac{i}{6v}\ln\det[D\gamma^{5T}D\gamma^{5T}+4\Delta\Delta^*\delta_{ij}(\delta_{\alpha\beta}-\delta_{\alpha3}\delta_{\beta3})],\end{aligned}\tag{12}$$

where $\Delta\equiv\Delta^3$. Note that the first term under the det symbol in Eq. (12) is a diagonal operator in the flavor space. One can easily see that the second term is also a diagonal operator, but this time in the flavor, color, spinor, as well as coordinate spaces.

[5]Clearly, $\langle\vec{\pi}\rangle\neq 0$ would yield spontaneous breaking of parity. In the theory of strong interactions parity is, however, a conserved quantum number, justifying our assumption that $\langle\vec{\pi}\rangle=0$. Nevertheless, note that at large densities a parity breaking diquark condensate could appear [20].

[6]Having background fields $A^a_\mu(x)$, given by Eq. (8) or (9), one can form three matrix fields $A_\mu(x)=A^a_\mu(x)(\lambda_a/2)$, $(a=1,2,3)$. Now it is easy to see that for vector-potential I (II) the corresponding fields $A_\mu(x)$ do not commute (commute) between themselves. Due to this reason, vector-potential I (II) is sometimes called a non-Abelian (Abelian) vector potential.

III. PHASE STRUCTURE AT ZERO EXTERNAL FIELD

If the external field vanishes, we have the evident relation $D\gamma^5 D\gamma^5 = (\sigma^2 + \partial^2)\delta_{ij}\delta_{\alpha\beta}$. Taking into account this formula as well as the usual one, $\det O = \exp(\mathrm{tr}\ln O)$, the determinant in Eq. (12) can be calculated straightforwardly leading to the following one-loop expression for the effective potential of the initial model (10) ($V_{\rm eff} \equiv V_0$):

$$V_0(\sigma,\Delta,\Delta^*) = \frac{\sigma^2}{2G_1} + \frac{\Delta\Delta^*}{G_2} + \frac{8i}{3}\int \frac{d^3k}{(2\pi)^3}\ln(\sigma^2 + 4|\Delta|^2 - k^2) + \frac{4i}{3}\int \frac{d^3k}{(2\pi)^3}\ln(\sigma^2 - k^2), \quad (13)$$

where we have used the momentum space representation. This expression has ultraviolet divergences. Hence, we need to regularize it by introducing in Eq. (13) Euclidean metric ($k^0 \to ik^0$) and cutting off the range of integration ($k^2 \leqslant \Lambda^2$). Next, by performing the integration and introducing renormalized coupling constants g and f, defined by

$$\frac{1}{G_1} - \frac{4\Lambda}{\pi^2} \equiv \frac{1}{g}, \quad \frac{1}{G_2} - \frac{16\Lambda}{3\pi^2} \equiv \frac{1}{f}, \quad (14)$$

we can express the effective potential in terms of ultraviolet-finite quantities

$$V_0(\sigma,\Delta,\Delta^*) = \frac{\sigma^2}{2g} + \frac{\Delta\Delta^*}{f} + \frac{4}{9\pi}(\sigma^2 + 4|\Delta|^2)^{3/2} + \frac{2}{9\pi}|\sigma|^3. \quad (15)$$

We shall now search for the global minimum point of the potential (15). Before doing this, let us introduce a set of more convenient parameters

$$\phi = 2|\Delta|, \quad A = 3\pi/g, \quad B = 3\pi/(2f), \quad (16)$$

in terms of which the effective potential (15) is given by

$$3\pi V_0(\sigma,\phi) = \frac{A\sigma^2}{2} + \frac{B\phi^2}{2} + \frac{4}{3}(\sigma^2 + \phi^2)^{3/2} + \frac{2}{3}|\sigma|^3. \quad (17)$$

By symmetry reasons, it is sufficient to study the function (17) in the region $\{\sigma \geqslant 0, \phi \geqslant 0\}$, where we have the following stationarity equations:

$$\frac{\partial V_0}{\partial \sigma} = 0 = \sigma\{A + 4\sqrt{\sigma^2 + \phi^2} + 2\sigma\},$$

$$\frac{\partial V_0}{\partial \phi} = 0 = \phi\{B + 4\sqrt{\sigma^2 + \phi^2}\}. \quad (18)$$

In order to find the global minimum point of the effective potential (17) one should search for all the solutions of the stationarity equations (18) and then find among them the single one, where the effective potential takes an absolute minimum value. Omitting the detailed calculations, we present at once the results in terms of the phase portrait shown in Fig. 1.

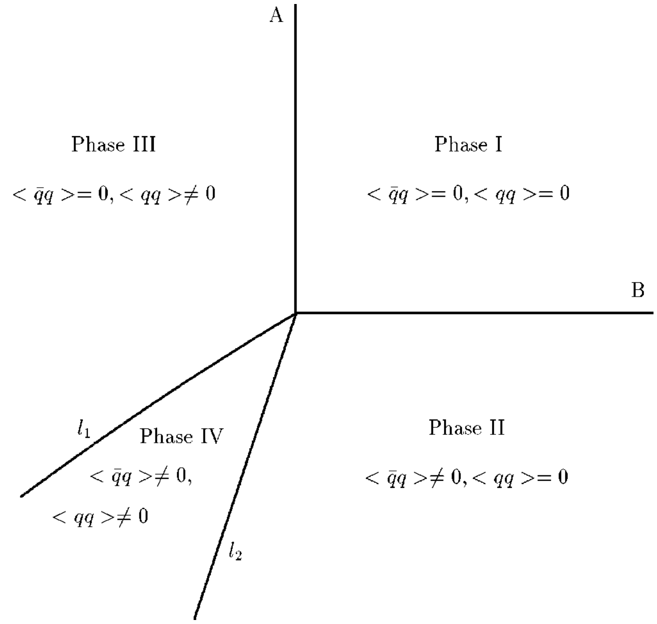

FIG. 1. Phase portrait of the model at vanishing chromomagnetic field $H=0$. The boundaries l_1, l_2 are defined by $l_1 = \{(A,B):\ A=B\}$, $l_2 = \{(A,B):\ 2A=3B\}$, where A and B are related to the interaction strengths G_1 and G_2 through Eqs. (14) and (16).

This figure shows the plane of parameters (A,B) divided into four domains corresponding to the four possible phases of the model. In domain I, the point $\sigma,\phi=0$ is the absolute minimum point of V_0, in II, it is at values $\phi=0$, $\sigma=-A/6$, in region III, the global minimum is located at the point $\sigma=0$, $\phi=-B/4$, and finally, in region IV the global minimum lies in the point $\sigma=(B-A)/2, \phi = \sqrt{B^2/16-(B-A)^2}/4$.

Recall that the coordinates of the global minimum point of the effective potential are the vacuum expectation values $\langle\sigma\rangle$, $\langle\phi\rangle$ of the fields σ,ϕ. Lagrangian (2) provides us with the equations of motion for σ,Δ from which one can easily get the following relations: $\langle\sigma\rangle \sim \langle\overline{q}q\rangle$, $\langle\phi\rangle \sim \langle qq\rangle$, i.e., the global minimum point of the effective potential gives us information about chiral and Cooper-type diquark condensates of the model. Hence, if the parameters (A,B) belong to region I, we have the symmetric phase, because of $\langle\sigma\rangle = \langle\overline{q}q\rangle = 0$, $\langle\phi\rangle = \langle qq\rangle = 0$ in this case. The phase with spontaneously broken chiral symmetry is situated in region II, since here the chiral condensate $\langle\overline{q}q\rangle \neq 0$. In region III the diquark condensate $\langle qq\rangle$ is nonzero, thus indicating the presence of the color superconductivity phase. Clearly, in this case the electromagnetic as well as color symmetries of the model are spontaneously broken. Finally, region IV corresponds to the mixed phase of the theory, where chiral, electromagnetic, and color symmetries are spontaneously broken down (here $\langle\overline{q}q\rangle$, $\langle qq\rangle \neq 0$).

IV. PHASE STRUCTURE OF THE MODEL FOR VECTOR-POTENTIAL I

Let us next study the influence of the external chromomagnetic field with constant vector potential (8) on the phase structure of the model (1). By using the momentum space representation for the operator under the det symbol in Eq. (12), we obtain instead of the differential operator an algebraic (24×24) matrix which has three different eigenvalues $E_i(p)(i=1,2,3)$,

$$E_{1,2}(p)=\overline{p^2}-p_0^2+\sigma^2+4|\Delta|^2+\frac{eH}{2}\pm\frac{1}{2}\sqrt{(eH)^2+4eH\overline{p^2}},$$

$$E_3(p)=\overline{p^2}-p_0^2+\sigma^2, \tag{19}$$

each of them having an eight-fold degeneracy, and $\overline{p^2}=p_1^2+p_2^2$. Taking into account this fact, one can easily obtain from Eqs. (10) and (12) the following expression for the effective potential of the model in the presence of an external vector potential (8) $(V_{\rm eff}\equiv V_{H_1})$:

$$V_{H_1}(\sigma,\Delta,\Delta^*)=\frac{\sigma^2}{2G_1}+\frac{\Delta\Delta^*}{G_2}+\frac{4i}{3}\sum_{i=1}^{3}\int\frac{d^3p}{(2\pi)^3}\ln E_i(p). \tag{20}$$

Integrating here first over p_0 and then over the space of $p_{1,2}$ variables and employing a suitable ultraviolet cutoff, one can get after adopting a renormalization procedure (similar calculations were performed in [15] for the model (1) in the case with $G_2=0$),

$$\begin{aligned}3\pi V_{H_1}(\sigma,\phi)=&\frac{A\sigma^2}{2}+\frac{B\phi^2}{2}+\frac{2|\sigma|^3}{3}+\frac{2}{3}\\&\times\Bigg[(\sigma^2+\phi^2)^{3/2}+(\sigma^2+\phi^2+eH)^{3/2}\\&-\frac{3eH}{4}(\sigma^2+\phi^2+eH)^{1/2}-\frac{3}{4}\\&\times(\sigma^2+\phi^2)\sqrt{eH}\ln\frac{\sqrt{eH}+\sqrt{\sigma^2+\phi^2+eH}}{\sqrt{\sigma^2+\phi^2}}\Bigg],\end{aligned} \tag{21}$$

where we have used the notations introduced in formula (16). Instead of the dimensional quantity (21), let us consider the dimensionless function $V_1(x,y)\equiv3\pi V_{H_1}(\sigma,\phi)/(eH)^{3/2}$, where $x=\sigma/\sqrt{eH}$, $y=\phi/\sqrt{eH}$. Evidently, we have

$$\begin{aligned}V_1(x,y)=&\frac{\tilde{A}x^2}{2}+\frac{\tilde{B}y^2}{2}+\frac{2}{3}|x|^3+\frac{2}{3}\\&\times\Bigg[(x^2+y^2)^{3/2}+(1+x^2+y^2)^{3/2}-\frac{3}{4}\sqrt{1+x^2+y^2}\\&-\frac{3}{4}(x^2+y^2)\ln\left(\frac{1+\sqrt{1+x^2+y^2}}{\sqrt{x^2+y^2}}\right)\Bigg],\end{aligned} \tag{22}$$

where $\tilde{A}=A/\sqrt{eH}$, $\tilde{B}=B/\sqrt{eH}$.

In contrast to the potential (21), which has three dimensional parameters A,B,eH, the function (22) depends only on two parameters $\tilde{A},\tilde{B}$. So, first of all we shall study the global minimum dependence of the potential V_1 on the parameters $\tilde{A},\tilde{B}$. Doing this, we get the possibility of discussing the phase structure of the model in the presence of the external vector potential (8). Since V_1 is symmetric under transformations $x\to-x$ or $y\to-y$, it is sufficient to look for its global minimum only in the region $x,y\geqslant0$. The stationarity equations then take the form

$$\frac{\partial V_1}{\partial x}\equiv x\{\tilde{A}+2x+F(z)|_{z=\sqrt{x^2+y^2}}\}=0,$$

$$\frac{\partial V_1}{\partial y}\equiv y\{\tilde{B}+F(z)|_{z=\sqrt{x^2+y^2}}\}=0, \tag{23}$$

where

$$F(z)=2z+2\sqrt{1+z^2}-\ln[(1+\sqrt{1+z^2})/z]. \tag{24}$$

We should find all the solutions of the equations (23) in the region $x,y\geqslant0$ and then select that one, where the function $V_1(x,y)$ takes its global minimum. Omitting here calculational details, we directly quote the results in the form of the phase portrait. The detailed investigation is given in Appendix B.

In Fig. 2 the phase portrait of the model (1) in the presence of a nonzero vector potential of type I is presented in terms of $\tilde{A}$ and $\tilde{B}$. Here one can see three regions. Above the line $\tilde{l}_1$ there is the phase III of the model, which corresponds to the $V_1(x,y)$ global minimum point of the form $(0,y_0(\tilde{B}))$. [The properties of the function $y_0(\tilde{B})$ as well as the functions $x_0(\tilde{A})$, $x_1(\tilde{A},\tilde{B})$, $y_1(\tilde{A},\tilde{B})$ considered below are given in Appendix B.] In this case $\langle\sigma\rangle=0$, $\langle\phi\rangle\neq0$. Below the curve $\tilde{l}_2$ the global minimum of the effective potential has the form $(x_0(\tilde{A}),0)$. Thus, in this region phase II is located, since for such values of $\tilde{A},\tilde{B}$ the model has a vacuum with $\langle\sigma\rangle\neq0$, $\langle\phi\rangle=0$. Finally, inside the Ω-domain there is a mixed phase IV, since here the global minimum point $(x_1(\tilde{A},\tilde{B}),y_1(\tilde{A},\tilde{B}))$ corresponds to the vacuum with $\langle\bar{q}q\rangle\neq0$ and $\langle qq\rangle\neq0$. It is necessary to emphasize that in the presence of such types of external vector potentials phase I is absent at all [obviously, only when $A_\mu^a(x)=0$, can this phase be realized in the model].

V. PHASE STRUCTURE OF THE MODEL FOR VECTOR-POTENTIAL II

Next, let us study the influence of a nonvanishing external chromomagnetic field with vector potential (9) on the phase structure of the model (1). In this case, after some calculations, the operator, which stands under the det symbol in Eq. (12), can be transformed to the following expression in the color space:

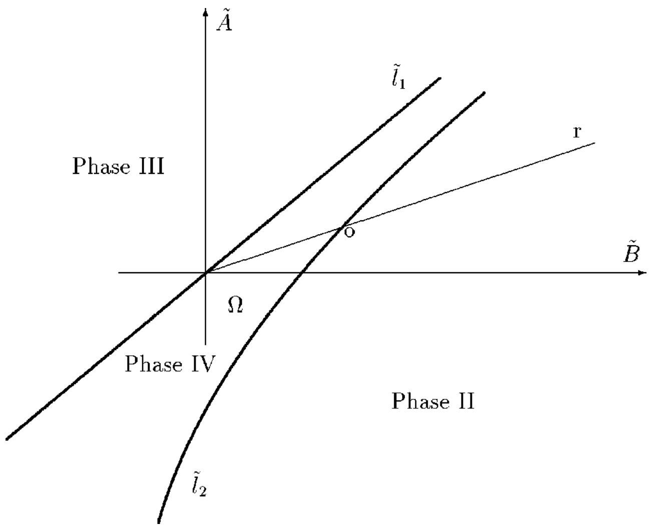

FIG. 2. Phase portrait of the model in the presence of a nonzero vector-potential I. The symmetric phase I is absent. The boundaries $\tilde{l}_1$ and $\tilde{l}_2$ of the region Ω are defined according to $\tilde{l}_1=\{(\tilde{A},\tilde{B}):\tilde{A}=\tilde{B}\}$, $\tilde{l}_2=\{(\tilde{A},\tilde{B}):x_0(\tilde{A})=y_0(\tilde{B})\}$. Here, $\tilde{A}$ and $\tilde{B}$ are defined as $\tilde{A}=A/\sqrt{eH}$, $\tilde{B}=B/\sqrt{eH}$. At $\tilde{B}\to+\infty$ ($\tilde{B}\to-\infty$) the curve $\tilde{l}_2$ approaches asymptotically the line $\tilde{l}_1$ (the line $2\tilde{A}=3\tilde{B}$). The curve $\tilde{l}_2$ intersects the axes $\tilde{A}$ and $\tilde{B}$ in the points (-0.36) and (0.3), respectively. For fixed A and B and varying H one moves along some ray r in the $(\tilde{A},\tilde{B})$ plane. At $H=H_c(A,B)$ there is a phase transition from phase II to the mixed phase IV, if the ray r intersects the line $\tilde{l}_2$ in some point o (see the detailed discussion in Sec. VI).

$$
\begin{aligned}
&D\gamma^5 D\gamma^5+4\Delta\Delta^*\delta_{ij}(\delta_{\alpha\beta}-\delta_{\alpha3}\delta_{\beta3})\\
&\quad=\mathrm{diag}[\delta_{ij}D_+,\delta_{ij}D_-,\delta_{ij}(\sigma^2+\partial^2)],
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
&D_\pm=\sigma^2+4|\Delta|^2-\Pi^\pm_\mu\Pi^{\pm\mu}\mp\frac{ie}{4}\gamma^\mu\gamma^\nu\overline{F}_{\mu\nu},\\
&\Pi^\pm_\mu=i\partial_\mu\pm e\overline{A}_\mu(x)/2,\\
&\overline{A}_\mu=H\delta_{\mu2}x_1.
\end{aligned}
\tag{26}
$$

Note that in Eqs. (25), (26) $D_\pm$ are operators in the coordinate and spinor spaces only. The same is true for the expression $(\sigma^2+\partial^2)$, which is the unit operator in the spinor space. Taking into account Eq. (25), one can easily find for the potential $\tilde{V}$ (12) the expression

$$
\tilde{V}=\frac{i}{3v}\mathrm{tr}\ln D_+ +\frac{i}{3v}\mathrm{tr}\ln D_- +\frac{i}{3v}\mathrm{tr}\ln(\sigma^2+\partial^2)
\tag{27}
$$

[the trace prescription in Eq. (27) is taken over coordinate as well as spinor spaces]. The last term in this formula was calculated in Sec. III of the present paper. Concerning the first two terms in Eq. (27), we should point out that $D_\pm=\tilde{D}_\pm\gamma^5\tilde{D}_\pm\gamma^5$, where $\tilde{D}_\pm=i\gamma^\mu\partial_\mu\pm e\gamma^\mu\overline{A}_\mu(x)/2+M$ are formally the Dirac operators for Fermi particles with electric charges $\pm e/2$ and effective mass $M=\sqrt{\sigma^2+4|\Delta|^2}$ in the presence of a constant external magnetic field H. Similar expressions were calculated in a lot of papers [see, e.g., [24], from which it follows that the first term in Eq. (27) is equal to the second one]. So, we omit details of $\mathrm{tr}\ln D_\pm$ calculations and present at once the corresponding effective potential of the model

$$
\begin{aligned}
V_{H_2}(\sigma,\Delta,\Delta^*)=&\frac{\sigma^2}{2G_1}+\frac{\Delta\Delta^*}{G_2}+\frac{eH}{6\pi^{3/2}}\int_0^\infty\frac{ds}{s^{3/2}}\\
&\times\exp(-s(\sigma^2+4|\Delta|^2))\coth(eHs/2)\\
&+\frac{4i}{3}\int\frac{d^3k}{(2\pi)^3}\ln(\sigma^2-k^2).
\end{aligned}
\tag{28}
$$

In this formula eH has a positive value. Both integrals in Eq. (28) are ultraviolet divergent ones. To renormalize the first integral one can act in the same manner, as it was done in [18] with the effective potential of the three-dimensional Gross-Neveu model in the presence of an external magnetic field. The second integral in Eq. (28) was already renormalized in Sec. III. Hence, the finite expression for the effective potential of the model (1) in an external chromomagnetic field of type II looks like

$$
\begin{aligned}
V_{H_2}(\sigma,\Delta,\Delta^*)=&V_0(\sigma,\Delta,\Delta^*)+\frac{eH}{6\pi^{3/2}}\int_0^\infty\frac{ds}{s^{3/2}}\\
&\times\exp(-s(\sigma^2+4|\Delta|^2))\\
&\times\left[\coth\left(\frac{eHs}{2}\right)-\frac{2}{eHs}\right],
\end{aligned}
\tag{29}
$$

where $V_0(\sigma,\Delta,\Delta^*)$ denotes the effective potential at $H=0$ [see Eq. (15)]. Integrating in this formula with the help of integral tables [25], one can get the following more compact expression for the effective potential:

$$
\begin{aligned}
V_{H_2}(\sigma,\Delta,\Delta^*)=&\frac{\sigma^2}{2g}+\frac{\Delta\Delta^*}{f}+\frac{2}{9\pi}|\sigma|^3+\frac{eH\sqrt{\sigma^2+4|\Delta|^2}}{3\pi}\\
&-\frac{2(eH)^{3/2}}{3\pi}\zeta\left(-\frac{1}{2},\frac{\sigma^2+4|\Delta|^2}{eH}\right),
\end{aligned}
\tag{30}
$$

where $\zeta(\nu,x)$ is the generalized Riemann zeta function [26]. As in the previous section, let us further introduce the dimensionless function $V_2(x,y)=3\pi(eH)^{-3/2}V_{H_2}(\sigma,\Delta,\Delta^*)$:

$$
\begin{aligned}
V_2(x,y)=&\frac{\tilde{A}x^2}{2}+\frac{\tilde{B}y^2}{2}+\frac{2}{3}|x|^3+\sqrt{x^2+y^2}\\
&-2\zeta(-1/2,x^2+y^2),
\end{aligned}
\tag{31}
$$

where $x=\sigma/\sqrt{eH}$, $y=2|\Delta|/\sqrt{eH}$ and $\tilde{A},\tilde{B}$ are the same parameters as in the relation (22). Clearly, this expression differs from the corresponding quantity (22) for a non-Abelian background field.

TABLE I. The A-dependence of $H_c(A,B)$ for some fixed values of the ratio B/A.

B/A	100	10	2	1.5	1.1	1.05
$eH_c(A,B)/A^2$	112925.63	967.55	15.804	5.1038	0.6593	0.3508

The reader, who is not interested in following the details of our investigation of a global minimum point for $V_2(x,y)$, can at once look at the phase portrait of the model in terms of $\tilde{A},\tilde{B}$. Qualitatively it is the same as the phase portrait for the function $V_1(x,y)$ (see Fig. 2), i.e., it contains only three different phases II, III, and IV. Details of calculations are quoted in Appendix C.

VI. CHROMOMAGNETIC CATALYSIS OF COLOR SUPERCONDUCTIVITY

Let us now analyze in more detail the phase portrait of the model (1), this time in terms of A,B,eH. In particular, we shall describe phase transitions which occur for arbitrary fixed A,B and with varying values of H. In general, our discussions concern both cases with vector potentials of types I and II simultaneously. However, where it is necessary, we indicate to what type of vector-potential the information is related. First of all a general remark: if A,B are fixed and H is varied from 0 to ∞, then in the plane $(\tilde{A},\tilde{B})$ one moves along some ray (which depends on A,B) from infinity to the origin (this fact simply follows from the definition of $\tilde{A},\tilde{B}$).

The case $A,B>0$. (In this case we have a weak coupling for both bare constants $G_{1,2}<G_c\sim\pi^2/\Lambda$.) At $H=0$ this choice of parameters corresponds to the unbroken phase I (see Fig. 1). If A,B are fixed in such a way, that $A>B$ (in terms of bare coupling constants this means $1/G_1>1/2G_2+4\Lambda/3\pi^2$), then at $H\neq 0$ we have in the $(\tilde{A},\tilde{B})$ plane of Fig. 2 a ray which is located above the line $\tilde{l}_1$, i.e., is in phase III. Hence, in this case the external chromomagnetic field induces (catalyzes) the dynamical generation of a nonzero diquark condensate. Here in the point $H_c=0_+$ one has a second order phase transition from phase I to the phase with color superconductivity. At varying values of H the diquark condensate behaves, e.g., in the case with vector-potential II, in the following way:[7]

$$\langle qq\rangle\sim feH \quad \text{at} \quad H\to 0, \quad \langle qq\rangle\sim\sqrt{eH} \quad \text{at} \quad H\to\infty. \tag{32}$$

(The chiral condensate in this case is identically zero.)

If $A<B$ and the external chromomagnetic field H varies in the interval $H\in(0,\infty)$, then points in the $(\tilde{A},\tilde{B})$ plane vary along a ray r (see Fig. 2). If $H\to 0_+$, we are at the infinite end of this ray, i.e., in phase II of the model (see Fig. 2). Hence, in the point $H_c=0_+$ the chromomagnetic field induces a dynamical chiral symmetry breaking phase transition (a second order phase transition), but a diquark condensate is not produced. These properties of the vacuum are not changed for sufficiently small values of H such that $H<H_c(A,B)$. The value $H=H_c(A,B)$ corresponds to point o (see Fig. 2) in which ray r crosses the line $\tilde{l}_2$ and passes to the Ω region where chiral and diquark condensates are both nonvanishing. So, with growing values of H in some point $H_c(A,B)$ one has a second order phase transition from phase II to the mixed phase IV. In Table I the results of a numerical investigation of $H_c(A,B)$ as a function of A are presented for some fixed values of B/A in the case of a vector potential of type I. One can see from this table that if the ratio B/A is fixed, then $H_c(A,B)$ increases with A as A^2. It is also clear that for each fixed value of A the quantity $H_c(A,B)$ is a growing function of B.

The behavior of condensates in the case $A<B$ and for a vector-potential II are the following:

$$\langle\bar{q}q\rangle\sim geH \quad \text{at} \quad H\to 0,$$
$$\langle\bar{q}q\rangle\sim\langle\sigma\rangle\equiv\frac{(B-A)}{2} \quad \text{at} \quad H_c(A,B)\leqslant H, \tag{33}$$

$$\langle qq\rangle\equiv 0 \quad \text{at} \quad H\leqslant H_c(A,B),$$
$$\langle qq\rangle\sim\sqrt{eH} \quad \text{at} \quad H\to\infty. \tag{34}$$

So, at $H\neq 0$ phase I of the model is completely absent in the phase structure of the model for both types of external chromomagnetic fields.

The case $A<0,2A<3B$. In this case at $H=0$ one has a phase II of the theory (see Fig. 1) with spontaneously broken chiral symmetry. If the external chromomagnetic field H varies in the interval $H\in(0,\infty)$, then in the $(\tilde{A},\tilde{B})$ plane there is a ray which crosses the line $\tilde{l}_2$ in some definite point. If $H\to 0_+$, we are in the infinite end of this ray, i.e., in the phase II of the model. However, in contrast to the previous case with $A,B>0$ and $A<B$, in the present case the value $H=0_+$ is no more the point of a phase transition. (At $A<0$, when the bare coupling constant G_1 takes a supercritical value $G_1>G_c$, the origin of chiral symmetry breaking is the rather strong supercritical quark-antiquark attraction, but not the chromomagnetic field. In this case the external chromomagnetic field only stabilizes the vacuum with chiral symmetry breaking [15,18].) If H increases, we move along this ray to the origin of the $(\tilde{A},\tilde{B})$ plane. Hence, starting from some value $H_c(A,B)$, we are in region Ω (see Fig. 2), where

[7]In order to find $\langle qq\rangle\sim\langle\Delta\rangle$ and $\langle\bar{q}q\rangle\sim\langle\sigma\rangle$, one should multiply the coordinates of the global minimum point of the functions $V_1(x,y)$ or $V_2(x,y)$ (see Appendixes A and B, correspondingly) by the quantity $\sqrt{eH}$. For example, in the case under consideration $\langle qq\rangle\sim\langle\Delta\rangle=y_0(\tilde{B})\sqrt{eH}/2$, where the function $y_0(\tilde{B})$ and some of its properties are presented in formula (B4) of Appendix B.

besides $\langle\bar{q}q\rangle\neq 0$ the diquark condensate is nonzero as well. So, at sufficiently high values of $H>H_c(A,B)$ phase II of the theory is transformed into a mixed phase IV.

The influence of vector-potential II on the chiral phase II of the model is realized in the following behavior of condensates:

$$\langle\bar{q}q\rangle\sim\langle\sigma\rangle=-\frac{A}{6} \quad \text{at} \quad H\to 0,$$

$$\langle\bar{q}q\rangle\sim\langle\sigma\rangle\equiv\frac{(B-A)}{2} \quad \text{at} \quad H_c(A,B)\leqslant H, \tag{35}$$

$$\langle qq\rangle\equiv 0 \quad \text{at} \quad H\leqslant H_c(A,B),$$

$$\langle qq\rangle\sim\sqrt{eH} \quad \text{at} \quad H\to\infty. \tag{36}$$

The case $B<0,A>B$. In this case at $H=0$ there is a perfect (not mixed) color superconducting phase III of the theory (see Fig. 1). One can easily show that now for all values of H only the diquark condensate $\langle qq\rangle$ is nonzero. This vacuum is chirally invariant, but the $U_{em}(1)$ as well as color $SU_c(3)$ symmetries are broken down. It is possible to show that in this case

$$\langle qq\rangle\sim\langle\Delta\rangle=-B/4 \quad \text{at} \quad H\to 0, \quad \langle qq\rangle\sim\sqrt{eH}, \quad \text{at} \quad H\to\infty, \tag{37}$$

i.e., the external chromomagnetic field even enhances the color superconductivity.

The case $A<B$, $2A>3B$. Analyzing here the behavior of the quark condensates in a similar way as in the previous cases, one can easily establish that the vacuum properties are not changed with growing values of H. Hence, at $H=0$ as well as at $H\neq 0$ there is a mixed phase IV with nonzero quark and diquark condensates. Remark that the action of an external chromomagnetic field on the mixed phase does not change the value of the chiral condensate; it is the same as at $H=0$, where $\langle\bar{q}q\rangle\sim\langle\sigma\rangle\equiv(B-A)/2$. However, the diquark condensate depends on the value of H,

$$\langle qq\rangle\sim\langle\Delta\rangle=\sqrt{B^2/16-(B-A)^2/4} \quad \text{at} \quad H\to 0,$$

$$\langle qq\rangle\sim\sqrt{eH}, \quad \text{at} \quad H\to\infty. \tag{38}$$

In conclusion, let us remark that for arbitrary fixed parameters A,B and in the presence of sufficiently large values of external chromomagnetic fields of both types there arises a nonzero diquark condensate $\langle qq\rangle\neq 0$, i.e., the color superconducting phase of the model is realized. If $A>3B/2$, $B<0$ (i.e., for sufficiently high values of $G_2>G_c$), then $\langle qq\rangle\neq 0$ even at $H=0$ (in this case the external chromomagnetic field enhances the CSC). However, for other regions of the (A,B) plane the nonzero external chromomagnetic fields catalyze the generation of $\langle qq\rangle\neq 0$. The critical value of H, at which color superconductivity is induced, may be 0_+ (if $A,B>0$, $A>B$), or some finite value $H_c(A,B)\neq 0$ (in the last case we have not a perfect, but mixed color superconducting phase in which the diquark condensate coexists with the chiral condensate $\langle\bar{q}q\rangle\neq 0$). Apart from this, in the presence of a chromomagnetic field, the phase portrait of the model does not contain the symmetric phase.

VII. SUMMARY AND CONCLUSIONS

In the present paper the phase structure of a $(2+1)$-dimensional four-fermionic NJL-type of model (1) with two coupling constants was investigated admitting nonzero background vector potentials of two nonequivalent types I and II [see Eqs. (8), (9)]. In the framework of such a model the external vector potential might be thought to simulate such a nonperturbative feature of the real QCD vacuum like a nonzero gluon condensate $\langle FF\rangle=H^2$. The structure of the Lagrangian (1) permits us, in particular, to consider then the competition between chiral $\langle\bar{q}q\rangle$ and diquark $\langle qq\rangle$ condensates and to get some insight into the role of the gluon condensate as a possible catalyst of color superconductivity.

It is well-known that color-superconducting quark matter with two quark flavors arises by the condensation of color antitriplet diquark Cooper pairs. The condensate breaks the $SU_c(3)$ symmetry down to $SU_c(2)$. Hence, the three gluon fields corresponding to the generators of unbroken $SU_c(2)$ stay massless and the remaining five gluon fields receive a mass by the Higgs mechanism (Meissner effect). We have studied the influence of the external chromomagnetic fields living in an unbroken $SU_c(2)$ subgroup of $SU_c(3)$, i.e., having a form $H^a=(H^1,H^2,H^3,0,\ldots,0)$, on the formation of the color diquark condensate. Using global $SU_c(2)$ color rotations one can bring this field to the form $H^a=(0,0,H,0,\ldots,0)$ which corresponds to the above-mentioned vector-potentials Eqs. (8), (9).

The main conclusion from our investigations is that at zero chemical potential the external chromomagnetic fields of these type are good catalysts of color superconductivity. Earlier, it was shown that external (chromo) magnetic fields catalyze dynamically the spontaneous breaking of chiral symmetry in some $(2+1)$-dimensional four-fermionic models [24,27,15,18]. It turns out that this is a particular manifestation of the so-called magnetic catalysis effect (see, e.g., [28,17,29]), which has a rather universal model independent character.

Indeed, we have shown that for sufficiently small bare coupling constants $G_{1,2}<G_c\sim\pi^2/\Lambda$, i.e., for such values of $G_{1,2}$ at which for $H=0$ one has a symmetric phase I of the theory (see Fig. 1), the pure CSC phase ($\langle\bar{q}q\rangle=0$, $\langle qq\rangle\neq 0$) is realized in the model at infinitesimally small values of the external chromomagnetic field H if $G_2>G_1$ (in terms of A,B that means $A>B$). If $G_2<G_1$ ($A<B$), then, first, a chiral breaking phase transition induced at $H=0_+$ (chromomagnetic catalysis of chiral symmetry breaking at which $\langle\bar{q}q\rangle\neq 0$, $\langle qq\rangle=0$) occurs. After that, with growing values of H, at some point $H=H_c$ there is a second phase transition to the mixed phase of the theory, where both condensates $\langle\bar{q}q\rangle$ and $\langle qq\rangle$ are nonzero (both phase transitions are continuous second order ones).

The action of an external chromomagnetic field on the chiral phase II of the theory (see Fig. 1) is to induce $\langle qq\rangle$

$\neq 0$ at some critical point $H=H_c\neq 0$, thus drastically changing the vacuum properties and transposing the system into a mixed phase IV.

Finally, we should mention that the ground states of phases III and IV are not changed under the influence of the above-mentioned external chromomagnetic field. So, the external chromomagnetic fields living in an unbroken $SU_c(2)$ subgroup of $SU_c(3)$ only enhances the CSC phenomenon. Notice that all the above-mentioned effects are observed in the presence of both types of vector-potentials I and II.

Let us recall that the chromomagnetic catalysis effect of dynamical chiral symmetry breaking occurs in $(2+1)$- as well as $(3+1)$-dimensional NJL models [15,17]. So, the very existence of this effect is not sensitive to the dimensionality of space-time. Moreover, there are many physical phenomena in QCD, in particular in low energy meson physics, that can be reasonably described both in the framework of $(3+1)$- and $(2+1)$-dimensional NJL type models [30,13]. On these grounds one might thus expect that chromomagnetic catalysis of color superconductivity is inherent to the four-dimensional version of model (1), and hence to the real two-flavor QCD_4, too. The proof of this fact is the subject of our nearest future considerations.

In our opinion there exists a deep connection between the chromomagnetic catalysis of color superconductivity and chiral symmetry breaking, induced by external chromomagnetic fields [15–18]. This assumption is based on the existence of the Pauli-Gürsey (PG) transformation [31], mixing quarks and antiquarks, due to which some phenomena in the $\bar{q}q$ channel can have its analogy in the qq channel. In particular, this suggests that diquark condensation might partly be understood as the properly PG-transformed chiral condensation. However, the detailed consideration of this question is not the subject of the present paper and will be investigated elsewhere.

Moreover, in the near future we are going to include into our consideration of the simple NJL model (1) a nonzero chemical potential μ in addition to the external chromomagnetic fields. Recently, in the framework of NJL models [32] the influence of μ and an external magnetic field on the chiral properties of the vacuum were considered. Apart from discovering different kinds of magnetic oscillations (relativistic van Alphen-de Haas effect) in the strongly interacting quark systems, it was also found that in the NJL model at nonzero baryon density the chiral symmetry must be restored at sufficiently large values of the magnetic field. By analogy with [32], we expect that the diquark condensate $\langle qq\rangle$ should disappear in the case of nonzero baryon density for a sufficiently strong external chromomagnetic field, i.e., at large values of the gluon condensate.

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APPENDIX A: ALGEBRA OF THE γ MATRICES AND FLAVOR SYMMETRY FOR $D=(2+1)$

The two-dimensional irreducible representation of the three-dimensional Lorentz group SO(2,1) is realized by the following 2×2 $\tilde{\gamma}$ matrices:

$$\tilde{\gamma}^0=\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \tilde{\gamma}^1=i\sigma_1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$
$$\tilde{\gamma}^2=i\sigma_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{A1}$$

acting on two-component Dirac spinors.

They have the properties

$$\mathrm{Tr}(\tilde{\gamma}^\mu\tilde{\gamma}^\nu)=2g^{\mu\nu},\quad [\tilde{\gamma}^\mu,\tilde{\gamma}^\nu]=-2i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha,$$
$$\tilde{\gamma}^\mu\tilde{\gamma}^\nu=-i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha+g^{\mu\nu}, \tag{A2}$$

where $g^{\mu\nu}=g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$, $\tilde{\gamma}_\alpha=g_{\alpha\beta}\tilde{\gamma}^\beta$, $\varepsilon^{012}=1$. There is also the relation

$$\mathrm{Tr}(\tilde{\gamma}^\mu\tilde{\gamma}^\nu\tilde{\gamma}^\alpha)=-2i\varepsilon^{\mu\nu\alpha}. \tag{A3}$$

Note that the definition of chiral symmetry is a bit unusual in three dimensions [here spin is a pseudoscalar rather than a (axial) vector]. The reason is simply that there exists no other 2×2 matrix anticommuting with the Dirac matrices $\tilde{\gamma}^\nu$ which would allow the introduction of a γ^5 matrix in the irreducible representation. The important concept of ''chiral'' symmetries and their breakdown by mass terms can nevertheless be realized also in the framework of $(2+1)$-dimensional quantum field theories by considering a four-component reducible representation for Dirac fields. In this case the Dirac spinors q have the following form:

$$q(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}, \tag{A4}$$

with ψ_1,ψ_2 being two-component spinors. In the reducible four-dimensional spinor representation one deals with (4×4) γ matrices $\gamma^\mu=\mathrm{diag}(\tilde{\gamma}^\mu,-\tilde{\gamma}^\mu)$, where $\tilde{\gamma}^\mu$ are given in Eq. (A1). One can easily show that $(\mu,\nu=0,1,2)$,

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu)=4g^{\mu\nu},\quad \gamma^\mu\gamma^\nu=\sigma^{\mu\nu}+g^{\mu\nu},$$
$$\sigma^{\mu\nu}=\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]$$
$$=\mathrm{diag}(-i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha,-i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha). \tag{A5}$$

In addition to the Dirac matrices γ^μ $(\mu=0,1,2)$ there exist two other matrices γ^3, γ^5 which anticommute with all γ^μ $(\mu=0,1,2)$ and with themselves,

$$\gamma^3=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3=i\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix},\qquad \text{(A6)}$$

with I being the unit 2×2 matrix. In the space of four-component spinors (A4) it is now possible to consider ''chiral'' transformations

$$q\to\exp(i\theta\gamma^3)q,\quad q\to\exp(i\omega\gamma^5)q,$$

as well as the discrete parity transformation P,

$$\mathrm{P}:\quad q(t,x,y)\to -i\gamma^5\gamma^1 q(t,-x,y).\qquad \text{(A7)}$$

In the present work the charge conjugation matrix C for four-component spinors was chosen to be γ^2. With this choice of C it is possible to show that quark bilinears in Eq. (1) obey the relations

$$\mathrm{P}[\bar{q}q]\mathrm{P}^{-1}(t,x,y)=[\bar{q}q](t,-x,y),$$

$$\mathrm{P}[\bar{q}\vec{\tau}\gamma^5 q]\mathrm{P}^{-1}(t,x,y)=-[\bar{q}\vec{\tau}\gamma^5 q](t,-x,y),\qquad \text{(A8)}$$

$$\mathrm{P}[i\overline{q^C}\varepsilon\epsilon^b\gamma^5 q]\mathrm{P}^{-1}(t,x,y)=[i\overline{q^C}\varepsilon\epsilon^b\gamma^5 q](t,-x,y).$$

It follows from Eqs. (3) and (A8) that σ and Δ^b are indeed scalar fields, whereas $\vec{\pi}$ fields are pseudoscalars.

Let us for a moment consider the case of one flavor, $\mathrm{N}_f=1$. The 4×4-matrices $\{\mathbf{1},\gamma^5,\gamma^3,\gamma^{3,5}=i\gamma^5\gamma^3\}$ are the unit matrix and Pauli matrices in block form and as such they generate the $\mathrm{U}(2)\simeq\mathrm{SU}(2)\times\mathrm{U}(1)$ group of transformations of the four-dimensional spinor (A4). Notice that with respect to SU(2) the set of quark bilinears $\{\bar{q}q,\ \bar{q}i\gamma^3 q,\ \bar{q}i\gamma^5 q\}$ transforms as a triplet, whereas $\bar{q}i\gamma^5\gamma^3 q$ is a SU(2) singlet. Analogously, diquarks form a triplet, $\{\overline{q^c}q,\ \overline{q^c}i\gamma^5 q,\ \overline{q^c}i\gamma^5\gamma^3 q\}$ (since $\mathrm{C}i\gamma^3$ is a symmetric matrix, the product with Grassman spinors vanishes, and a singlet is excluded). Obviously, for N_f flavors any of the N_f flavor components of q is a Dirac 4 spinor, and the total flavor group is $\mathrm{U}(2\mathrm{N}_f)$. Its respective Lie algebra is given by (direct) products of generators of $\mathrm{U}(\mathrm{N}_f)$ and U(2). In particular, for the case of two-flavors $\mathrm{N}_f=2$ considered in the text, the symmetry group of the kinetic part of the Lagrangian is U(4). Finally, let us mention that in $\mathrm{D}=(2+1)$ there might arise two possible mass terms, $m\bar{q}q$ and $\tilde{m}\bar{q}i\gamma^5\gamma^3 q$. The first (standard) one is P invariant, but breaks the total U(4) flavor symmetry down according to $\mathrm{U}(4)\to\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$, with generators of the residual group given by $\frac{1}{2}\tau^a\mathbf{1},\ \frac{1}{2}\tau^a i\gamma^5\gamma^3$ $(a=0,1,2,3)$. Note that this symmetry breaking scenario includes the breaking of the ''usual chiral'' γ^5 invariance, as considered in corresponding $\mathrm{D}=(3+1)$ models. Contrary to this, the other possible mass term leaves the flavor symmetry unbroken, but instead violates parity and will thus not be considered here.

APPENDIX B: INVESTIGATION OF THE GLOBAL MINIMUM POINT OF $V_1(x,y)$

It follows from Eq. (24) in the text that the function $F(z)$ monotonically increases on the interval $z\in(0,\infty)$ and $F(0_+)=-\infty$, $F(+\infty)=+\infty$. Hence, for arbitrary fixed values of $\tilde{A},\tilde{B}$ there exist only two real numbers $x_0(\tilde{A})>0$, $y_0(\tilde{B})>0$, such that the two pairs $(x_0,0)$, $(0,y_0)$ as well as the trivial one (0,0) are solutions for the system of stationarity equations (23). [The $x_0(\tilde{A})$ and $y_0(\tilde{B})$ are zeros of the functions which are located inside the first (at $y=0$) and second (at $x=0$) pair of braces in Eq. (23), respectively.] Furthermore, one can easily see that $\partial V_1/\partial x<0$ if $y=0$ and $x\in(0,x_0(\tilde{A}))$ as well as $\partial V_1/\partial y<0$ if $x=0$ and $y\in(0,y_0(\tilde{B}))$. This means that the quantities $V_1(x_0,0)$ and $V_1(0,y_0)$ are smaller than $V_1(0,0)$. So, for arbitrary finite real values $\tilde{A},\tilde{B}$ the global minimum point of the function $V_1(x,y)$ cannot lie in (0,0). Due to this reason the symmetric phase is absent in the phase structure of the model.

In the next formulas some properties of $x_0(\tilde{A})$ and $y_0(\tilde{B})$ are presented:

$$x_0(\tilde{A})\cong 2e^{-\tilde{A}-2}\quad\text{at}\quad \tilde{A}\to+\infty,$$

$$x_0(0)=0.147\ldots,\qquad \text{(B1)}$$

$$x_0(\tilde{A})\cong-\tilde{A}/6\quad\text{at}\quad \tilde{A}\to-\infty,$$

$$y_0(\tilde{B})\cong 2e^{-\tilde{B}-2}\quad\text{at}\quad \tilde{B}\to+\infty,$$

$$y_0(0)=0.183\ldots,\qquad \text{(B2)}$$

$$y_0(\tilde{B})\cong-\tilde{B}/4\quad\text{at}\quad \tilde{B}\to-\infty.$$

It follows from Eq. (23) that there may exist (but not for all values of $\tilde{A},\tilde{B}$) one more solution $(x_1(\tilde{A},\tilde{B}),y_1(\tilde{A},\tilde{B}))$ of the stationarity equations, where $x_1>0, y_1>0$. [For this solution the functions which are inside both braces in Eq. (23) take zero values.] Evidently, we have

$$x_1(\tilde{A},\tilde{B})=(\tilde{B}-\tilde{A})/2,\quad y_1(\tilde{A},\tilde{B})=\sqrt{y_0^2(\tilde{B})-(\tilde{B}-\tilde{A})^2/4}.\qquad \text{(B3)}$$

From Eq. (B3) one can easily see that this type of solution for Eqs. (23) exists inside the region Ω of the $(\tilde{A},\tilde{B})$ plane (see also Fig. 2),

$$\Omega=\{(\tilde{A},\tilde{B}):\tilde{B}-2y_0(\tilde{B})<\tilde{A}<\tilde{B}\}.\qquad \text{(B4)}$$

Using Eqs. (23) one can find the following values of the potential (22) in its stationary points:

$$V_1(x_0(\tilde{A}),0)=\frac{\sqrt{1+x_0^2(\tilde{A})}}{6}[1-2x_0^2(\tilde{A})]-\frac{2}{3}x_0^3(\tilde{A}),\qquad \text{(B5)}$$

$$V_1(0,y_0(\tilde{B}))=\frac{\sqrt{1+y_0^2(\tilde{B})}}{6}[1-2y_0^2(\tilde{B})]-\frac{1}{3}y_0^3(\tilde{B}), \tag{B6}$$

$$V_1(x_1(\tilde{A},\tilde{B}),y_1(\tilde{A},\tilde{B})) = \frac{\sqrt{1+y_0^2(\tilde{B})}}{6}[1-2y_0^2(\tilde{B})] - \frac{1}{3}y_0^3(\tilde{B})-\frac{(\tilde{B}-\tilde{A})^3}{24}. \tag{B7}$$

On the line $\tilde{l}_1=\{(\tilde{A},\tilde{B}):\tilde{A}=\tilde{B}\}$, which is a part of the boundary for the region Ω, we have $(x_1,y_1)\equiv(0,y_0)$. Hence, on this line $V_1(0,y_0)\equiv V_1(x_1,y_1)$. Comparing Eqs. (B6) and (B7), we see that inside the Ω-region $V_1(0,y_0)>V_1(x_1,y_1)$.

The other part of the boundary for the region Ω is the line $\tilde{l}_2=\{(\tilde{A},\tilde{B}):\tilde{A}=\tilde{B}-2y_0(\tilde{B})\}$. With the help of the stationarity equations it is possible to show that on this line the following relations are also fulfilled: $\tilde{B}=\tilde{A}+2x_0(\tilde{A})$, $x_0(\tilde{A})=y_0(\tilde{B})$. As a consequence, we have $(x_1,y_1)\equiv(x_0,0)$ as well as $V_1(x_0,0)\equiv V_1(x_1,y_1)$ on the line $\tilde{l}_2$.

Numerical investigations show that inside the region Ω there is a line on which $V_1(x_0,0)=V_1(0,y_0)$. Further, it is important to remark that the derivative of the function $V_1(0,y_0(\tilde{B}))$ with respect to $\tilde{B}$ as well as the corresponding partial derivative of the function $V_1(x_1(\tilde{A},\tilde{B}),y_1(\tilde{A},\tilde{B}))$ are positively defined quantities in their regions of definition. Now, taking into account all the above mentioned facts, it is possible to assert that in Fig. 2 the phase portrait of the model (1) in the presence of a nonzero vector potential (8) is presented in terms of $\tilde{A}$ and $\tilde{B}$. This means that above the line $\tilde{l}_1$ there is phase III of the model, which corresponds to the $V_1(x,y)$ global minimum point of the form $(0,y_0)$. (In this case $\langle\sigma\rangle=0$, $\langle\phi\rangle\neq0$.) Below the curve $\tilde{l}_2$ the effective potential global minimum has the form $(x_0,0)$. So in this region phase II is located, since for such values of $\tilde{A},\tilde{B}$ the model has a vacuum with $\langle\sigma\rangle\neq0$, $\langle\phi\rangle=0$. Finally, inside the Ω domain there is a mixed phase IV, since here the global minimum point (x_1,y_1) corresponds to the vacuum with both nonzero condensates $\langle\bar{q}q\rangle$ and $\langle qq\rangle$.

APPENDIX C: INVESTIGATION OF THE GLOBAL MINIMUM POINT OF $V_2(x,y)$

In this appendix we present the search of the global minimum point for the potential $V_2(x,y)$ as well as its dependence on the parameters $\tilde{A},\tilde{B}$. Since this function is symmetric under two discrete transformations $x\to-x$ and $y\to-y$, it is sufficient to study it only in the region $x,y\geq0$. The stationarity equations for $V_2(x,y)$ take the form

$$\partial V_2(x,y)/\partial x\equiv x\{\tilde{A}+2x+(x^2+y^2)^{-1/2} -2\zeta(1/2,x^2+y^2)\}=0, \tag{C1}$$

$$\partial V_2(x,y)/\partial y\equiv y\{\tilde{B}+(x^2+y^2)^{-1/2} -2\zeta(1/2,x^2+y^2)\}=0. \tag{C2}$$

One can see from Eqs. (C1) and (C2) that the first derivatives of V_2 do not exist in the point (0,0). [This means that if the point (x,y) tends to the origin along different ways, the resulting expressions for the partial derivatives at the point (0,0) do not coincide.] In contrast, the function $V_1(x,y)$ is differentiable in the point (0,0). So, we need a special investigation of this point. Let us put $y=0$ in the equation (C1). Then, using properties of the $\zeta(\nu,x)$ function [26], it is easily seen that at $y=0$ and $x\to0_+$ the partial derivative $\partial V_2/\partial x$ tends to (-1). Analogously, at $x=0$ and $y\to0_+$ the derivative $\partial V_2/\partial y$ (C2) tends to (-1) as well. This fact means that for arbitrary values of $\tilde{A},\tilde{B}$ the point (0,0) cannot be a global minimum for the potential $V_2(x,y)$. So, in contrast to the case with $H=0$, the ground state with intact initial symmetry is no more possible in the model (1) at $H\neq0$. Such a property of the effective potential is a characteristic feature for a phenomenon which is called (chromo) magnetic catalysis of dynamical symmetry breaking. According to this effect the external (chromo)magnetic field promotes to a great extent the spontaneous breaking of initial symmetry of the theory (for more details, see the last section of the present paper).

Similar to the case with non-Abelian vector-potential of type I, in the present consideration it is possible to show that for arbitrary values of $\tilde{A},\tilde{B}$ the stationarity equations (C1),(C2) have two solutions of the form $(x_0(\tilde{A}),0)$ and $(0,y_0(\tilde{B}))$, where

$$x_0(\tilde{A})\cong1/\tilde{A}\ \text{ at }\ \tilde{A}\to+\infty,\quad x_0(\tilde{A})\cong-\tilde{A}/6\ \text{ at }\ \tilde{A}\to-\infty, \tag{C3}$$

$$y_0(\tilde{B})\cong1/\tilde{B}\ \text{ at }\ \tilde{B}\to+\infty,\quad y_0(\tilde{B})\cong-\tilde{B}/4\ \text{ at }\ \tilde{B}\to-\infty. \tag{C4}$$

[Here and in the following discussions of the present appendix we use the same notations $x_0(\tilde{A})$ and $y_0(\tilde{B})$ for the solutions of stationarity equations as in the previous Appendix B. But one should remember that these functions have quite different numerical values than similar functions had in Appendix B.]

From Eqs. (C1),(C2) it follows that only for $(\tilde{A},\tilde{B})\in\Omega$, where Ω is defined formally in Eq. (B4), there is a solution of the form $(x_1(\tilde{A},\tilde{B}),y_1(\tilde{A},\tilde{B}))$, where $x_1>0$, $y_1>0$. These functions are given in Eq. (B3). There are no other solutions of the stationarity equations.

Using numerical and analytical methods, it is now possible to compare the values of the effective potential in its stationary points and thus to find the global minimum point of V_2 as well as its dependence on the parameters $\tilde{A},\tilde{B}$ of the theory. We omit the details of this investigation and present only the results, which can be formulated in the form of a phase portrait of the model in terms of $\tilde{A},\tilde{B}$.

It turns out that Fig. 2, which is the phase portrait of the model for nonzero vector-potential I, formally may serve as a phase portrait of the model for nonzero external gauge fields of type II as well. In both cases the line $\tilde{l}_1$ separates the color superconducting phase III from the mixed phase IV. Further, in both cases, phase IV is separated from the chiral phase II by the line $\tilde{l}_2$, which has the same analytic definition through $x_0(\tilde{A})$ and $y_0(\tilde{B})$ (see the figure caption to Fig. 2). Moreover, the leading asymptotic behaviors of the $\tilde{l}_2$ curves are in both cases identical (at $\tilde{A}\to\infty$ we have instead the line $\tilde{l}_1$, at $\tilde{A}\to-\infty$ it is the line $2\tilde{A}=3\tilde{B}$). Of course, since for type I and type II vector potentials the functions $x_0(\tilde{A})$ and $y_0(\tilde{B})$ obey different stationarity equations, line $\tilde{l}_2$ of case I does not coincide with line $\tilde{l}_2$ of case II. Finally, we should stress again that for both types of nonzero vector-potentials I and II, the symmetric phase I of the theory, which is present in the phase structure of the model at $H=0$ (see Fig. 1), is absent at all.

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